New theoretical perspectives on black holes

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based on a collaboration with José Luis Jaramillo

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Mexico, 5 December 2006

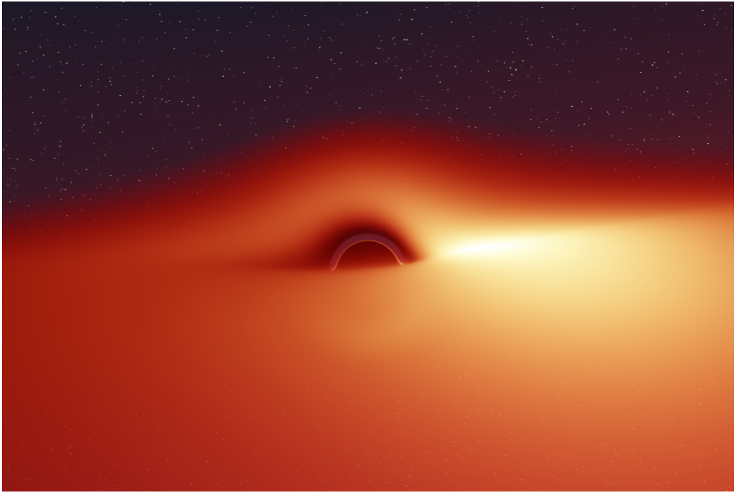
Plan

1. Review of "classical" black holes
2. New approaches to black holes
3. Geometry of hypersurface foliations by spacelike 2-surfaces
4. A Navier-Stokes-like equation
5. Area evolution and energy equation

Outline

What is a black hole ?

... for the astrophysicist: a very deep gravitational potential well



[J.A. Marck, CQG **13**, 393 (1996)]

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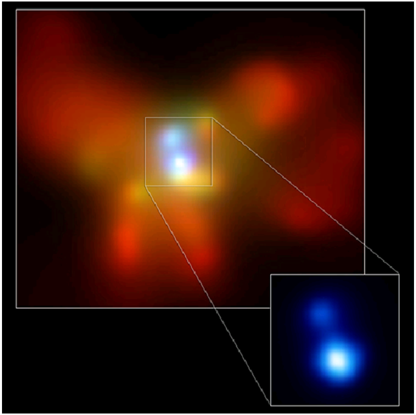
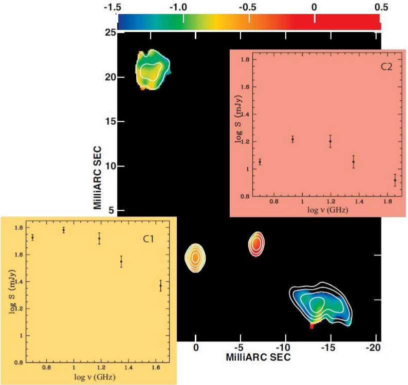


Binary BH in galaxy NGC 6240

$d = 1.4$ kpc

[Komossa et al., ApJ **582**, L15 (2003)]



Binary BH in radio galaxy 0402+379

$d = 7.3$ pc

[Rodriguez et al., ApJ in press, astro-ph/0604042]

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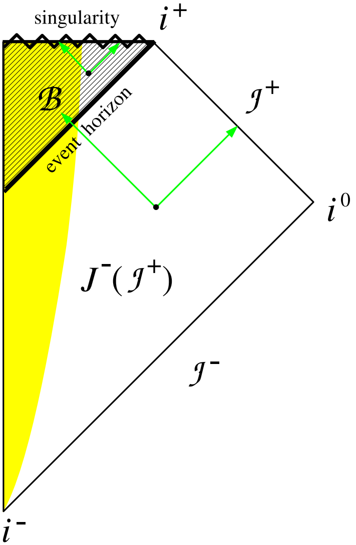

... for the mathematical physicist:

$$\mathcal{B} := \mathscr{M} - J^-(\mathscr{I}^+)$$

i.e. the region of spacetime where light rays cannot escape to infinity

- $\mathscr{M}$ = asymptotically flat manifold
- $\mathscr{I}^+$ = future null infinity
- $J^-(\mathscr{I}^+)$ = causal past of $\mathscr{I}^+$

event horizon: $\mathcal{H} := \dot{J}^-(\mathscr{I}^+)$
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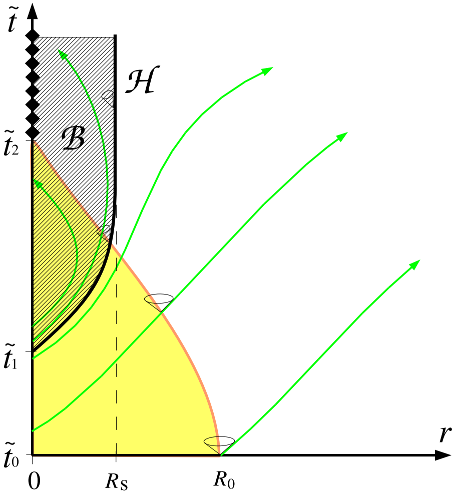

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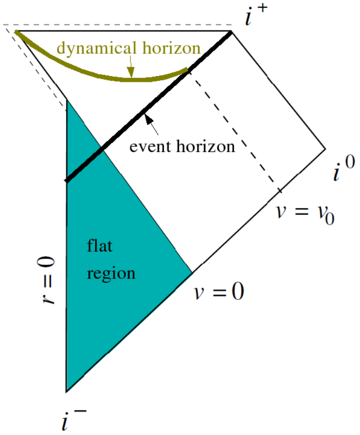
The determination of the boundary of $J^-(\mathscr{I}^+)$ requires the knowledge of the entire future null infinity. Moreover this is not locally linked with the notion of strong gravitational field:



Example of event horizon in a **flat** region of spacetime:

Vaidya metric, describing incoming radiation from infinity:

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right) dv^2 + 2dv\,dr + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

with
$m(v) = 0$ for $v < 0$
$dm/dv > 0$ for $0 \le v \le v_0$
$m(v) = M_0$ for $v > v_0$

[Ashtekar & Krishnan, LRR **7**, 10 (2004)]

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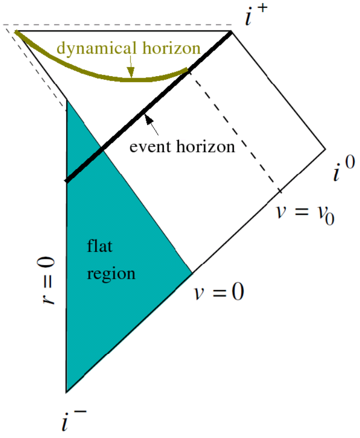

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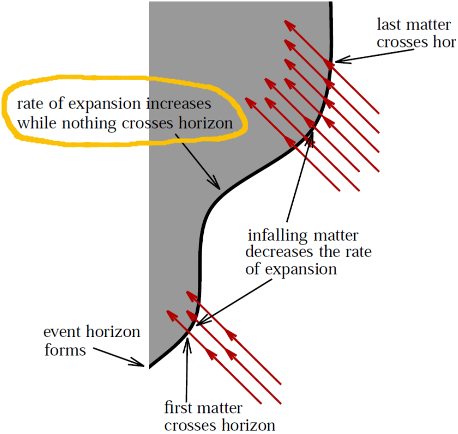
$\Rightarrow$ no local physical experiment whatsoever can locate the event horizon

Another non-local feature: teleological nature of event horizons



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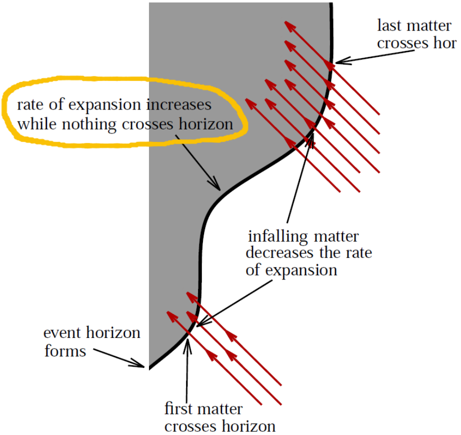

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To deal with black holes as physical objects, a local definition would be desirable

Outline

Local characterizations of black holes

Recently a **new paradigm** appeared in the theoretical approach of black holes: instead of *event horizons*, black holes are described by

- trapping horizons (Hayward 1994)
- isolated horizons (Ashtekar et al. 1999)
- dynamical horizons (Ashtekar and Krishnan 2002)

All these concepts are **local** and are based on the notion of trapped surfaces

Motivations: quantum gravity, numerical relativity

What is a trapped surface ?

1/ Expansion of a surface along a normal vector field

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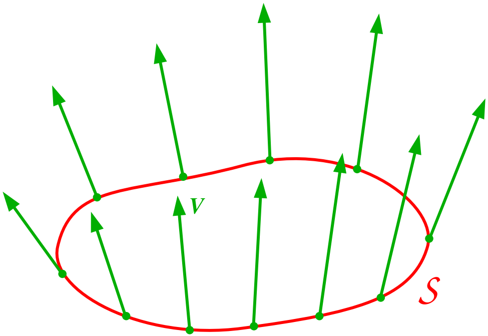

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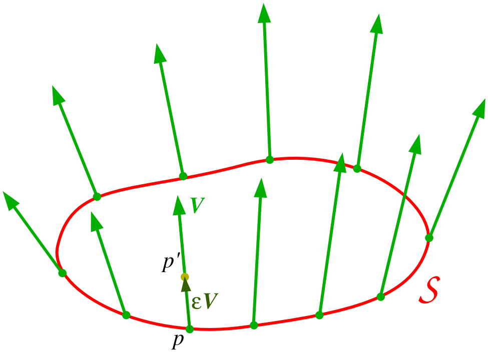

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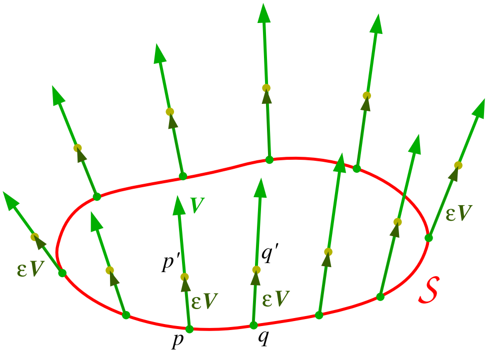

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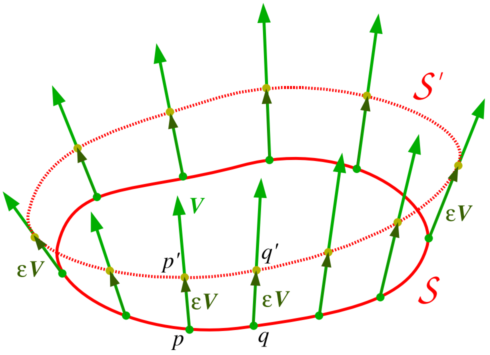

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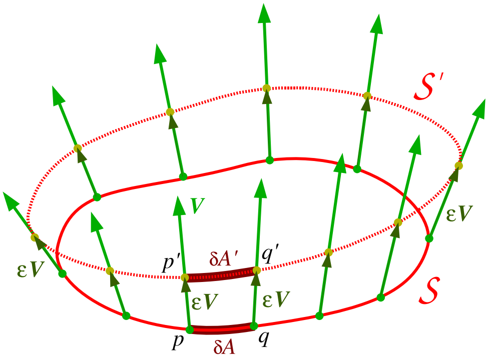

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At each point, the **expansion of $\mathcal{S}$ along $\boldsymbol{v}$** is defined from the relative change in the area element δA:

$$\theta^{(\boldsymbol{v})} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \frac{\delta A' - \delta A}{\delta A} = \boldsymbol{\mathcal{L}}_{\boldsymbol{v}} \ln \sqrt{q} = q^{\mu\nu} \nabla_\mu v_\nu$$

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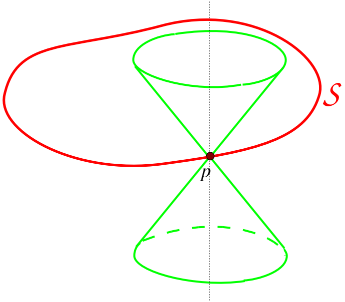

Being spacelike, $\mathcal{S}$ lies outside the light cone

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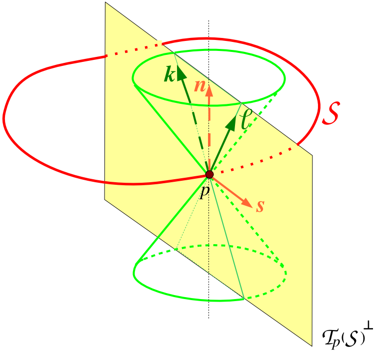

Being spacelike, $\mathcal{S}$ lies outside the light cone

$\exists$ two future-directed null directions orthogonal to $\mathcal{S}$:

$\boldsymbol{\ell}$ = outgoing, expansion $\theta^{(\boldsymbol{\ell})}$

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In flat space, $\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} > 0$

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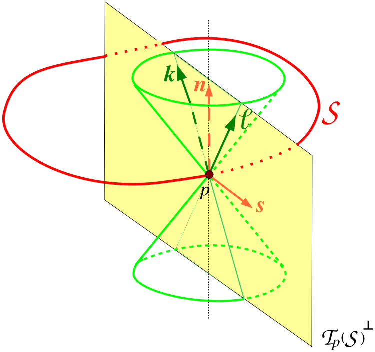

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$\mathcal{S}$ is trapped $\iff$ $\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} < 0$ [Penrose 1965]

$\mathcal{S}$ is marginally trapped $\iff$ $\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} = 0$

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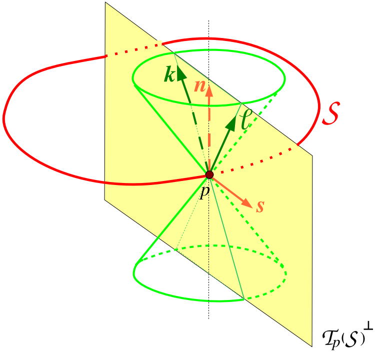

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trapped surface = **local** concept characterizing very strong gravitational fields

Link with apparent horizons

A closed spacelike 2-surface $\mathcal{S}$ is said to be outer trapped (resp. marginally outer trapped (MOTS)) iff [Hawking & Ellis 1973]

- the notions of *interior* and *exterior* of $\mathcal{S}$ can be defined (for instance spacetime asymptotically flat) $\Rightarrow$ $\boldsymbol{\ell}$ is chosen to be the *outgoing* null normal and $\boldsymbol{k}$ to be the *ingoing* one
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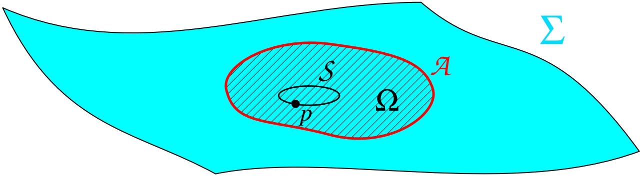


Σ: spacelike hypersurface extending to spatial infinity (Cauchy surface)

outer trapped region of Σ : Ω = set of points $p \in \Sigma$ through which there is a outer trapped surface $\mathcal{S}$ lying in Σ

apparent horizon in Σ: $\mathcal{A}$ = connected component of the boundary of Ω

Proposition [Hawking & Ellis 1973]: $\mathcal{A}$ smooth $\Longrightarrow$ $\mathcal{A}$ is a MOTS

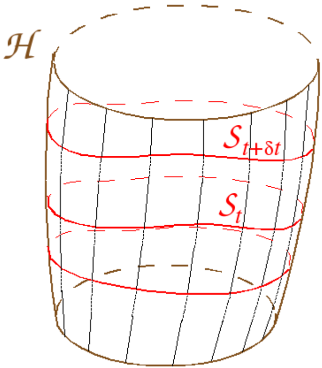
Connection with singularities and black holes

Proposition [Penrose (1965)]:
provided that the weak energy condition holds,
$\exists$ a trapped surface $\mathcal{S} \Longrightarrow \exists$ a singularity in $(\mathscr{M}, \boldsymbol{g})$ (in the form of a future inextendible null geodesic)

Proposition [Hawking & Ellis (1973)]:
provided that the cosmic censorship conjecture holds,
$\exists$ a trapped surface $\mathcal{S} \Longrightarrow \exists$ a black hole $\mathcal{B}$ and $\mathcal{S} \subset \mathcal{B}$

Local definitions of "black holes"

A hypersurface $\mathcal{H}$ of $(\mathscr{M}, \boldsymbol{g})$ is said to be



- a **future outer trapping horizon (FOTH)** iff
  (i) $\mathcal{H}$ foliated by marginally trapped 2-surfaces
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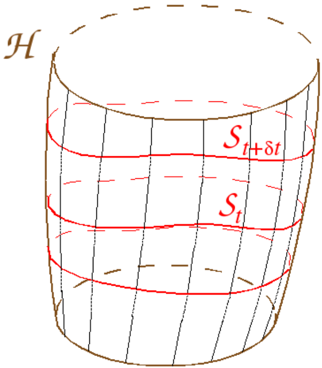

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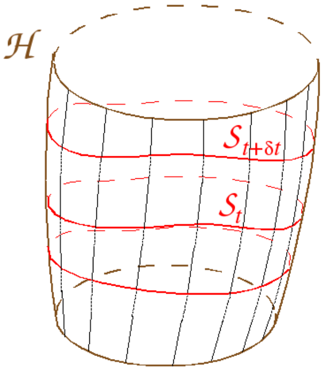

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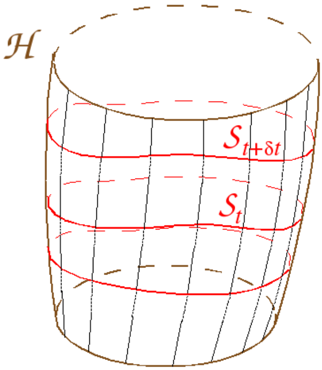

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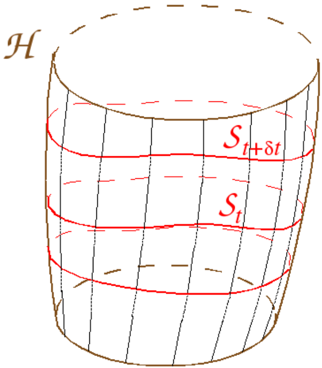

BH in equilibrium (e.g. Kerr) = IH
BH out of equilibrium = DH
generic BH = FOTH

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Dynamics of these new horizons

The *trapping horizons* and *dynamical horizons* have their **own dynamics**, ruled by Einstein equations.
In particular, one can establish for them

- existence and (partial) uniqueness theorems
  [Andersson, Mars & Simon, PRL **95**, 111102 (2005)],
  [Ashtekar & Galloway, Adv. Theor. Math. Phys. **9**, 1 (2005)]
- first and second laws of black hole mechanics
  [Ashtekar & Krishnan, PRD **68**, 104030 (2003)], [Hayward, PRD **70**, 104027 (2004)]
- a viscous fluid bubble analogy ("membrane paradigm" as for the event horizon), leading to a Navier-Stokes-like equation and a **positive** bulk viscosity *(event horizon = negative bulk viscosity)*
  [Gourgoulhon, PRD **72**, 104007 (2005)], [Gourgoulhon & Jaramillo, PRD **74**, 087502 (2006)]

Reviews: [Ashtekar & Krishnan, Liv. Rev. Relat. **7**, 10 (2004)], [Booth, Can. J. Phys. **83**, 1073 (2005)], [Gourgoulhon & Jaramillo, Phys. Rep. **423**, 159 (2006)]

Outline

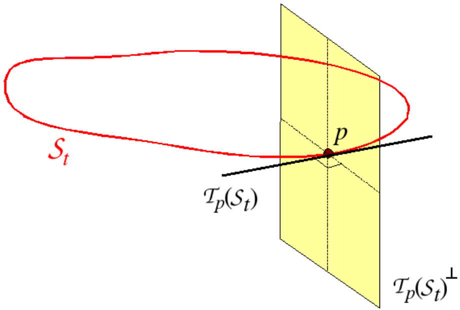
Closed spacelike surfaces

$\mathcal{S}$: closed (i.e. compact without boundary) spacelike 2-dimensional surface embedded in spacetime $(\mathscr{M}, \boldsymbol{g})$

$\mathcal{S}$ spacelike $\iff$ metric $\boldsymbol{q}$ induced by $\boldsymbol{g}$ is positive definite

$\boldsymbol{q}$ not degenerate $\Longrightarrow$ orthogonal decomposition of the tangent space at any $p \in \mathscr{M}$:

$$\mathcal{T}_p(\mathscr{M}) = \mathcal{T}_p(\mathcal{S}) \oplus \mathcal{T}_p(\mathcal{S})^{\perp}$$



$\boldsymbol{q}$: induced metric on $\mathcal{S}$, components: $q_{\alpha\beta}$

$\vec{\boldsymbol{q}}$: orthogonal projector onto $\mathcal{S}$, components: $q^{\alpha}{}_{\beta}$

Projection operator $\vec{q}^*$

$\boldsymbol{A}$: tensor of covariance type (m, n)
$\vec{\boldsymbol{q}}^* \boldsymbol{A}$: tensor of same covariance type, defined by

$$(\vec{q}^* A)^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n} := q^{\alpha_1}{}_{\mu_1} \cdots q^{\alpha_m}{}_{\mu_m} q^{\nu_1}{}_{\beta_1} \cdots q^{\nu_n}{}_{\beta_n} A^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n}$$

Remark: for a vector: $\vec{\boldsymbol{q}}^* \boldsymbol{v} = \vec{\boldsymbol{q}}(\boldsymbol{v})$
for a 1-form, $\vec{\boldsymbol{q}}^* \boldsymbol{\omega} = \boldsymbol{\omega} \circ \vec{\boldsymbol{q}}$

Definition: a tensor $\boldsymbol{A}$ is *tangent to* $\mathcal{S}$ iff $\vec{\boldsymbol{q}}^* \boldsymbol{A} = \boldsymbol{A}$.

Expansion and shear along normal vectors

Let $\boldsymbol{v}$ be a vector field on $\mathscr{M}$, defined at least at $\mathcal{S}$ and everywhere normal to $\mathcal{S}$.
NB: $\boldsymbol{v}$ is not assumed to be null

Deformation tensor of $\mathcal{S}$ along v: $\boldsymbol{\Theta}^{(v)} := \vec{\boldsymbol{q}}^* \boldsymbol{\nabla} \underline{\boldsymbol{v}}$ or $\Theta^{(v)}_{\alpha\beta} := \nabla_\nu v_\mu \, q^\mu{}_\alpha q^\nu{}_\beta$

$\boldsymbol{v}$ normal to a 2-surface $(\mathcal{S}) \Longrightarrow \boldsymbol{\Theta}^{(v)}$ is a symmetric bilinear form

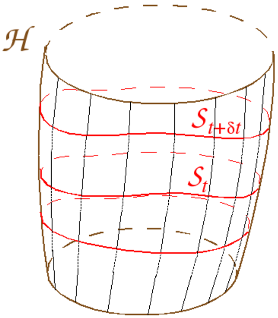
Prop: $\boldsymbol{\Theta}^{(v)} = \frac{1}{2} \vec{\boldsymbol{q}}^* \boldsymbol{\mathcal{L}}_{\boldsymbol{v}} \, \boldsymbol{q}$

Decomposition into traceless part (**shear** $\boldsymbol{\sigma}^{(v)}$) and trace part (**expansion** $\theta^{(v)}$):

$\boldsymbol{\Theta}^{(v)} = \boldsymbol{\sigma}^{(v)} + \frac{1}{2} \theta^{(v)} \, \boldsymbol{q}$ with $\theta^{(v)} := q^{\mu\nu} \Theta^{(v)}_{\mu\nu} = \mathcal{L}_{\boldsymbol{v}} \ln \sqrt{q}$, $q := \det q_{ab}$

Prop: $\mathcal{L}_{\boldsymbol{v}} \, {}^{\mathcal{S}}\boldsymbol{\epsilon} = \theta^{(v)} \, {}^{\mathcal{S}}\boldsymbol{\epsilon}$ with ${}^{\mathcal{S}}\boldsymbol{\epsilon}$ surface element of $(\mathcal{S}, \boldsymbol{q})$: ${}^{\mathcal{S}}\boldsymbol{\epsilon} = \sqrt{q} \, \mathbf{d}x^2 \wedge \mathbf{d}x^3$
$\Longrightarrow$ hence the name *expansion*

Foliation of a hypersurface by spacelike 2-surfaces



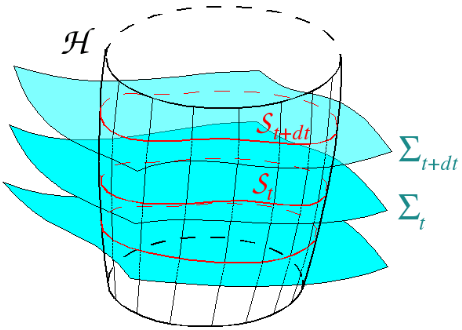
hypersurface $\mathcal{H}$ = submanifold of spacetime $(\mathscr{M}, \boldsymbol{g})$ of codimension 1

$$\mathcal{H} \text{ can be } \begin{cases} \text{spacelike} \\ \text{null} \\ \text{timelike} \end{cases}$$

$$\mathcal{H} = \bigcup_{t \in \mathbb{R}} \mathcal{S}_t$$

$\mathcal{S}_t$ = spacelike 2-surface

Foliation of a hypersurface by spacelike 2-surfaces



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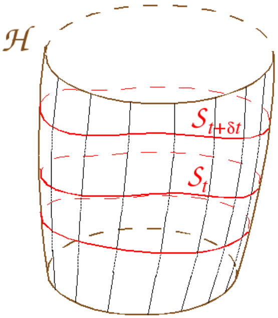
$\Longleftarrow$ 3+1 perspective

Foliation of a hypersurface by spacelike 2-surfaces



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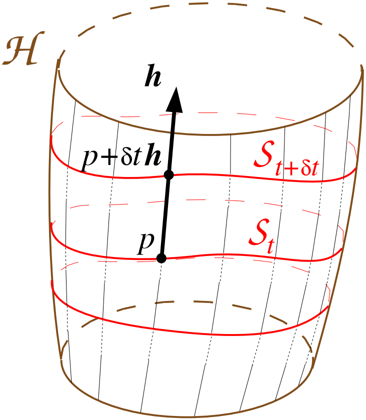
$$\mathcal{H} = \bigcup_{t\in\mathbb{R}} \mathcal{S}_t$$

$\mathcal{S}_t$ = spacelike 2-surface

intrinsic viewpoint adopted here (i.e. not relying on extra-structure such as a 3+1 foliation)

Evolution vector on the horizon



Vector field $\boldsymbol{h}$ on $\mathcal{H}$ defined by

- (i) $\boldsymbol{h}$ is tangent to $\mathcal{H}$
- (ii) $\boldsymbol{h}$ is orthogonal to $\mathcal{S}_t$
- (iii) $\mathcal{L}_{\boldsymbol{h}}\, t = h^\mu \partial_\mu t = \langle \mathbf{d}t, \boldsymbol{h} \rangle = 1$

NB: (iii) $\Longrightarrow$ the 2-surfaces $\mathcal{S}_t$ are Lie-dragged by $\boldsymbol{h}$

Lie derivatives along $\boldsymbol{h}$

Since the 2-surfaces $\mathcal{S}_t$ are Lie-dragged by $\boldsymbol{h}$, so are their tangent vectors:

$$\forall \boldsymbol{v} \in \mathcal{T}(\mathcal{S}_t),\ \mathcal{L}_{\boldsymbol{h}}\, \boldsymbol{v} \in \mathcal{T}(\mathcal{S}_t)$$

i.e. $\mathcal{L}_{\boldsymbol{h}}$ = internal operator on $\mathcal{T}(\mathcal{S}_t)$

Extension to 1-forms in $\mathcal{T}^*(\mathcal{S}_t)$:

$$\forall \boldsymbol{v} \in \mathcal{T}(\mathcal{S}_t), \quad \langle \mathcal{L}_{\boldsymbol{h}}\, \boldsymbol{\omega}, \boldsymbol{v}\rangle := \mathcal{L}_{\boldsymbol{h}} \langle \boldsymbol{\omega}, \boldsymbol{v}\rangle - \langle \boldsymbol{\omega}, \mathcal{L}_{\boldsymbol{h}}\, \boldsymbol{v}\rangle.$$

Extension to any tensor $\boldsymbol{A}$ tangent to $\mathcal{S}_t$ by tensor products

Definition:

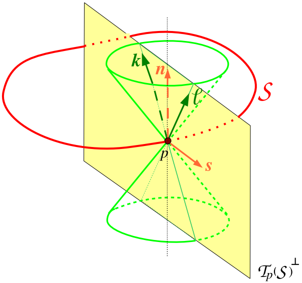
$$^{\mathcal{S}}\mathcal{L}_{\boldsymbol{h}}\, \boldsymbol{A} := \vec{\boldsymbol{q}}^{*} \mathcal{L}_{\boldsymbol{h}}\, \boldsymbol{A} = \vec{\boldsymbol{q}}^{*} \mathcal{L}_{\boldsymbol{h}}\, \vec{\boldsymbol{q}}^{*} \boldsymbol{A}$$

Norm of $\boldsymbol{h}$ and type of $\mathcal{H}$

Definition: $C := \frac{1}{2}\boldsymbol{h}\cdot\boldsymbol{h}$

$$\begin{array}{lclcl} \mathcal{H} \text{ is spacelike} & \iff & C > 0 & \iff & \boldsymbol{h} \text{ is spacelike} \\ \mathcal{H} \text{ is null} & \iff & C = 0 & \iff & \boldsymbol{h} \text{ is null} \\ \mathcal{H} \text{ is timelike} & \iff & C < 0 & \iff & \boldsymbol{h} \text{ is timelike.} \end{array}$$

Frames normal to $\mathcal{S}_t$



Two natural types of choice for a vector basis of $\mathcal{T}_p(\mathcal{S}_t)^\perp$:

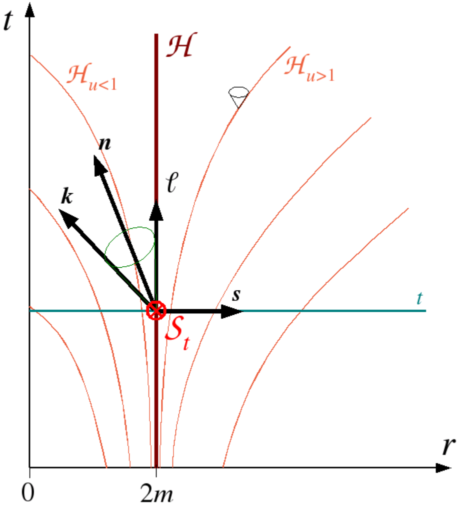
1. an orthonormal basis $(\boldsymbol{n}, \boldsymbol{s})$ ($\boldsymbol{n}$ = timelike, $\boldsymbol{s}$ = spacelike):
   $\boldsymbol{n}\cdot\boldsymbol{n} = -1, \quad \boldsymbol{s}\cdot\boldsymbol{s} = 1, \quad \boldsymbol{n}\cdot\boldsymbol{s} = 0$
2. a pair linearly independent future-directed null vectors $(\boldsymbol{\ell}, \boldsymbol{k})$:
   $\boldsymbol{\ell}\cdot\boldsymbol{\ell} = 0, \quad \boldsymbol{k}\cdot\boldsymbol{k} = 0, \quad \boldsymbol{\ell}\cdot\boldsymbol{k} =: -e^{\sigma}$

Degrees of freedom:

1. boost : $\begin{cases} \boldsymbol{n}' = \cosh\eta\,\boldsymbol{n} + \sinh\eta\,\boldsymbol{s} \\ \boldsymbol{s}' = \sinh\eta\,\boldsymbol{n} + \cosh\eta\,\boldsymbol{s} \end{cases}, \quad \eta \in \mathbb{R}$
2. rescaling : $\begin{cases} \boldsymbol{\ell}' = \lambda\,\boldsymbol{\ell}, & \lambda > 0 \\ \boldsymbol{k}' = \mu\,\boldsymbol{k}, & \mu > 0 \end{cases}$

Orthogonal projector: $\vec{\boldsymbol{q}} = \mathbf{1} + \langle \underline{\boldsymbol{n}}, .\rangle\,\boldsymbol{n} - \langle \underline{\boldsymbol{s}}, .\rangle\,\boldsymbol{s} = \mathbf{1} + e^{-\sigma}\langle \underline{\boldsymbol{k}}, .\rangle\,\boldsymbol{\ell} + e^{-\sigma}\langle \underline{\boldsymbol{\ell}}, .\rangle\,\boldsymbol{k}$

Example of normal frames



$\mathcal{H}$ = event horizon of Schwarzschild black hole
$\mathcal{S}_t$ = slice of constant Eddington-Finkelstein time

Second fundamental tensor of $\mathcal{S}_t$

Tensor $\boldsymbol{\mathcal{K}}$ of type $(1,2)$ relating the covariant derivative of a vector tangent to $\mathcal{S}_t$ taken by the spacetime connection $\boldsymbol{\nabla}$ to that taken by the connection $\boldsymbol{\mathcal{D}}$ in $\mathcal{S}_t$ compatible with the induced metric $\boldsymbol{q}$:

$$\forall(\boldsymbol{u},\boldsymbol{v}) \in \mathcal{T}(\mathcal{S}_t)^2, \quad \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} = \boldsymbol{\mathcal{D}}_{\boldsymbol{u}}\boldsymbol{v} + \boldsymbol{\mathcal{K}}(\boldsymbol{u},\boldsymbol{v})$$

Prop:

$\mathcal{K}^{\alpha}{}_{\beta\gamma} = \nabla_\mu q^\alpha{}_\nu \, q^\mu{}_\beta q^\nu{}_\gamma$

$\mathcal{K}^{\alpha}{}_{\beta\gamma} = n^\alpha \, \Theta^{(\boldsymbol{n})}_{\beta\gamma} - s^\alpha \, \Theta^{(\boldsymbol{s})}_{\beta\gamma} = e^{-\sigma} \left(k^\alpha \, \Theta^{(\boldsymbol{\ell})}_{\beta\gamma} + \ell^\alpha \, \Theta^{(\boldsymbol{k})}_{\beta\gamma} \right)$

Remark: for a hypersurface of normal $\boldsymbol{n}$ and extrinsic curvature $\boldsymbol{K}$,
$\mathcal{K}^{\alpha}{}_{\beta\gamma} = -n^\alpha K_{\beta\gamma}$

Normal fundamental forms

Extrinsic geometry of $\mathcal{S}_t$ not entirely specified by $\mathcal{K}$ (contrary to the hypersurface case)

$\mathcal{K}$ involves only the deformation tensors $\Theta^{(\cdot)}$ of the normals to $\mathcal{S}_t \Longrightarrow \mathcal{K}$ encodes only the part of the variation of $\mathcal{S}_t$'s normals which is parallel to $\mathcal{S}_t$

Variation of the two normals with respect to each other: encoded by the **normal fundamental forms** (also called *external rotation coefficients* or *connection on the normal bundle*, or if $\mathcal{H}$ is null, *Hájíček 1-form*):

1. $\boldsymbol{\Omega}^{(\boldsymbol{n})} := \boldsymbol{s} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{n}$ or $\Omega^{(\boldsymbol{n})}_{\alpha} := s_\mu \nabla_\nu n^\mu \, q^\nu{}_\alpha$

   $\boldsymbol{\Omega}^{(\boldsymbol{s})} := \boldsymbol{n} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{s}$
2. $\boldsymbol{\Omega}^{(\boldsymbol{\ell})} := \dfrac{1}{\boldsymbol{k} \cdot \boldsymbol{\ell}} \boldsymbol{k} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{\ell}$ or $\Omega^{(\boldsymbol{\ell})}_{\alpha} := \dfrac{1}{k_\rho \ell^\rho} k_\mu \nabla_\nu \ell^\mu \, q^\nu{}_\alpha$

   $\boldsymbol{\Omega}^{(\boldsymbol{k})} := \dfrac{1}{\boldsymbol{k} \cdot \boldsymbol{\ell}} \boldsymbol{\ell} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{k}$

Basic properties of the normal fundamental forms

From the definition: $\boldsymbol{\Omega}^{(\boldsymbol{s})} = -\boldsymbol{\Omega}^{(\boldsymbol{n})}$ and $\boldsymbol{\Omega}^{(\boldsymbol{k})} = -\boldsymbol{\Omega}^{(\boldsymbol{\ell})} + \boldsymbol{\mathcal{D}}\sigma$

Relation between the $(\boldsymbol{n}, \boldsymbol{s})$-type and the $(\boldsymbol{\ell}, \boldsymbol{k})$-type:
$\boldsymbol{\Omega}^{(\boldsymbol{\ell})} = \boldsymbol{\Omega}^{(\boldsymbol{n})}$ $[\boldsymbol{\ell} = \boldsymbol{n} + \boldsymbol{s}]$ and $\boldsymbol{\Omega}^{(\boldsymbol{k})} = -\boldsymbol{\Omega}^{(\boldsymbol{n})}$ $[\boldsymbol{k} = \boldsymbol{n} - \boldsymbol{s}]$

The normal fundamental forms are not unique

(contrary to the second fundamental tensor $\mathcal{K}$)
Dependence of the normal frame

1. $(\boldsymbol{n}, \boldsymbol{s}) \mapsto (\boldsymbol{n}', \boldsymbol{s}') \Longrightarrow$ $\boxed{\boldsymbol{\Omega}^{(\boldsymbol{n}')} = \boldsymbol{\Omega}^{(\boldsymbol{n})} + \boldsymbol{\mathcal{D}}\eta}$
2. $(\boldsymbol{\ell}, \boldsymbol{k}) \mapsto (\boldsymbol{\ell}', \boldsymbol{k}') \Longrightarrow$ $\boxed{\boldsymbol{\Omega}^{(\boldsymbol{\ell}')} = \boldsymbol{\Omega}^{(\boldsymbol{\ell})} + \boldsymbol{\mathcal{D}} \ln \lambda}$

"Surface-gravity" 1-forms

If the vector fields $(\boldsymbol{\ell}, \boldsymbol{k})$ are extended away from $\mathcal{S}_t$, define the 1-form

$$\boxed{\boldsymbol{\kappa}^{(\boldsymbol{\ell})} := \frac{1}{\boldsymbol{k}\cdot\boldsymbol{\ell}}\, \boldsymbol{k}\cdot\boldsymbol{\nabla}_{\boldsymbol{p}}\,\boldsymbol{\ell}} \qquad \text{or} \quad \kappa^{(\ell)}_\alpha := \frac{1}{k_\rho \ell^\rho} k_\mu \nabla_\nu \ell^\mu \, p^\nu{}_\alpha$$

where $\boldsymbol{p}$ is the orthogonal projector complementary to $\vec{\boldsymbol{q}}$: $\mathbf{1} = \vec{\boldsymbol{q}} + \boldsymbol{p}$.

NB: Since $\boldsymbol{p}$ is a projector in a direction transverse to $\mathcal{S}_t$, the 1-form $\boldsymbol{\kappa}^{(\boldsymbol{\ell})}$ is not intrinsic to the 2-surface $\mathcal{S}_t$: it depends on the choice of $\boldsymbol{\ell}$ away from $\mathcal{S}_t$

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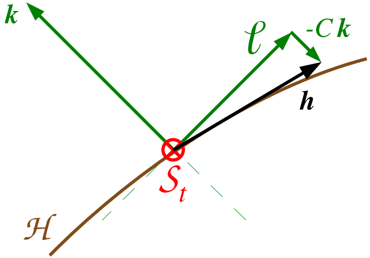
If $\boldsymbol{\ell}$ is extended along one of the two families of light rays emanating radially from $\mathcal{S}_t$, then $\boldsymbol{\ell}$ is pre-geodesic: $\boldsymbol{\nabla}_{\boldsymbol{\ell}}\,\boldsymbol{\ell} = \nu_{(\boldsymbol{\ell})}\,\boldsymbol{\ell}$, with the *inaffinity parameter* (*surface gravity* if $\boldsymbol{\ell}$ = null Killing vector of Kerr spacetime) given by the 1-form $\boldsymbol{\kappa}^{(\boldsymbol{\ell})}$ applied to $\boldsymbol{\ell}$:

$$\nu_{(\boldsymbol{\ell})} = \langle \boldsymbol{\kappa}^{(\boldsymbol{\ell})}, \boldsymbol{\ell} \rangle$$

Normal null frame associated with the evolution vector



The foliation $(\mathcal{S}_t)_{t\in\mathbb{R}}$ entirely fixes the ambiguities in the choice of the null normal frame $(\boldsymbol{\ell}, \boldsymbol{k})$, via the evolution vector $\boldsymbol{h}$: there exists a unique normal null frame $(\boldsymbol{\ell}, \boldsymbol{k})$ such that

$$\boldsymbol{h} = \boldsymbol{\ell} - C\boldsymbol{k} \qquad \text{and} \qquad \boldsymbol{\ell}\cdot\boldsymbol{k} = -1$$

Outline

Concept of black hole viscosity

- Hartle and Hawking (1972, 1973): introduced the concept of **black hole viscosity** when studying the response of the *event horizon* to external perturbations
- Damour (1979): 2-dimensional **Navier-Stokes** like equation for the event horizon $\Longrightarrow$ *shear viscosity* and *bulk viscosity*
- Thorne and Price (1986): **membrane paradigm** for black holes

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Shall we restrict the analysis to the event horizon ?

Can we extend the concept of viscosity to the local characterizations of black hole recently introduced, i.e. future outer trapping horizons and dynamical horizons ?

NB: *event horizon* = null hypersurface
future outer trapping horizon = null or spacelike hypersurface
dynamical horizon = spacelike hypersurface

Navier-Stokes equation in Newtonian fluid dynamics

$$\rho\left(\frac{\partial v^i}{\partial t} + v^j \nabla_j v^i\right) = -\nabla^i P + \mu \Delta v^i + \left(\zeta + \frac{\mu}{3}\right) \nabla^i (\nabla_j v^j) + f^i$$

or, in terms of fluid momentum density $\pi_i := \rho v_i$,

$$\boxed{\frac{\partial \pi_i}{\partial t} + v^j \nabla_j \pi_i + \theta \pi_i = -\nabla_i P + 2\mu \nabla^j \sigma_{ij} + \zeta \nabla_i \theta + f_i}$$

where θ is the fluid expansion:

$$\theta := \nabla_j v^j$$

and σ_{ij} the velocity shear tensor:

$$\sigma_{ij} := \frac{1}{2}(\nabla_i v_j + \nabla_j v_i) - \frac{1}{3}\theta\, \delta_{ij}$$

P is the pressure, μ the shear viscosity, ζ the bulk viscosity and f_i the density of external forces

Original Damour-Navier-Stokes equation

Hyp: $\mathcal{H}$ = null hypersurface (particular case: black hole **event horizon**)
Then $\boldsymbol{h} = \boldsymbol{\ell}$ ($C = 0$) ◂ reminder

Damour (1979) has derived from Einstein equation the relation

$$\boxed{{}^{\mathcal{S}}\mathcal{L}_{\boldsymbol{\ell}}\,\boldsymbol{\Omega}^{(\ell)} + \theta^{(\ell)}\boldsymbol{\Omega}^{(\ell)} = \boldsymbol{\mathcal{D}}\nu_{(\ell)} - \boldsymbol{\mathcal{D}}\cdot\vec{\boldsymbol{\sigma}}^{(\ell)} + \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\ell)} + 8\pi\vec{\boldsymbol{q}}^{*}\boldsymbol{T}\cdot\boldsymbol{\ell}}$$

or equivalently

$$\boxed{{}^{\mathcal{S}}\mathcal{L}_{\boldsymbol{\ell}}\,\boldsymbol{\pi} + \theta^{(\ell)}\boldsymbol{\pi} = -\boldsymbol{\mathcal{D}}P + 2\mu\boldsymbol{\mathcal{D}}\cdot\vec{\boldsymbol{\sigma}}^{(\ell)} + \zeta\boldsymbol{\mathcal{D}}\theta^{(\ell)} + \boldsymbol{f}}$$

with $\boldsymbol{\pi} := -\dfrac{1}{8\pi}\boldsymbol{\Omega}^{(\ell)}$ momentum surface density

$P := \dfrac{\nu_{(\ell)}}{8\pi}$ pressure

$\mu := \dfrac{1}{16\pi}$ shear viscosity

$\zeta := -\dfrac{1}{16\pi}$ bulk viscosity

$\boldsymbol{f} := -\vec{\boldsymbol{q}}^{*}\boldsymbol{T}\cdot\boldsymbol{\ell}$ external force surface density ($\boldsymbol{T}$ = stress-energy tensor)

Original Damour-Navier-Stokes equation (con't)

Introducing a coordinate system (t, x^1, x^2, x^3) such that

- t is compatible with $\boldsymbol{\ell}$: $\mathcal{L}_{\boldsymbol{\ell}}\, t = 1$
- $\mathcal{H}$ is defined by $x^1 = \text{const}$, so that $x^a = (x^2, x^3)$ are coordinates spanning $\mathcal{S}_t$

then

$$\boldsymbol{\ell} = \frac{\partial}{\partial t} + \boldsymbol{V}$$

with $\boldsymbol{V}$ tangent to $\mathcal{S}_t$: velocity of $\mathcal{H}$'s null generators with respect to the coordinates x^a [Damour, PRD **18**, 3598 (1978)].

Then

$$\theta^{(\boldsymbol{\ell})} = \mathcal{D}_a V^a + \frac{\partial}{\partial t} \ln \sqrt{q} \qquad q := \det q_{ab}$$

$$\sigma^{(\boldsymbol{\ell})}_{ab} = \frac{1}{2}\left(\mathcal{D}_a V_b + \mathcal{D}_b V_a\right) - \frac{1}{2}\theta^{(\boldsymbol{\ell})}\, q_{ab} + \frac{1}{2}\frac{\partial q_{ab}}{\partial t}$$

compare

Negative bulk viscosity of event horizons

From the Damour-Navier-Stokes equation, $\boxed{\zeta = -\frac{1}{16\pi} < 0}$

This negative value would yield to a *dilation or contraction instability* in an ordinary fluid

It is in agreement with the tendency of a null hypersurface to continually contract or expand

The event horizon is stabilized by the teleological condition imposing its expansion to vanish in the far future (equilibrium state reached)

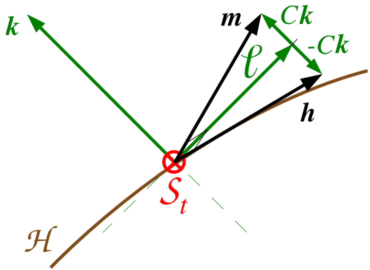
Generalization to the non-null case

Starting remark: in the null case, ℓ plays two different roles:

- evolution vector along $\mathcal{H}$ (e.g. term ${}^{\mathcal{S}}\!\boldsymbol{\mathcal{L}}_{\ell}$)
- normal to $\mathcal{H}$ (e.g. term $\vec{q}^{*}\boldsymbol{T}\cdot\boldsymbol{\ell}$)

When $\mathcal{H}$ is no longer null, these two roles have to be taken by two different vectors:

- evolution vector: obviously $\boldsymbol{h}$ reminder
- vector normal to $\mathcal{H}$: a natural choice is $\boxed{\boldsymbol{m} := \boldsymbol{\ell} + C\boldsymbol{k}}$



Generalized Damour-Navier-Stokes equation

Starting point of the calculation: contracted Ricci identity applied to the vector $\boldsymbol{m}$ and projected onto $\mathcal{S}_t$:

$$\left(\nabla_\mu \nabla_\nu m^\mu - \nabla_\nu \nabla_\mu m^\mu\right) q^\nu{}_\alpha = R_{\mu\nu} m^\mu q^\nu{}_\alpha$$

Final result:

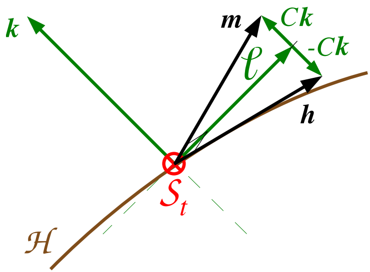
$$^{\mathcal{S}}\!\mathcal{L}_{\boldsymbol{h}}\, \boldsymbol{\Omega}^{(\boldsymbol{\ell})} + \theta^{(\boldsymbol{h})}\, \boldsymbol{\Omega}^{(\boldsymbol{\ell})} = \boldsymbol{\mathcal{D}} \langle \boldsymbol{\kappa}^{(\boldsymbol{\ell})}, \boldsymbol{h} \rangle - \boldsymbol{\mathcal{D}} \cdot \vec{\boldsymbol{\sigma}}^{(\boldsymbol{m})} + \frac{1}{2} \boldsymbol{\mathcal{D}} \theta^{(\boldsymbol{m})} - \theta^{(\boldsymbol{k})} \boldsymbol{\mathcal{D}} C + 8\pi \vec{\boldsymbol{q}}^{*} \boldsymbol{T} \cdot \boldsymbol{m}$$

- $\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$: normal fundamental form of $\mathcal{S}_t$ associated with null normal $\boldsymbol{\ell}$ reminder
- $\theta^{(\boldsymbol{h})}$, $\theta^{(\boldsymbol{m})}$ and $\theta^{(\boldsymbol{k})}$: expansion scalars of $\mathcal{S}_t$ along the vectors $\boldsymbol{h}$, $\boldsymbol{m}$ and $\boldsymbol{k}$ respectively reminder
- $\boldsymbol{\mathcal{D}}$: covariant derivative within $(\mathcal{S}_t, \boldsymbol{q})$
- $\boldsymbol{\kappa}^{(\boldsymbol{\ell})}$: "surface-gravity" 1-form associated with the null vector $\boldsymbol{\ell}$ reminder
- $\boldsymbol{\sigma}^{(\boldsymbol{m})}$: shear tensor of $\mathcal{S}_t$ along the vector $\boldsymbol{m}$ reminder
- C : half the scalar square of $\boldsymbol{h}$ reminder

Null limit

In the null limit,

$$\boldsymbol{h} = \boldsymbol{m} = \boldsymbol{\ell} \qquad \text{and} \qquad C = 0$$



and we recover the original Damour-Navier-Stokes equation:

$$\boxed{{}^{\mathcal{S}}\!\mathcal{L}_{\boldsymbol{\ell}}\, \boldsymbol{\Omega}^{(\ell)} + \theta^{(\ell)} \boldsymbol{\Omega}^{(\ell)} = \boldsymbol{\mathcal{D}} \nu_{(\ell)} - \boldsymbol{\mathcal{D}} \cdot \vec{\boldsymbol{\sigma}}^{(\ell)} + \frac{1}{2} \boldsymbol{\mathcal{D}} \theta^{(\ell)} + 8\pi \vec{\boldsymbol{q}}^{*} \boldsymbol{T} \cdot \boldsymbol{\ell}}$$

Case of future trapping horizons

Definition [Hayward, PRD **49**, 6467 (1994)] : $\mathcal{H}$ is a **future trapping horizon** iff $\theta^{(\ell)} = 0$ and $\theta^{(k)} < 0$.

The generalized Damour-Navier-Stokes equation reduces then to

$$ {}^{\mathcal{S}}\!\mathcal{L}_{h}\, \boldsymbol{\Omega}^{(\ell)} + \theta^{(h)}\, \boldsymbol{\Omega}^{(\ell)} = \boldsymbol{\mathcal{D}}\langle \boldsymbol{\kappa}^{(\ell)}, \boldsymbol{h}\rangle - \boldsymbol{\mathcal{D}} \cdot \vec{\boldsymbol{\sigma}}^{(m)} - \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(h)} - \theta^{(k)}\boldsymbol{\mathcal{D}}C + 8\pi \vec{\boldsymbol{q}}^{*}\boldsymbol{T} \cdot \boldsymbol{m} $$

NB: Notice the change of sign in the $-\frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(h)}$ term with respect to the original Damour-Navier-Stokes equation compare

Viscous fluid form

$$ {}^{\mathcal{S}}\!\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\,\boldsymbol{\pi} + \theta^{(\boldsymbol{h})}\boldsymbol{\pi} = -\boldsymbol{\mathcal{D}}P + \frac{1}{8\pi}\boldsymbol{\mathcal{D}}\cdot\vec{\boldsymbol{\sigma}}^{(\boldsymbol{m})} + \zeta\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{h})} + \boldsymbol{f} $$

with $\boldsymbol{\pi} := -\frac{1}{8\pi}\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$ momentum surface density

$P := \frac{\kappa}{8\pi}$ pressure

$\frac{1}{8\pi}\boldsymbol{\sigma}^{(\boldsymbol{m})}$ shear stress tensor

$\zeta := \frac{1}{16\pi}$ bulk viscosity

$\boldsymbol{f} := -\vec{\boldsymbol{q}}^{*}\boldsymbol{T}\cdot\boldsymbol{m} + \frac{\theta^{(\boldsymbol{k})}}{8\pi}\boldsymbol{\mathcal{D}}C$ external force surface density

Similar to the Damour-Navier-Stokes equation for an event horizon ◂ hyperlink, except for

- no Newtonian-fluid relation between *stress* and *strain*: $\boldsymbol{\sigma}^{(\boldsymbol{m})} \neq 2\mu\boldsymbol{\sigma}^{(\boldsymbol{h})}$
- positive bulk viscosity

This positive value of bulk viscosity shows that FOTHs and DHs behave as "ordinary" physical objects, in perfect agreement with their local nature

Generalized angular momentum

Definition [Booth & Fairhurst, CQG **22**, 4545 (2005)]: Let φ be a vector field on $\mathcal{H}$ which

- is tangent to $\mathcal{S}_t$
- has closed orbits
- has vanishing divergence with respect to the induced metric: $\boldsymbol{\mathcal{D}} \cdot \varphi = 0$

For dynamical horizons, $\theta^{(\boldsymbol{h})} \neq 0$ and there is a unique choice of φ as the generator (conveniently normalized) of the curves of constant $\theta^{(\boldsymbol{h})}$ [Hayward, PRD **74**, 104013 (2006)]

The *generalized angular momentum associated with* φ is then defined by

$$J(\varphi) := -\frac{1}{8\pi} \oint_{\mathcal{S}_t} \langle \boldsymbol{\Omega}^{(\ell)}, \varphi \rangle \, {}^{\mathcal{S}}\!\epsilon,$$

Remark 1: does not depend upon the choice of null vector $\boldsymbol{\ell}$, thanks to the divergence-free property of φ

Remark 2:

- coincides with Ashtekar & Krishnan's definition for a dynamical horizon
- coincides with Brown-York angular momentum if $\mathcal{H}$ is timelike and φ a Killing vector

Angular momentum flux law

Under the supplementary hypothesis that φ is transported along the evolution vector $\boldsymbol{h}$: $\mathcal{L}_{\boldsymbol{h}}\,\boldsymbol{\varphi} = 0$, the generalized Damour-Navier-Stokes equation leads to

$$\frac{d}{dt}J(\varphi) = -\oint_{\mathcal{S}_t} \boldsymbol{T}(\boldsymbol{m},\boldsymbol{\varphi})\,{}^{\mathcal{S}}\!\boldsymbol{\epsilon} - \frac{1}{16\pi}\oint_{\mathcal{S}_t}\left[\vec{\boldsymbol{\sigma}}^{(\boldsymbol{m})} : \mathcal{L}_{\boldsymbol{\varphi}}\,\boldsymbol{q} - 2\theta^{(\boldsymbol{k})}\boldsymbol{\varphi}\cdot\boldsymbol{\mathcal{D}}C\right]{}^{\mathcal{S}}\!\boldsymbol{\epsilon}$$

[Gourgoulhon, PRD **72**, 104007 (2005)]

Angular momentum flux law

Under the supplementary hypothesis that φ is transported along the evolution vector $\boldsymbol{h}$: $\mathcal{L}_{\boldsymbol{h}}\,\boldsymbol{\varphi} = 0$, the generalized Damour-Navier-Stokes equation leads to

$$\frac{d}{dt}J(\varphi) = -\oint_{\mathcal{S}_t} \boldsymbol{T}(\boldsymbol{m},\boldsymbol{\varphi})\,{}^{\mathcal{S}}\!\boldsymbol{\epsilon} - \frac{1}{16\pi}\oint_{\mathcal{S}_t}\left[\vec{\vec{\boldsymbol{\sigma}}}^{(\boldsymbol{m})} : \mathcal{L}_{\boldsymbol{\varphi}}\,\boldsymbol{q} - 2\theta^{(\boldsymbol{k})}\boldsymbol{\varphi}\cdot\mathcal{D}C\right]{}^{\mathcal{S}}\!\boldsymbol{\epsilon}$$

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- $\mathcal{H}$ = future trapping horizon :

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Outline

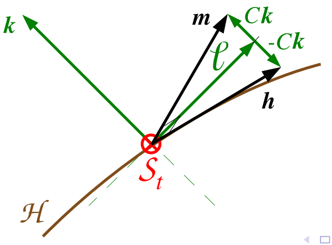
Starting point

From the Einstein equation, one can derive the following evolution law for any foliated hypersurface $\mathcal{H}$ [Gourgoulhon & Jaramillo, PRD **74**, 087502 (2006)] :

$$\begin{aligned} \mathcal{L}_{\boldsymbol{h}}\, \theta^{(\boldsymbol{m})} &= \kappa\, \theta^{(\boldsymbol{h})} - \frac{1}{2}\theta^{(\boldsymbol{h})}\theta^{(\boldsymbol{m})} - \boldsymbol{\sigma}^{(\boldsymbol{h})} : \boldsymbol{\sigma}^{(\boldsymbol{m})} - 8\pi \boldsymbol{T}(\boldsymbol{m}, \boldsymbol{h}) \\ & \quad + \theta^{(\boldsymbol{k})}\mathcal{L}_{\boldsymbol{h}}\, C + \boldsymbol{\mathcal{D}} \cdot \left(2C\vec{\boldsymbol{\Omega}}^{(\boldsymbol{\ell})} - \vec{\boldsymbol{\mathcal{D}}}C\right) \end{aligned}$$

where κ is the component along $\boldsymbol{\ell}$ of the "acceleration" of $\boldsymbol{h}$ in the decomposition

$$\boldsymbol{\nabla}_{\boldsymbol{h}}\boldsymbol{h} = \kappa\,\boldsymbol{\ell} + (C\kappa - \mathcal{L}_{\boldsymbol{h}}\, C)\boldsymbol{k} - \boldsymbol{\mathcal{D}}C$$



Two special cases

- **null hypersurface (event horizon)** : $\boldsymbol{h} = \boldsymbol{m} = \boldsymbol{\ell}$ and $C = 0$:

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, \theta^{(\boldsymbol{\ell})} + (\theta^{(\boldsymbol{\ell})})^2 - \kappa\, \theta^{(\boldsymbol{\ell})} = \frac{1}{2}(\theta^{(\boldsymbol{\ell})})^2 - \boldsymbol{\sigma}^{(\boldsymbol{\ell})} : \boldsymbol{\sigma}^{(\boldsymbol{\ell})} - 8\pi \boldsymbol{T}(\boldsymbol{\ell}, \boldsymbol{\ell})$$

$\longrightarrow$ this is the *null Raychaudhuri equation*

- **FOTH** : $\theta^{(\boldsymbol{\ell})} = 0 \Rightarrow \theta^{(\boldsymbol{m})} = -\theta^{(\boldsymbol{h})}$:

$$\begin{aligned} \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \theta^{(\boldsymbol{h})} + (\theta^{(\boldsymbol{h})})^2 + \kappa\, \theta^{(\boldsymbol{h})} \quad &= \quad \frac{1}{2}(\theta^{(\boldsymbol{h})})^2 + \boldsymbol{\sigma}^{(\boldsymbol{h})} : \boldsymbol{\sigma}^{(\boldsymbol{m})} + 8\pi \boldsymbol{T}(\boldsymbol{m}, \boldsymbol{h}) \\ &\quad -\theta^{(\boldsymbol{k})} \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, C + \boldsymbol{\mathcal{D}} \cdot \left(\vec{\boldsymbol{\mathcal{D}}} C - 2C \vec{\boldsymbol{\Omega}}^{(\boldsymbol{\ell})} \right) \end{aligned}$$

Notice the change of signs between the two cases

Energy equation

For a event horizon, Price and Thorne (1986) have defined the surface energy density as $\varepsilon := -\frac{1}{8\pi}\theta^{(\ell)}$

By analogy, define the surface energy density of a FOTH as $\boxed{\varepsilon := -\frac{1}{8\pi}\theta^{(m)}}$

Then the above evolution equation becomes

$$\boxed{\mathcal{L}_{h}\,\varepsilon + (\varepsilon + P)\theta^{(h)} = \frac{1}{8\pi}\boldsymbol{\sigma}^{(h)}{:}\boldsymbol{\sigma}^{(m)} + \zeta(\theta^{(h)})^2 - \boldsymbol{\mathcal{D}}\cdot\boldsymbol{Q} + \mathcal{R}}$$

[Gourgoulhon & Jaramillo, PRD **74**, 087502 (2006)]

with $P := \frac{\kappa}{8\pi}$ pressure, $\frac{1}{8\pi}\boldsymbol{\sigma}^{(m)}$ shear stress tensor

$\boldsymbol{\sigma}^{(h)}$ shear strain tensor, $\boxed{\zeta := \frac{1}{16\pi} > 0}$ bulk viscosity

$\boldsymbol{Q} := \frac{1}{4\pi}\left[C\vec{\boldsymbol{\Omega}}^{(\ell)} - \frac{1}{2}\vec{\boldsymbol{\mathcal{D}}}C\right]$ heat flux

$\mathcal{R} = \boldsymbol{T}(\boldsymbol{m},\boldsymbol{h}) - \frac{\theta^{(\boldsymbol{k})}}{8\pi}\mathcal{L}_{\boldsymbol{h}}\,C$ external energy production rate

We recover the positiveness of the bulk viscosity for a FOTH