Testing general relativity with the Galactic central black hole

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based on a collaboration with
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Séminaire
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Outline

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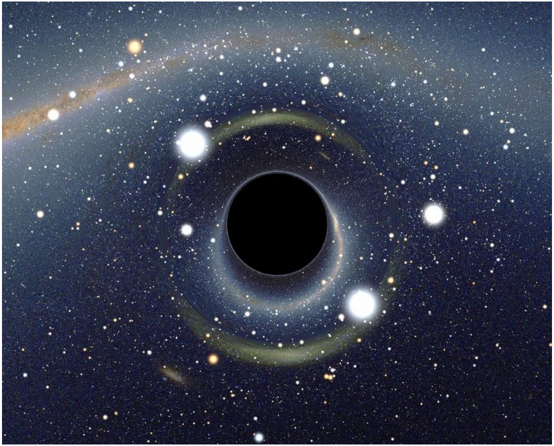
What is a black hole ?



[Alain Riazuelo, 2007]

... for the layman :

A **black hole** is a region of spacetime from which nothing, not even light, can escape.

The (immaterial) boundary between the black hole interior and the rest of the Universe is called the **event horizon**.

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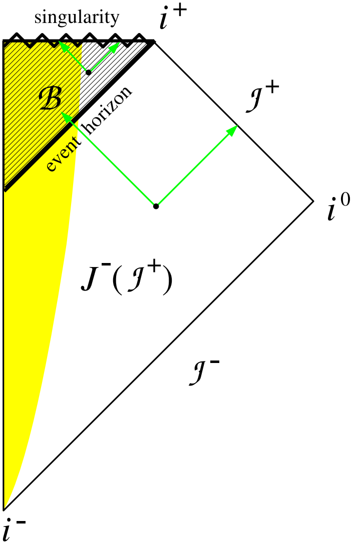

Textbook definition [Hawking & Ellis (1973)]

black hole : $\mathcal{B} := \mathscr{M} - J^-(\mathscr{I}^+)$

where

- $(\mathscr{M}, \boldsymbol{g})$ = asymptotically flat manifold
- $\mathscr{I}^+$ = (complete) future null infinity
- $J^-(\mathscr{I}^+)$ = causal past of $\mathscr{I}^+$

i.e. black hole = region of spacetime from which light rays cannot escape to infinity

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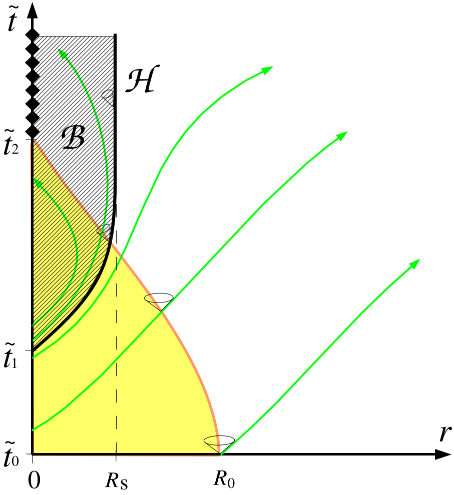

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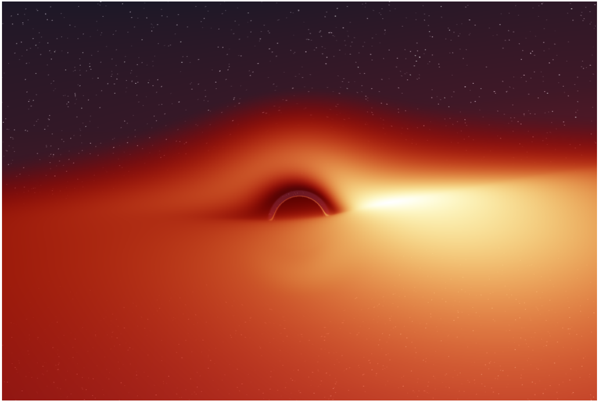
What is a black hole ?

... for the astrophysicist : a very deep gravitational potential well

Release of potential gravitational energy by **accretion** on a black hole : up to 42% of the mass-energy mc^2 of accreted matter !

NB : thermonuclear reactions release less than 1% mc^2



Matter falling in a black hole forms an **accretion disk** [Lynden-Bell (1969), Shakura & Sunayev (1973)]

[J.-A. Marck (1996)]

Main properties of black holes (1/3)

- In general relativity, a black hole contains a region where the spacetime curvature diverges : the singularity *(NB : this is not the primary definition of a black hole)*. The singularity is inaccessible to observations, being hidden by the event horizon.

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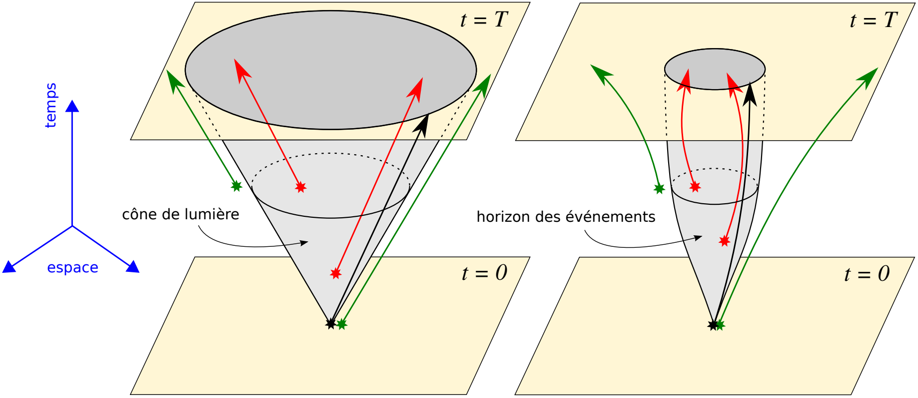
- The singularity marks the limit of validity of general relativity : to describe it, a quantum theory of gravitation would be required.
- The event horizon $\mathcal{H}$ is a global structure of spacetime : no physical experiment whatsoever can detect the crossing of $\mathcal{H}$.

Main properties of black holes (2/3)

The event horizon as a null cone



espace-temps plat

espace-temps avec trou noir

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- A black hole is not an infinitely dense object : on the contrary it is made of vacuum (except maybe at the singularity) ; if one defines its "mean density" by $\bar{\rho} = M/(4/3\pi R^3)$, then
  – for the Galactic centre BH (Sgr A*) : $\bar{\rho} \sim 10^6 \ \mathrm{kg\,m^{-3}} \sim 2\ 10^{-4}\ \rho_{\text{white dwarf}}$
  – for the BH at the centre of M87 : $\bar{\rho} \sim 2\ \mathrm{kg\,m^{-3}} \sim 2\ 10^{-3}\ \rho_{\text{water}}$!

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  $\Longrightarrow$ a black hole is a compact object : $\dfrac{M}{R}$ large, not $\dfrac{M}{R^3}$!
- Due to the non-linearity of general relativity, black holes can form in spacetimes without any matter, by collapse of gravitational wave packets.

Outline

The Kerr solution

Roy Kerr (1963)

$$g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = -\left(1-\frac{2GMr}{c^2\rho^2}\right)c^2\mathrm{d}t^2 - \frac{4GMar\sin^2\theta}{c^2\rho^2}\,c\,\mathrm{d}t\,\mathrm{d}\varphi + \frac{\rho^2}{\Delta}\,\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \left(r^2+a^2+\frac{2GMa^2r\sin^2\theta}{c^2\rho^2}\right)\sin^2\theta\,\mathrm{d}\varphi^2$$

where

$\rho^2 := r^2 + a^2\cos^2\theta$, $\Delta := r^2 - \frac{2GM}{c^2}r + a^2$ and $r \in (-\infty, \infty)$

→ spacetime manifold : $\mathscr{M} = \mathbb{R}^2 \times \mathbb{S}^2 \setminus \{r = 0\ \&\ \theta = \pi/2\}$

→ 2 parameters : M : gravitational mass ; $a := \frac{J}{cM}$ reduced angular momentum

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→ Schwarzschild solution as the subcase $a = 0$:

$$g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = -\left(1-\frac{2GM}{c^2r}\right)c^2\mathrm{d}t^2 + \left(1-\frac{2GM}{c^2r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2\right)$$

Basic properties of Kerr metric

- Asymptotically flat ($r \to \pm\infty$)
- Stationary : metric components independent from t
- Axisymmetric : metric components independent from φ
- Not static when $a \neq 0$
- Contains a black hole $\iff$ $\boxed{0 \leq a \leq m}$, where $m := GM/c^2$

  event horizon : $r = r_+ := m + \sqrt{m^2 - a^2}$
- Contains a curvature singularity at $\rho = 0 \iff r = 0$ and $\theta = \pi/2$

Physical meaning of the parameters M and J

- mass M : *not* a measure of the "amount of matter" inside the black hole, but rather a *characteristic of the external gravitational field*
  $\rightarrow$ measurable from the orbital period of a test particle in far circular orbit around the black hole *(Kepler's third law)*

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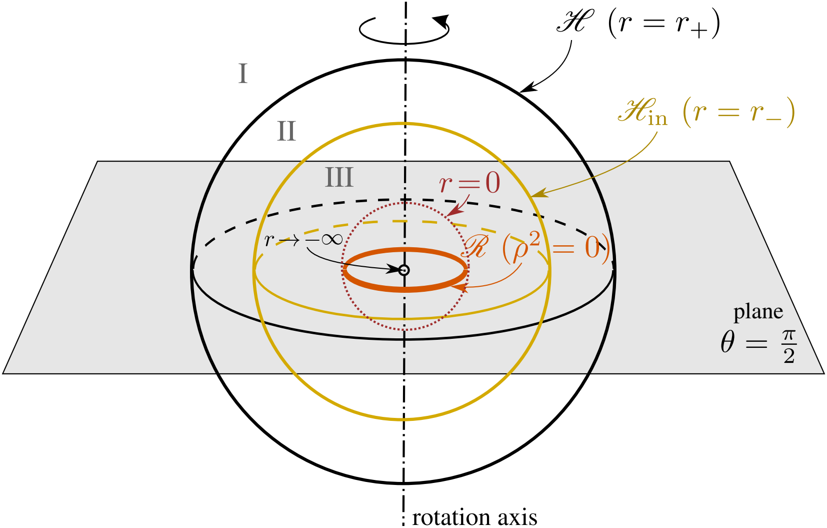
Remark : the radius of a black hole is not a well defined concept : it *does not* correspond to some distance between the black hole "centre" and the event horizon. A well defined quantity is the area of the event horizon, A.
The radius can be then defined from it : for a Schwarzschild black hole :

$$R := \sqrt{\frac{A}{4\pi}} = \frac{2GM}{c^2} \simeq 3 \left(\frac{M}{M_\odot} \right) \text{ km}$$

Kerr spacetime



Slice $t = \text{const}$ of the Kerr spacetime viewed in O'Neill coordinates (R, θ, φ), with $R := e^r$, $r \in (-\infty, +\infty)$

The no-hair theorem

Dorochkevitch, Novikov & Zeldovitch (1965), Israel (1967), Carter (1971), Hawking (1972)

Within 4-dimensional general relativity, a *stationary* black hole in an otherwise empty universe is necessarily a Kerr-Newmann black hole, which is an electro-vacuum solution of Einstein equation described by only 3 numbers :

- the total mass M
- the total specific angular momentum $a = J/(Mc)$
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Astrophysical black holes have to be electrically neutral :

- $Q = 0$: Kerr solution (1963)

Other special cases :

- $a = 0$: Reissner-Nordström solution (1916, 1918)
- $a = 0$ and $Q = 0$: Schwarzschild solution (1916)
- $a = 0$, $Q = 0$ and $M = 0$: Minkowski metric (1907)

The no-hair theorem : a precise mathematical statement

Any spacetime $(\mathscr{M}, \boldsymbol{g})$ that

- is 4-dimensional
- is asymptotically flat
- is pseudo-stationary
- is a solution of the vacuum Einstein equation : $\mathrm{Ric}(\boldsymbol{g}) = 0$
- contains a black hole with a connected regular horizon
- has no closed timelike curve in the domain of outer communications
- is analytic

has a domain of outer communications that is isometric to the domain of outer communications of the Kerr spacetime.

domain of outer communications : black hole exterior

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Possible improvements : remove the hypotheses of analyticity and non-existence of closed timelike curves (analyticity removed recently but only for slowly rotating black holes [Alexakis, Ionescu & Klainerman, Duke Math. J. **163**, 2603 (2014)])

The Kerr metric is specific to black holes

Spherically symmetric (non-rotating) bodies :

Birkhoff theorem

Within 4-dimensional general relativity, the spacetime outside any spherically symmetric body is described by Schwarzschild metric

$\Longrightarrow$ No possibility to distinguish a non-rotating black hole from a non-rotating dark star by monitoring orbital motion or fitting accretion disk spectra

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Rotating axisymmetric bodies :

No Birkhoff theorem

Moreover, no "reasonable" matter source has ever been found for the Kerr metric (the only known source consists of two counter-rotating thin disks of collisionless particles [Bicak & Ledvinka, PRL **71**, 1669 (1993)])

$\Longrightarrow$ The Kerr metric is specific to rotating black holes
(in 4-dimensional general relativity)

Lowest order no-hair theorem : quadrupole moment

Asymptotic expansion (large r) of the metric in terms of multipole moments $(\mathcal{M}_k, \mathcal{J}_k)_{k\in\mathbb{N}}$ [Geroch (1970), Hansen (1974)] :

- $\mathcal{M}_k$: mass 2^k-pole moment
- $\mathcal{J}_k$: angular momentum 2^k-pole moment

$\Longrightarrow$ For the Kerr metric, all the multipole moments are determined by (M, a) :

- $\mathcal{M}_0 = M$
- $\mathcal{J}_1 = aM = J/c$
- $$\boxed{\mathcal{M}_2 = -a^2 M = -\frac{J^2}{c^2 M}} \qquad (1)$$ $\leftarrow$ mass quadrupole moment
- $\mathcal{J}_3 = -a^3 M$
- $\mathcal{M}_4 = a^4 M$
- $\cdots$

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- $\cdots$

Measuring the three quantities M, J, $\mathcal{M}_2$ provides a compatibility test w.r.t. the Kerr metric, by checking (1)

Theoretical alternatives to the Kerr black hole

Within general relativity

The compact object is not a black hole but

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- a gravastar
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Beyond general relativity

The compact object is a black hole but in a theory that differs from 4-dimensional GR :

- Horndeski theories (scalar-tensor)
- Chern-Simons gravity
- Hořava-Lifshitz gravity
- Higher-dimensional GR
- ...

Viable scalar-tensor theories after GW170817

$$\text{GW170817} \Longrightarrow \left| \frac{c_{\rm gw} - c}{c} \right| < 5\ 10^{-16}$$

	$c_g = c$	$c_g \neq c$
Horndeski	General Relativity quintessence/k-essence [42] Brans-Dicke/$f(R)$ [43, 44] Kinetic Gravity Braiding [46]	quartic/quintic Galileons [13, 14] Fab Four [15, 16] de Sitter Horndeski [45] $G_{\mu\nu}\phi^\mu\phi^\nu$ [47], Gauss-Bonnet
beyond H.	Derivative Conformal (20) [18] Disformal Tuning (22) DHOST with $A_1 = 0$	quartic/quintic GLPV [19] DHOST [20, 48] with $A_1 \neq 0$
	Viable after GW170817	Non-viable after GW170817

[Ezquiaga & Zumalacárregui, PRL **119**, 251304 (2017)]

Testing the Kerr black hole hypothesis

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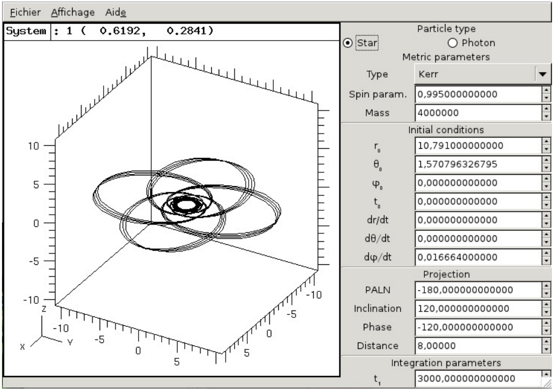
Need for a good and versatile **geodesic integrator**
to compute timelike geodesics (orbits) and null geodesics (ray-tracing) in any kind of metric

Gyoto code

Main developers : T. Paumard & F. Vincent



- Integration of geodesics in Kerr metric
- Integration of geodesics in any numerically computed 3+1 metric
- Radiative transfer included in optically thin media
- Very modular code (C++)
- Yorick and Python interfaces
- Free software (GPL) : http://gyoto.obspm.fr/

[Vincent, Paumard, Gourgoulhon & Perrin, CQG **28**, 225011 (2011)]

[Vincent, Gourgoulhon & Novak, CQG **29**, 245005 (2012)]

Geodesics with the free computer algebra system SageMath

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Timelike geodesic in Schwarzschild spacetime

This Jupyter/SageMath worksheet presents the numerical computation of a timelike geodesic in Schwarzschild spacetime, given an initial point and tangent vector. It uses the `integrated_geodesic` functionality introduced by Karim Van Aelst in **SageMath 8.1**, in the framework of the SageManifolds project.

A version of SageMath at least equal to 8.1 is required to run this worksheet:

In [1]:
```
version()
```

Out[1]: 'SageMath version 8.1.rc0, Release Date: 2017-11-08'

In [2]:
```
%display latex  # LaTeX rendering turned on
```

We define first the spacetime manifold M and the standard Schwarzschild-Droste coordinates on it:

In [3]:
```
M = Manifold(4, 'M')
X.<t,r,th,ph> = M.chart(r't r:(0,+oo) th:(0,pi):\theta ph:\phi')
X
```

Out[3]: $(M,(t,r,\theta,\phi))$

http://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Worksheets/v1.1/SM_simple_geod_Schwarz.ipynb

Geodesics with the free computer algebra system SageMath

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X.<t,r,th,ph> = M.chart(r't r:(0,+oo) th:(0,pi):\theta ph:\phi')
X
```

Out[3]: $(M, (t, r, \theta, \phi))$

For graphical purposes, we introduce $\mathbb{R}^3$ and some coordinate map $M \to \mathbb{R}^3$:

In [4]:
```
R3 = Manifold(3, 'R^3', latex_name=r'\mathbb{R}^3')
X3.<x,y,z> = R3.chart()
to_R3 = M.diff_map(R3, {(X, X3): [r*sin(th)*cos(ph),
                                  r*sin(th)*sin(ph), r*cos(th)]})
to_R3.display()
```

Out[4]:

$$\begin{array}{llll} M & \longrightarrow & \mathbb{R}^3 & \\ (t, r, \theta, \phi) & \longmapsto & (x, y, z) = (r\cos(\phi)\sin(\theta), r\sin(\phi)\sin(\theta), r\cos(\theta)) \end{array}$$

Then, we define the Schwarzschild metric:

In [5]:
```
g = M.lorentzian_metric('g')
m = var('m'); assume(m >= 0)
g[0,0], g[1,1] = -(1-2*m/r), 1/(1-2*m/r)
g[2,2], g[3,3] = r^2, (r*sin(th))^2
g.display()
```

Out[5]:

$$g = \left(\frac{2\,m}{r} - 1\right) \mathrm{d}t \otimes \mathrm{d}t + \left(-\frac{1}{\frac{2\,m}{r} - 1}\right) \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\theta \otimes \mathrm{d}\theta + r^2 \sin(\theta)^2 \mathrm{d}\phi \otimes \mathrm{d}\phi$$

We pick an initial point and an initial tangent vector:

In [6]:
```
p0 = M.point((0, 8*m, pi/2, 1e-12), name='p_0')
v0 = M.tangent_space(p0)((1.297513, 0, 0, 0.0640625/m), name='v_0')
v0.display()
```

Out[6]:

$$v_0 = 1.29751300000000 \frac{\partial}{\partial t} + \frac{0.0640625000000000}{m} \frac{\partial}{\partial \phi}$$

Geodesics with the free computer algebra system SageMath

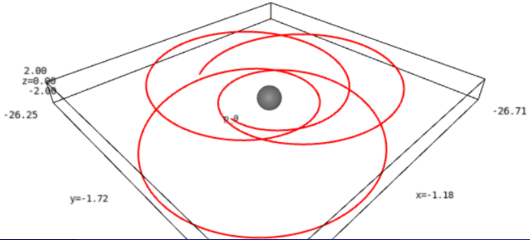
We declare a geodesic with such initial conditions, denoting by s the affine parameter (proper time), with $(s_{\min}, s_{\max}) = (0, 1500\, m)$:

In [7]:
```
s = var('s')
geod = M.integrated_geodesic(g, (s, 0, 1500), v0); geod
```

Out[7]: Integrated geodesic in the 4-dimensional differentiable manifold M

We ask for the numerical integration of the geodesic, providing some numerical value for the parameter m, and then plot it in terms of the Cartesian chart X3 of $\mathbb{R}^3$:

In [8]:
```
sol = geod.solve(parameters_values={m: 1})  # numerical integration
interp = geod.interpolate()                 # interpolation of the solution for the plot
graph = geod.plot_integrated(chart=X3, mapping=to_R3, plot_points=500,
                             thickness=2, label_axes=False)          # the geodesic
graph += p0.plot(chart=X3, mapping=to_R3, size=4, parameters={m: 1}) # the starting point
graph += sphere(size=2, color='grey')                                # the event horizon
show(graph, viewer='threejs', online=True)
```



Geodesics with the free computer algebra system SageMath

In [11]:
```
g.christoffel_symbols_display()
```

Out[11]:

$$\begin{array}{lcl}
\Gamma^{t}{}_{t\,r} & = & -\frac{m}{2\,mr-r^2} \\
\Gamma^{r}{}_{t\,t} & = & -\frac{2\,m^2-mr}{r^3} \\
\Gamma^{r}{}_{r\,r} & = & \frac{m}{2\,mr-r^2} \\
\Gamma^{r}{}_{\theta\,\theta} & = & 2\,m-r \\
\Gamma^{r}{}_{\phi\,\phi} & = & (2\,m-r)\sin(\theta)^2 \\
\Gamma^{\theta}{}_{r\,\theta} & = & \frac{1}{r} \\
\Gamma^{\theta}{}_{\phi\,\phi} & = & -\cos(\theta)\sin(\theta) \\
\Gamma^{\phi}{}_{r\,\phi} & = & \frac{1}{r} \\
\Gamma^{\phi}{}_{\theta\,\phi} & = & \frac{\cos(\theta)}{\sin(\theta)}
\end{array}$$

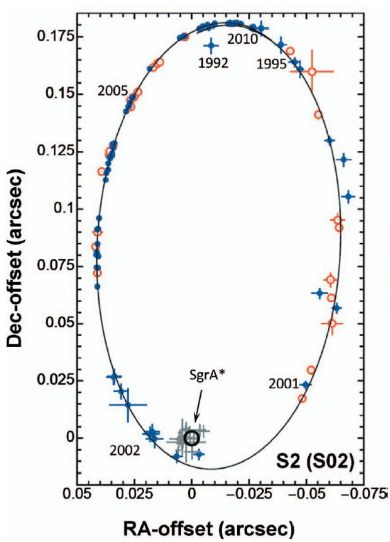
In [12]:
```
g.riemann().display_comp()
```

Out[12]:

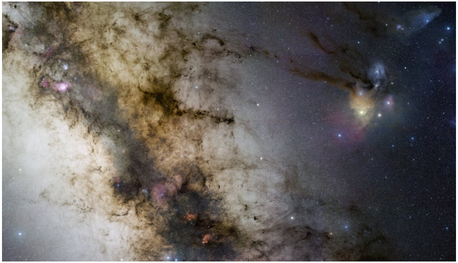
$$\begin{array}{lcl}
\mathrm{Riem}(g)^{t}{}_{r\,t\,r} & = & -\frac{2\,m}{2\,mr^2-r^3} \\
\mathrm{Riem}(g)^{t}{}_{r\,r\,t} & = & \frac{2\,m}{2\,mr^2-r^3} \\
\mathrm{Riem}(g)^{t}{}_{\theta\,t\,\theta} & = & -\frac{m}{r} \\
\mathrm{Riem}(g)^{t}{}_{\theta\,\theta\,t} & = & \frac{m}{r} \\
\mathrm{Riem}(g)^{t}{}_{\phi\,t\,\phi} & = & -\frac{m\sin(\theta)^2}{r} \\
\mathrm{Riem}(g)^{t}{}_{\phi\,\phi\,t} & = & \frac{m\sin(\theta)^2}{r}
\end{array}$$

Outline

The black hole at the centre of our galaxy : Sgr A*



movie : [ESO / L. Calçada]



[ESO (2009)]

Mass of Sgr A* black hole deduced from stellar dynamics :

$$M_{\rm BH} = 4.3 \times 10^6 \, M_\odot$$

← Orbit of the star S2 around Sgr A*

$P = 16\,\mathrm{yr}, \quad r_{\rm per} = 120\,\mathrm{UA} = 1400\,R_{\rm S},$
$V_{\rm per} = 0.02\,c$

[Genzel, Eisenhauer & Gillessen, RMP **82**, 3121 (2010)]

Next periastron passage : mid 2018

Can we see it from the Earth ?

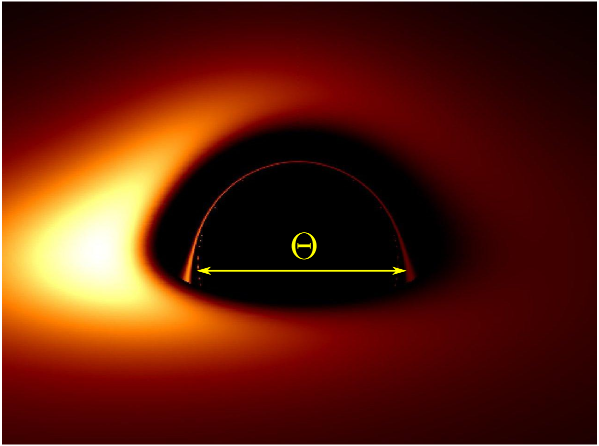


Image of a thin accretion disk around a Schwarzschild BH
[Vincent, Paumard, Gourgoulhon & Perrin, CQG **28**, 225011 (2011)]

Angular diameter of the silhouette of a Schwarzschild BH of mass M seen from a distance d :

$$\Theta = 6\sqrt{3}\,\frac{GM}{c^2 d} \simeq 2.60\,\frac{2R_{\rm S}}{d}$$

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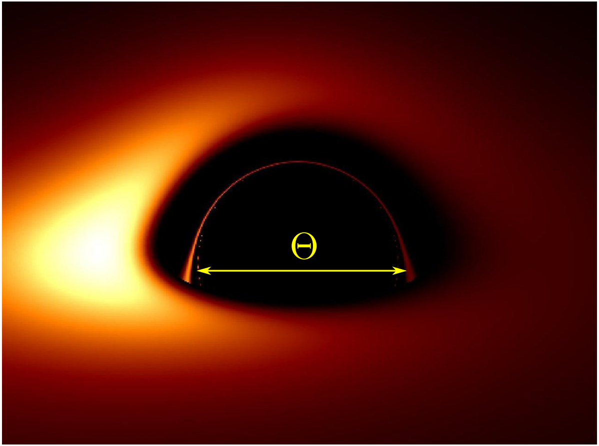

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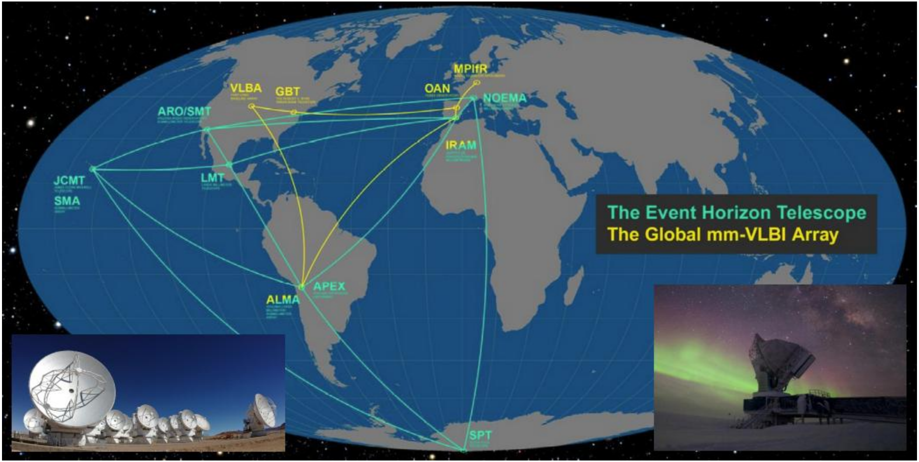
Largest black holes in the Earth's sky :

Sgr A* : $\Theta = 53\ \mu\mathrm{as}$
M87 : $\Theta = 21\ \mu\mathrm{as}$
M31 : $\Theta = 20\ \mu\mathrm{as}$

Remark : black holes in X-ray binaries are $\sim 10^5$ times smaller, for $\Theta \propto M/d$

Reaching μas resolution : the Event Horizon Telescope



http://eventhorizontelescope.org/

Very Large Baseline Interferometry (VLBI) at $\lambda = 1.3$ mm

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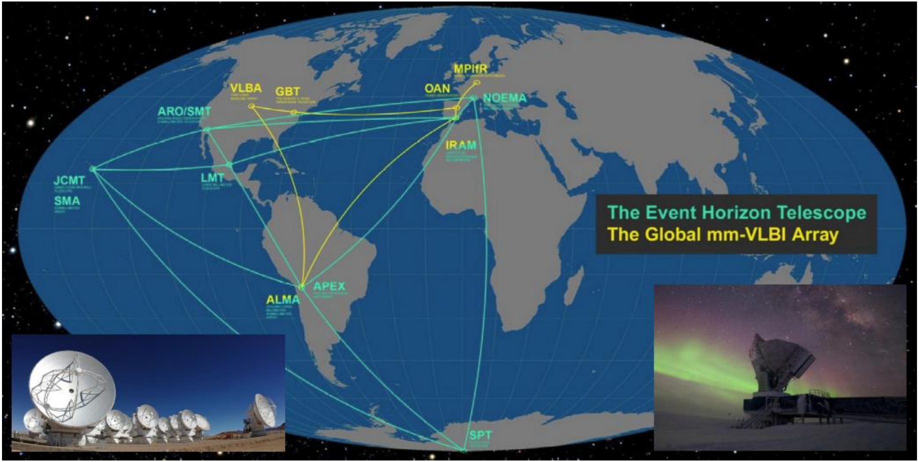

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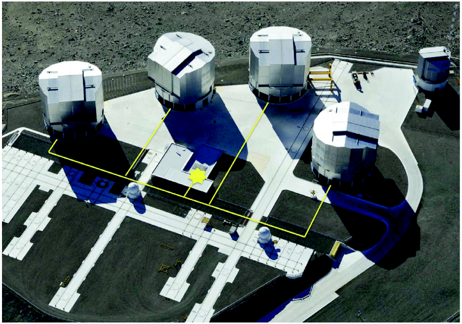
Very Large Baseline Interferometry (VLBI) at $\lambda = 1.3$ mm

April 2017 : large observation campaign $\Longrightarrow$ first image soon ?

Near-infrared optical interferometry : GRAVITY

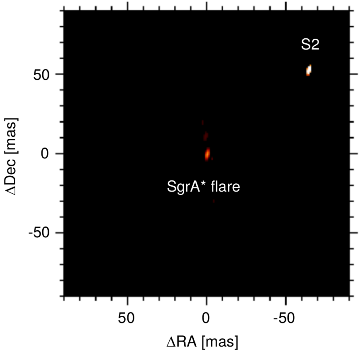
GRAVITY instrument at VLTI (start : 2016) : beam combiner (the four 8 m telescopes + four auxiliary telescopes)

Astrometric precision on orbits : 10 μas



[Gillessen et al. 2010]

First observation :



[Abuter et al., A&A **602**, A94 (2017)]

Outline

Boson stars

Boson star = localized configurations of a self-gravitating complex scalar field Φ
$\equiv$ "Klein-Gordon geons" [Bonazzola & Pacini (1966), Kaup (1968), Ruffini & Bonazzola (1969)]

- Minimally coupled scalar field : $\mathcal{L} = \frac{1}{16\pi} R - \frac{1}{2}\left[\nabla_\mu \bar{\Phi} \nabla^\mu \Phi + V(|\Phi|^2)\right]$
- Scalar field equation : $\nabla_\mu \nabla^\mu \Phi = V'(|\Phi|^2)\,\Phi$
- Einstein equation : $R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 8\pi T_{\alpha\beta}$

  with $T_{\alpha\beta} = \nabla_{(\alpha} \bar{\Phi} \nabla_{\beta)} \Phi - \frac{1}{2}\left[\nabla_\mu \bar{\Phi} \nabla^\mu \Phi + V(|\Phi|^2)\right] g_{\alpha\beta}$

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Examples :

- free field : $V(|\Phi|^2) = \frac{m^2}{\hbar^2} |\Phi|^2$, m : boson mass

  $\Longrightarrow$ field equation = Klein-Gordon equation : $\nabla_\mu \nabla^\mu \Phi = \frac{m^2}{\hbar^2} \Phi$
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Boson stars as black-hole mimickers

Boson stars can be very compact and are the less exotic alternative to black holes : they require only a scalar field and since 2012 we know that at least one fundamental scalar field exists in Nature : the Higgs boson !

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Maximum mass

- Free field : $M_{\rm max} = \alpha \frac{\hbar}{m} = \alpha \frac{m_{\rm P}^2}{m}$, with $\alpha \sim 1$
- Self-interacting field : $M_{\rm max} \sim \left(\frac{\lambda}{4\pi}\right)^{1/2} \frac{m_{\rm P}^2}{m} \times \frac{m_{\rm P}}{m}$

$m_{\rm P} = \sqrt{\hbar} = \sqrt{\hbar c/G} = 2.18\ 10^{-8}\ {\rm kg}$: Planck mass

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m	$M_{\rm max}$ (free field)	$M_{\rm max}$ (self-interacting field, $\lambda = 1$)
125 GeV (Higgs)	$2\ 10^{9}$ kg	$2\ 10^{26}$ kg
1 GeV	$3\ 10^{11}$ kg	$2\ M_\odot$
0.5 MeV	$3\ 10^{14}$ kg	$5\ 10^{6}\ M_\odot$

Rotating boson stars

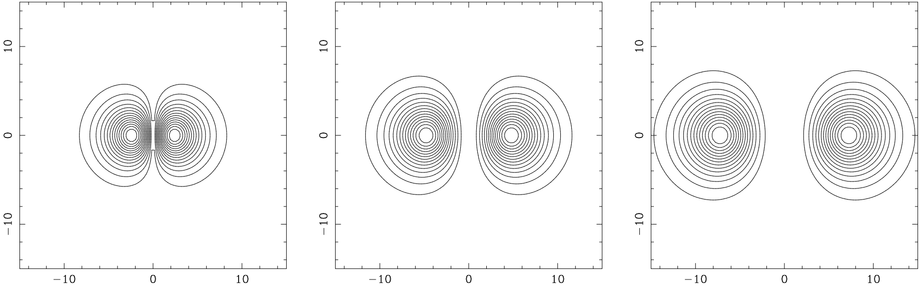
Solutions computed by means of Kadath [Grandclément, JCP **229**, 3334 (2010)]

http://kadath.obspm.fr/

→ *see Philippe Grandclément's talk*

Isocontours of $\Phi_0(r,\theta)$ in the plane $\varphi = 0$ for $\omega = 0.8\frac{m}{\hbar}$:

$k = 1$ $k = 2$ $k = 3$



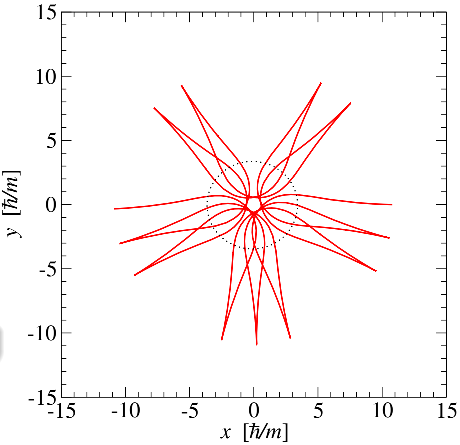
[Granclément, Somé & Gourgoulhon, PRD **90**, 024068 (2014)]

Initially-at-rest orbits around rotating boson stars

Orbit with a rest point around a rotating boson star based on the scalar field $\Phi = \Phi_0(r,\theta)e^{i(\omega t + k\varphi)}$ with $k = 2$ and $\omega = 0.75\, m/\hbar$

Orbit = timelike geodesic computed by means of Gyoto

$\Longrightarrow$ strong Lense-Thirring effect



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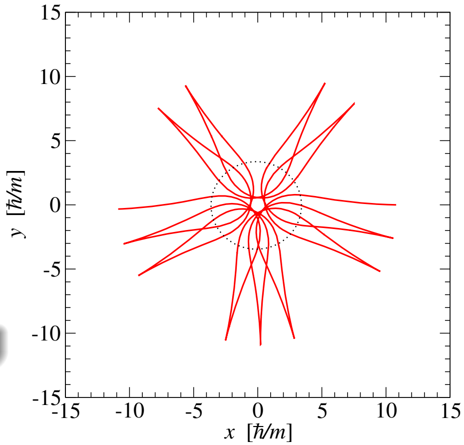
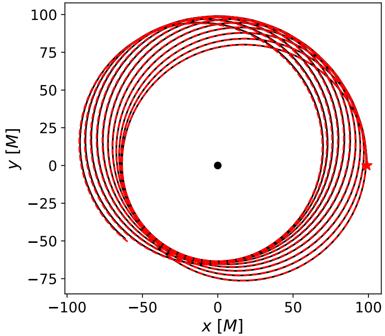


[Grandclément, Somé & Gourgoulhon, PRD **90**, 024068 (2014)]

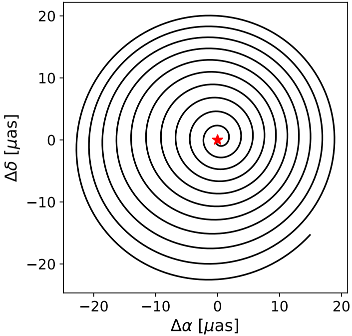
No equivalent in Kerr spacetime

Comparing orbits with a Kerr BH

Same reduced spin for the boson star and the Kerr BH : $a = 0.802\,M$
Boson star (BS) : $k = 1$ and $\omega = 0.8\,m/\hbar$
Orbit with pericenter of $60\,M$ and apocenter of $100\,M$



The two orbits

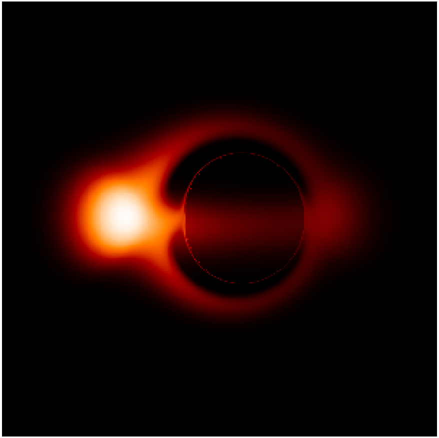


Difference between the BS orbit and the BH one for Sgr A*

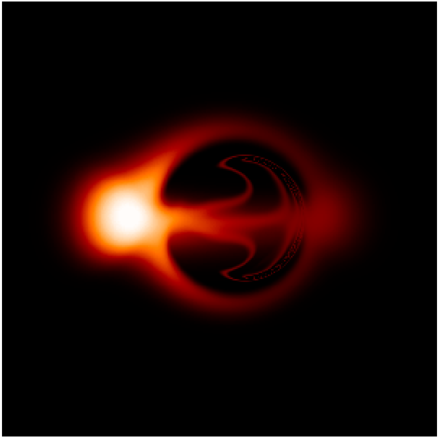
[Grould, Meliani, Vincent, Grandclément & Gourgoulhon, CQG **34**, 215007 (2017)]

Image of an accretion torus : comparing with a Kerr BH

Kerr BH $a/M = 0.9$



Boson star $k = 1$, $\omega = 0.70\, m/\hbar$



[Vincent, Meliani, Grandclément, Gourgoulhon & Straub, CQG **33**, 105015 (2016)]

Hairy black holes

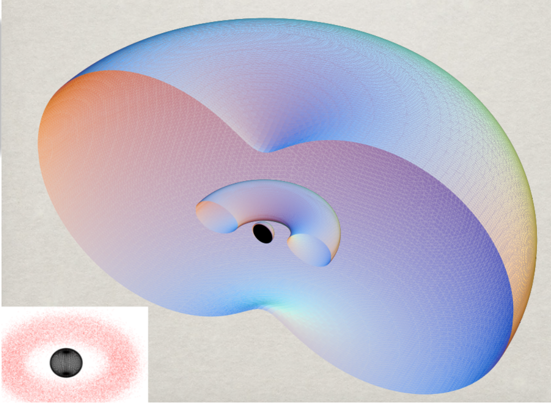
Herdeiro & Radu discovery (2014)

A black hole can have a complex scalar hair

Stationary axisymmetric configuration with a self-gravitating massive complex scalar field Φ and an event horizon

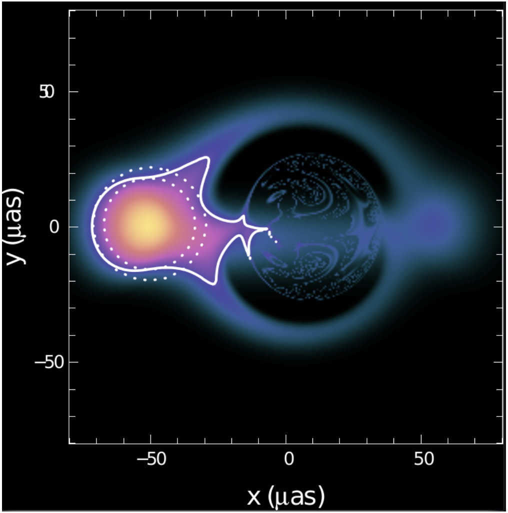
$\Phi(t, r, \theta, \varphi) = \Phi_0(r, \theta) e^{i(\omega t + k\varphi)}$

$\omega = k\Omega_{\rm H}, \quad k \in \mathbb{N}$



[Herdeiro & Radu, PRL **112**, 221101 (2014)]

Hairy black hole



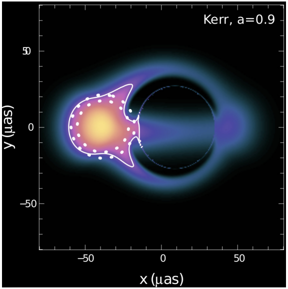
Accretion torus around a scalar-field-hairy rotating black hole

[Vincent, Gourgoulhon, Herdeiro & Radu, Phys. Rev. D **94**, 084045 (2016)]

Alternatives to the Kerr black hole

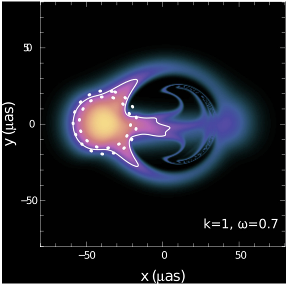
Kerr black hole

$a/M = 0.9$



boson star [[1]]

$k=1,\ \omega=0.7\, m/\hbar$

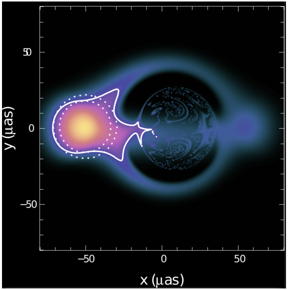


Kadath → metric

Gyoto → ray-tracing

hairy black hole [[2]]

$a/M = 0.9$



HR code → metric

(via Lorene)

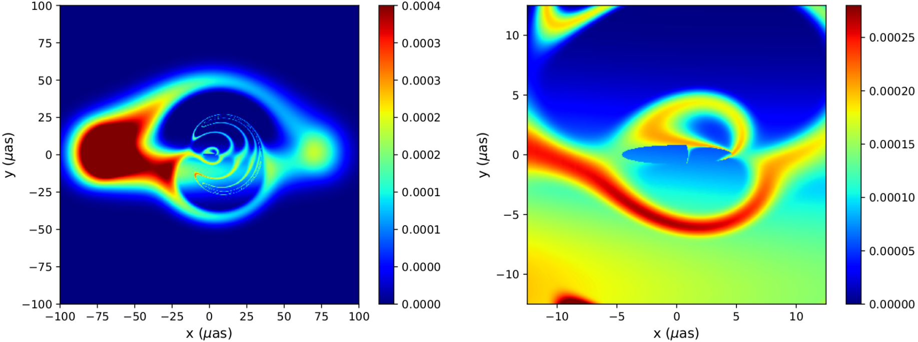
Gyoto → ray-tracing

[[1] Vincent, Meliani, Grandclément, Gourgoulhon & Straub, Class. Quantum Grav. **33**, 105015 (2016)]

[[2] Vincent, Gourgoulhon, Herdeiro & Radu, Phys. Rev. D **94**, 084045 (2016)]

A more exotic alternative : naked rotating wormhole

Regular (singularity-free) spacetime with wormhole topology ($\mathbb{R}^2 \times \mathbb{S}^2$), sustained by exotic matter, asymptotically close a to Kerr spacetime with a naked singularity ($a > M$).



[Lamy, Gourgoulhon, Paumard & Vincent, arXiv:1802.01635]

Conclusions and perspectives

After a century marked by the Golden Age (1965-1975), the first astronomical discoveries and the ubiquity of black holes in high-energy astrophysics, black hole physics is very much alive.

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It is actually entering a new observational era, with the advent of high-angular resolution telescopes and gravitational wave detectors, which provide unique opportunities to test general relativity in the strong field regime, notably by searching for some violation of the *no-hair theorem*.

To conduct these tests, it is necessary to perform studies of possible theoretical alternatives to the Kerr black hole, like *boson stars*, *hairy black holes* or *black holes in some extensions of general relativity*.