Introduction to spectral methods

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Plan

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1

Basic principles

Solving a partial differential equation

Consider the PDE with boundary condition

$$Lu(\boldsymbol{x}) = s(\boldsymbol{x}), \qquad \boldsymbol{x} \in U \subset \mathbb{R}^d \tag{1}$$

$$Bu(\boldsymbol{y}) = 0, \qquad \boldsymbol{y} \in \partial U, \tag{2}$$

where L and B are linear differential operators.

Question: What is a numerical solution of (1)-(2) ?

Answer: It is a function $\bar{u}$ which satisfies (2) and makes the residual

$$R := L\bar{u} - s$$

small.

What do you mean by "small" ?

Answer in the framework of

Method of Weighted Residuals (MWR):

Search for solutions $\bar{u}$ in a finite-dimensional sub-space $\mathcal{P}_N$ of some Hilbert space $\mathcal{W}$ (typically a L^2 space).

Expansion functions = *trial functions* : basis of $\mathcal{P}_N$: $(\phi_0, \ldots, \phi_N)$

$\bar{u}$ is expanded in terms of the trial functions: $\bar{u}(\boldsymbol{x}) = \sum_{n=0}^{N} \tilde{u}_n \, \phi_n(\boldsymbol{x})$

Test functions : family of functions $(\chi_0, \ldots, \chi_N)$ to define the smallness of the residual R, by means of the Hilbert space scalar product:

$$\forall\, n \in \{0, \ldots, N\}, \quad (\chi_n, R) = 0$$

Various numerical methods

Classification according to the trial functions ϕ_n:

Finite difference: trial functions $=$ overlapping local polynomials of low order

Finite element: trial functions $=$ local smooth functions (polynomial of fixed degree which are non-zero only on subdomains of U)

Spectral methods : trial functions $=$ global smooth functions (*example:* Fourier series)

Various spectral methods

All spectral method: trial functions (ϕ_n) = complete family (basis) of smooth global functions

Classification according to the test functions χ_n:

Galerkin method: test functions = trial functions: $\chi_n = \phi_n$ and each ϕ_n satisfy the boundary condition : $B\phi_n(\boldsymbol{y}) = 0$

tau method: (Lanczos 1938) test functions = (most of) trial functions: $\chi_n = \phi_n$ but the ϕ_n do not satisfy the boundary conditions; the latter are enforced by an additional set of equations.

collocation or **pseudospectral method:** test functions = delta functions at special points, called *collocation points*: $\chi_n = \delta(\boldsymbol{x} - \boldsymbol{x}_n)$.

Solving a PDE with a Galerkin method

Let us return to Equation (1).

Since $\chi_n = \phi_n$, the smallness condition for the residual reads, for all $n \in \{0, \ldots, N\}$,

$$
\begin{aligned}
(\phi_n, R) = 0 \quad &\Longleftrightarrow \quad (\phi_n, L\bar{u} - s) = 0 \\
&\Longleftrightarrow \quad \left(\phi_n, L \sum_{k=0}^{N} \tilde{u}_k \, \phi_k\right) - (\phi_n, s) = 0 \\
&\Longleftrightarrow \quad \sum_{k=0}^{N} \tilde{u}_k \, (\phi_n, L\phi_k) - (\phi_n, s) = 0 \\
&\Longleftrightarrow \quad \boxed{\sum_{k=0}^{N} L_{nk} \, \tilde{u}_k = (\phi_n, s)}, \qquad (3)
\end{aligned}
$$

where L_{nk} denotes the matrix $L_{nk} := (\phi_n, L\phi_k)$.

$\rightarrow$ Solving for the linear system (3) leads to the $(N+1)$ coefficients $\tilde{u}_k$ of $\bar{u}$

Solving a PDE with a tau method

Here again $\chi_n = \phi_n$, but the ϕ_n's do not satisfy the boundary condition: $B\phi_n(\boldsymbol{y}) \neq 0$. Let (g_p) be an orthonormal basis of $M+1 < N+1$ functions on the boundary ∂U and let us expand $B\phi_n(\boldsymbol{y})$ upon it:

$$B\phi_n(\boldsymbol{y}) = \sum_{p=0}^{M} b_{pn}\, g_p(\boldsymbol{y})$$

The boundary condition (2) then becomes

$$Bu(\boldsymbol{y}) = 0 \Longleftrightarrow \sum_{k=0}^{N} \sum_{p=0}^{M} \tilde{u}_k\, b_{pk}\, g_p(\boldsymbol{y}) = 0,$$

hence the $M+1$ conditions:

$$\sum_{k=0}^{N} b_{pk}\, \tilde{u}_k = 0 \qquad 0 \leq p \leq M$$

Solving a PDE with a tau method (cont'd)

The system of linear equations for the $N+1$ coefficients $\tilde{u}_n$ is then taken to be the $N-M$ first raws of the Galerkin system (3) plus the $M+1$ equations above:

$$\begin{aligned} \sum_{k=0}^{N} L_{nk}\,\tilde{u}_k &= (\phi_n, s) & \qquad 0 \le n \le N-M-1 \\ \sum_{k=0}^{N} b_{pk}\,\tilde{u}_k &= 0 & \qquad 0 \le p \le M \end{aligned}$$

The solution $(\tilde{u}_k)$ of this system gives rise to a function $\bar{u} = \sum_{k=0}^{N} \tilde{u}_k\,\phi_k$ such that

$$L\bar{u}(\boldsymbol{x}) = s(\boldsymbol{x}) + \sum_{p=0}^{M} \tau_p\,\phi_{N-M+p}(\boldsymbol{x})$$

Solving a PDE with a pseudospectral (collocation) method

This time: $\chi_n(\boldsymbol{x}) = \delta(\boldsymbol{x} - \boldsymbol{x}_n)$, where the $(\boldsymbol{x}_n)$ constitute the collocation points. The smallness condition for the residual reads, for all $n \in \{0, \ldots, N\}$,

$$\begin{aligned}(\chi_n, R) = 0 \quad &\Longleftrightarrow \quad (\delta(\boldsymbol{x} - \boldsymbol{x}_n), R) = 0 \Longleftrightarrow R(\boldsymbol{x}_n) = 0 \Longleftrightarrow Lu(\boldsymbol{x}_n) = s(\boldsymbol{x}_n) \\ &\Longleftrightarrow \quad \sum_{k=0}^{N} L\phi_k\,(\boldsymbol{x}_n)\tilde{u}_k = s(\boldsymbol{x}_n) \end{aligned} \tag{4}$$

The boundary condition is imposed as in the tau method. One then drops $M+1$ raws in the linear system (4) and solve the system

$$\boxed{\begin{array}{lcll} \displaystyle\sum_{k=0}^{N} L\phi_k(\boldsymbol{x}_n)\,\tilde{u}_k & = & s(\boldsymbol{x}_n) & \qquad 0 \leq n \leq N - M - 1 \\ \displaystyle\sum_{k=0}^{N} b_{pk}\,\tilde{u}_k & = & 0 & \qquad 0 \leq p \leq M \end{array}}$$

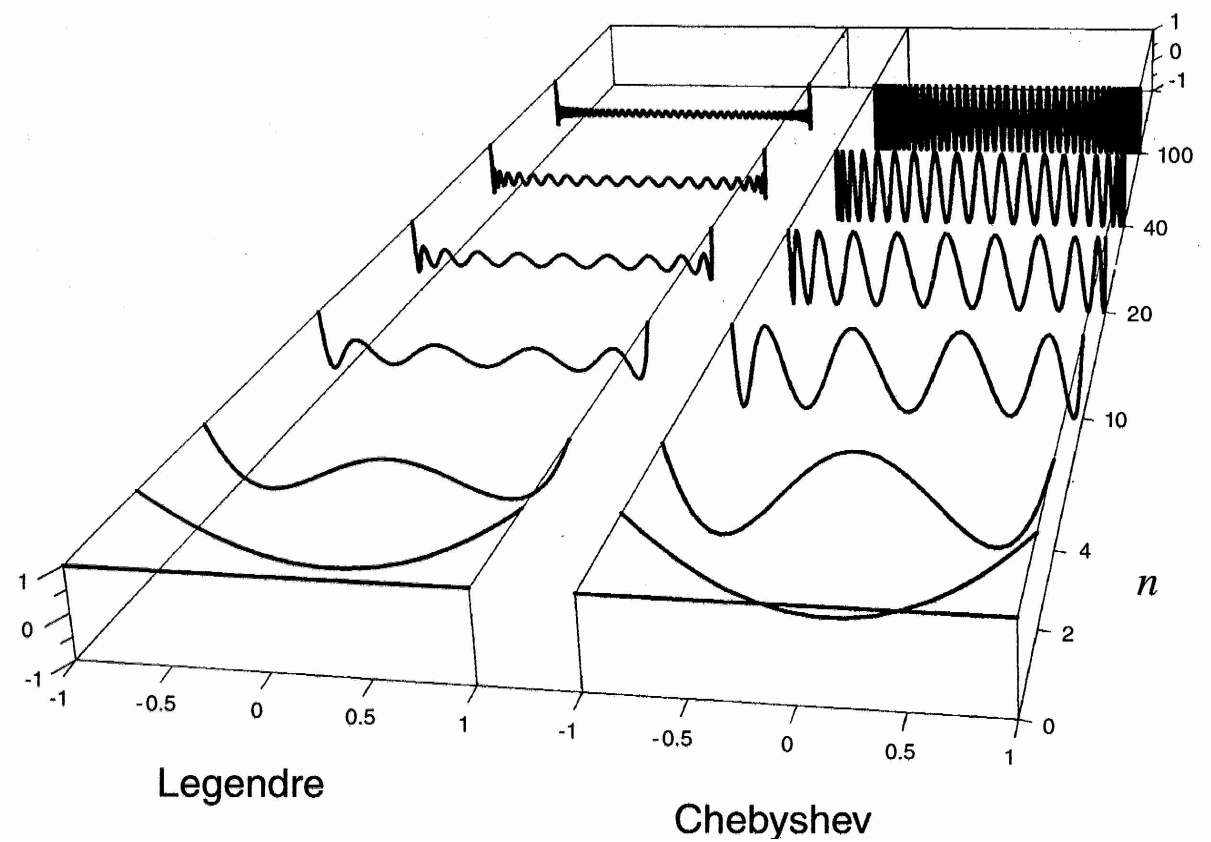
What choice for the trial functions ϕ_n ?

Periodic problem : $\phi_n =$ trigonometric polynomials (Fourier series)

Non-periodic problem : $\phi_n =$ orthogonal polynomials

2
Legendre and Chebyshev expansions

Legendre and Chebyshev polynomials



[from Fornberg (1998)]

Families of orthogonal polynomials on $[-1, 1]$:

Legendre polynomials:

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{2}{2n+1} \delta_{mn}$$

Chebyshev polynomials:

$$\int_{-1}^{1} T_m(x) T_n(x)\, \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}(1 + \delta_{0n})\, \delta_{mn}$$

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$ $\qquad$ $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$

Both Legendre and Chebyshev polynomials are a subclass of Jacobi polynomials

Properties of Chebyshev polynomials

Definition: $\cos n\theta = T_n(\cos\theta)$

Recurrence relation : $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

Eigenfunctions of the singular Sturm-Liouville problem:

$$\frac{d}{dx}\left(\sqrt{1-x^2}\,\frac{dT_n}{dx}\right) = -\frac{n^2}{\sqrt{1-x^2}}\,T_n(x)$$

Orthogonal family in the Hilbert space $L^2_w[-1,1]$, equiped with the weight $w(x) = (1-x^2)^{-1/2}$:

$$(f,g) := \int_{-1}^{1} f(x)\,g(x)\,w(x)\,dx$$

Polynomial interpolation of functions

Given a set of $N+1$ nodes $(x_i)_{0\leq i\leq N}$ in $[-1,1]$, the Lagrangian interpolation of a function $u(x)$ is defined by the N-th degree polynomial:

$$I_N u(x) = \sum_{i=0}^{N} u(x_i) \prod_{\substack{j=0\\ j\neq i}}^{N} \left(\frac{x - x_j}{x_i - x_j} \right)$$

Cauchy theorem: there exists $x_0 \in [-1,1]$ such that

$$u(x) - I_N u(x) = \frac{1}{(N+1)!} u^{(N+1)}(x_0) \prod_{i=0}^{N} (x - x_i)$$

Minimize $u(x) - I_N u(x)$ independently of $u \iff$ minimize $\prod_{i=0}^{N} (x - x_i)$

Chebyshev interpolation of functions

Note that $\prod_{i=0}^{N}(x - x_i)$ is a polynomial of degree $N+1$ of the type $x^{N+1} + a_N x^N + \cdots$ (leading coefficient $= 1$).

Characterization of Chebyshev polynomials: Among all the polynomials of degree n with leading coefficient 1, the unique polynomial which has the smallest maximum on $[-1, 1]$ is the n-th Chebyshev polynomial divided by 2^{n-1} : $T_n(x)/2^{n-1}$.

$\Longrightarrow$ take the nodes x_i to be the $N+1$ zeros of the Chebyshev polynomial $T_{N+1}(x)$:

$$\prod_{i=0}^{N}(x - x_i) = \frac{1}{2^N} T_{N+1}(x)$$

$$x_i = -\cos\left(\frac{2i+1}{2(N+1)}\pi\right) \qquad 0 \le i \le N$$

Spectral expansions : continuous (exact) coefficients

Case where the trial functions are orthogonal polynomials ϕ_n in $L^2_w[-1,1]$ for some weight $w(x)$ (e.g. Legendre $(w(x)=1)$ or Chebyshev $(w(x)=(1-x^2)^{-1/2})$ polynomials).

The spectral representation of any function u is its orthogonal projection on the space of polynomials of degree $\leq N$:

$$P_N\, u(x) = \sum_{n=0}^{N} \tilde{u}_n \phi_n(x)$$

where the coefficients $\tilde{u}_n$ are given by the scalar product:

$$\tilde{u}_n = \frac{1}{(\phi_n, \phi_n)}\, (\phi_n, u) \qquad \text{with} \quad (\phi_n, u) := \int_{-1}^{1} \phi_n(x)\, u(x)\, w(x)\, dx \tag{5}$$

The integral (5) cannot be computed exactly...

Spectral expansions : discrete coefficients

The most precise way of numerically evaluating the integral (5) is given by
Gauss integration :

$$\int_{-1}^{1} f(x)\, w(x)\, dx = \sum_{i=0}^{N} w_i\, f(x_i) \tag{6}$$

where the x_i's are the $N+1$ zeros of the polynomial ϕ_{N+1} and the coefficients w_i are the solutions of the linear system $\sum_{j=0}^{N} x_j^i\, w_j = \int_{-1}^{1} x^i\, w(x)\, dx$.

Formula (6) is exact for any polynomial $f(x)$ of degree $\leq 2N+1$

Adaptation to include the boundaries of $[-1,1]$ *:* $x_0 = -1,\ x_1, \ldots, x_{N-1},\ x_N = 1$
$\Rightarrow$ **Gauss-Lobatto integration :** $x_i =$ zeros of the polynomial $P = \phi_{N+1} + \lambda\phi_N + \mu\phi_{N-1}$, with λ and μ such that $P(-1) = P(1) = 0$. Exact for any polynomial $f(x)$ of degree $\leq 2N-1$.

Spectral expansions : discrete coefficients (con't)

Define the discrete coefficients $\hat{u}_n$ to be the Gauss-Lobatto approximations of the integrals (5) giving the $\tilde{u}_n$'s :

$$\boxed{\hat{u}_n := \frac{1}{(\phi_n, \phi_n)} \sum_{i=0}^{N} w_i \, \phi_n(x_i) \, u(x_i)} \tag{7}$$

The actual numerical representation of a function u is then the polynomial formed from the discrete coefficients:

$$\boxed{I_N \, u(x) := \sum_{n=0}^{N} \hat{u}_n \phi_n(x)},$$

instead of the orthogonal projection $P_N u$ involving the $\tilde{u}_n$.

Note: if $(\phi_n) =$ Chebyshev polynomials, the coefficients $(\hat{u}_n)$ can be computed by means of a FFT [i.e. in $\sim N \ln N$ operations instead of the $\sim N^2$ operations of the matrix product (7)].

Aliasing error

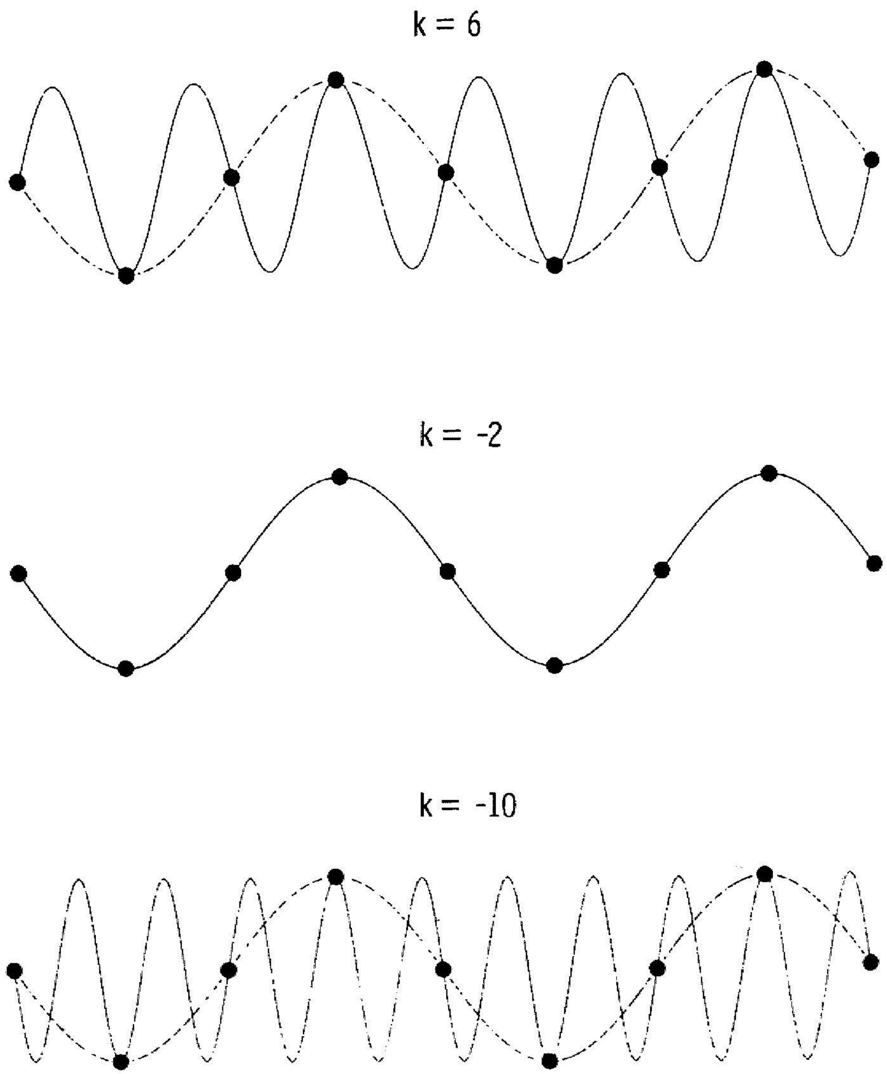
Proposition: $I_N\, u(x)$ is the interpolating polynomial of u through the $N+1$ nodes $(x_i)_{0\leq i\leq N}$ of the Gauss-Lobatto quadrature: $I_N\, u(x_i) = u(x_i) \quad 0 \leq i \leq N$

On the contrary the orthogonal projection $P_N u$ does not necessarily pass through the points (x_i).

The difference between $I_N\, u$ and $P_N\, u$, i.e. between the coefficients $\hat{u}_n$ and $\tilde{u}_n$, is called the **aliasing error**.

It can be seen as a contamination of $\hat{u}_n$ by the high frequencies $\tilde{u}_k$ with $k > N$, when performing the Gauss-Lobato integration (7).

Illustrating the aliasing error: case of Fourier series



Alias of a $\sin(-2x)$ wave by a $\sin(6x)$ wave

Alias of a $\sin(-2x)$ wave by a $\sin(-10x)$ wave

[from Canuto et al. (1998)]

Convergence of Legendre and Chebyshev expansions

Hyp.: u sufficiently regular so that all derivatives up to some order $m \geq 1$ exist.

Legendre: truncation error :

$$\|P_N u - u\|_{L^2} \leq \frac{C}{N^m} \sum_{k=0}^{m} \|u^{(k)}\|_{L^2}$$

$$\|P_N u - u\|_{\infty} \leq \frac{C}{N^{m-1/2}} V(u^{(m)})$$

interpolation error :

$$\|I_N u - u\|_{L^2} \leq \frac{C}{N^{m-1/2}} \sum_{k=0}^{m} \|u^{(k)}\|_{L^2}$$

Chebyshev: truncation error :

$$\|P_N u - u\|_{L^2_w} \leq \frac{C}{N^m} \sum_{k=0}^{m} \|u^{(k)}\|_{L^2_w}$$

$$\|P_N u - u\|_{\infty} \leq \frac{C(1+\ln N)}{N^m} \sum_{k=0}^{m} \|u^{(k)}\|_{\infty}$$

interpolation error :

$$\|I_N u - u\|_{L^2_w} \leq \frac{C}{N^m} \sum_{k=0}^{m} \|u^{(k)}\|_{L^2_w}$$

$$\|I_N u - u\|_{\infty} \leq \frac{C}{N^{m-1/2}} \sum_{k=0}^{m} \|u^{(k)}\|_{L^2_w}$$

Evanescent error

From the above decay rates, we conclude that for a C^∞ function, the error in the spectral expansion decays more rapidly than any power of $1/N$. In practice, it is an exponential decay.

Such a behavior is a key property of spectral methods and is called **evanescent error**.

(Remember that for a finite difference method of order k, the error decays only as $1/N^k$).

3
An example

... at last !

A simple differential equation with boundary conditions

Let us consider the 1-D second-order linear (P)DE

$$\frac{d^2u}{dx^2} - 4\frac{du}{dx} + 4u = e^x + C, \qquad x \in [-1, 1] \tag{8}$$

with the Dirichlet boundary conditions

$$u(-1) = 0 \qquad \text{and} \qquad u(1) = 0 \tag{9}$$

and where C is a constant: $C = -4e/(1 + e^2)$.

The exact solution of the system (8)-(9) is

$$u(x) = e^x - \frac{\sinh 1}{\sinh 2}\, e^{2x} + \frac{C}{4}$$

Resolution by means of a Chebyshev spectral method

Let us search for a numerical solution of (8)-(9) by means of the five first Chebyshev polynomials: $T_0(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$ and $T_4(x)$, i.e. we adopt $N = 4$.

Let us first expand the source $s(x) = e^x + C$ onto the Chebyshev polynomials:

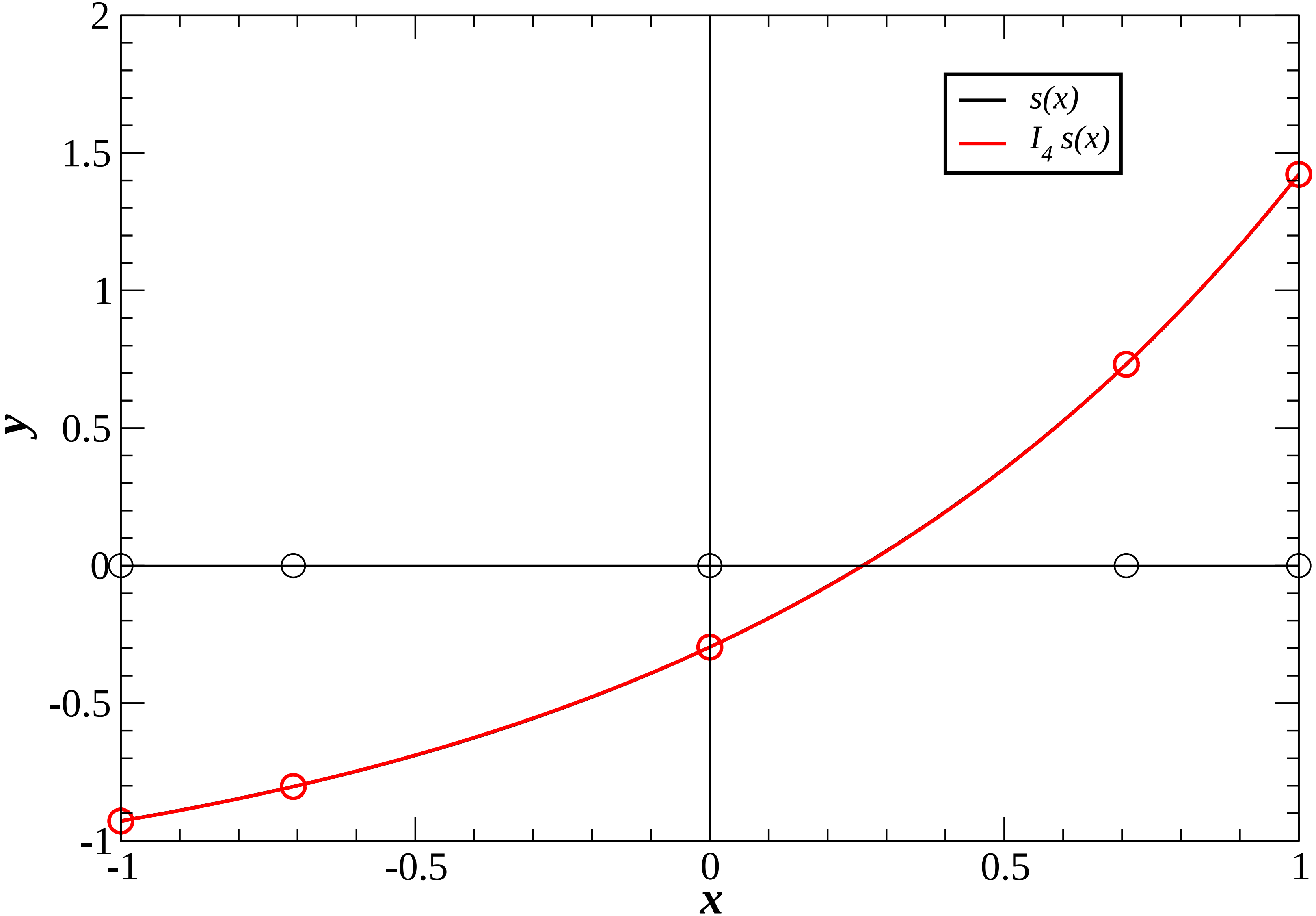
$$P_4\, s(x) = \sum_{n=0}^{4} \tilde{s}_n\, T_n(x) \qquad \text{and} \qquad I_4\, s(x) = \sum_{n=0}^{4} \hat{s}_n\, T_n(x)$$

with

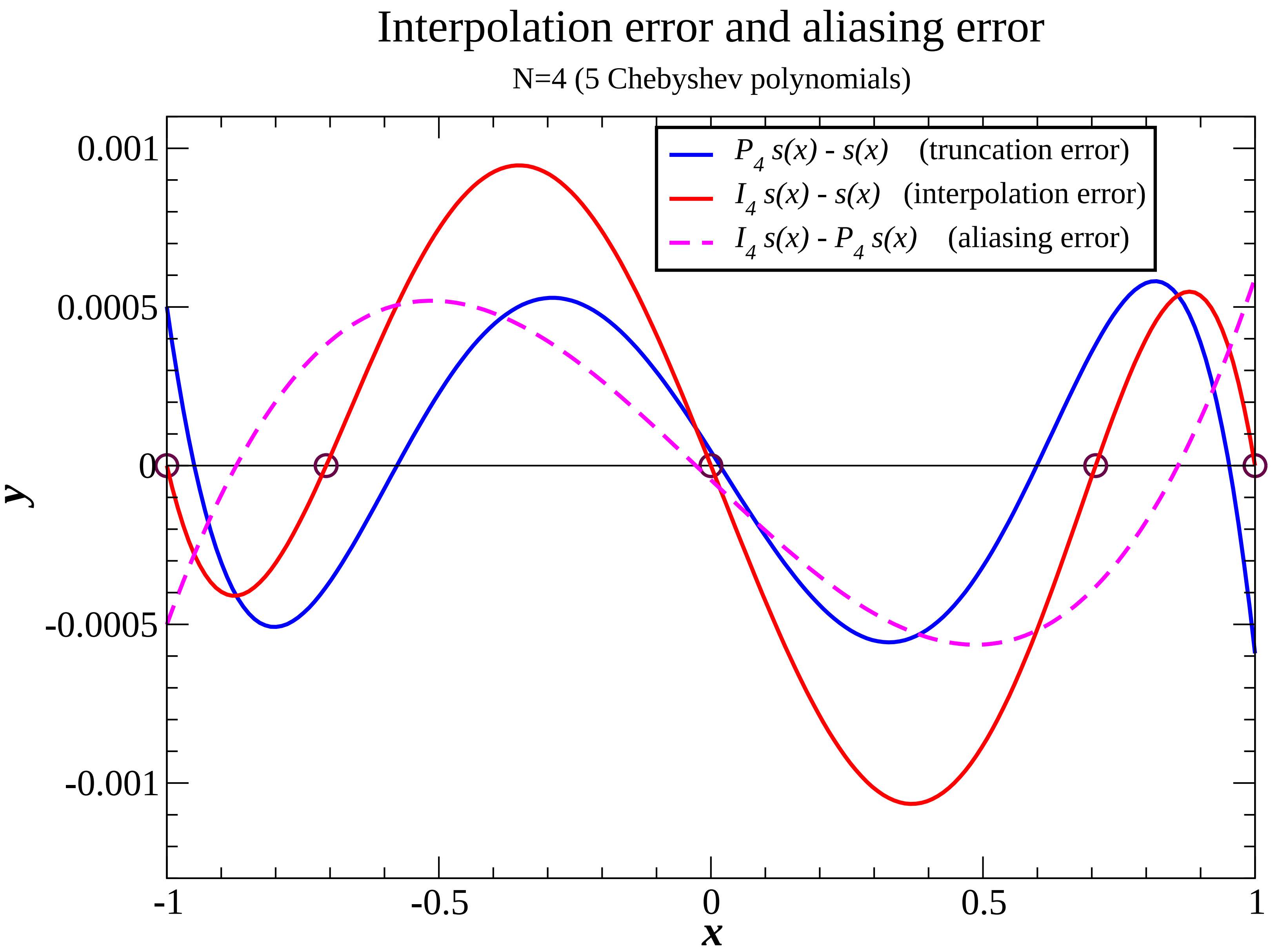
$$\tilde{s}_n = \frac{2}{\pi(1+\delta_{0n})} \int_{-1}^{1} T_n(x) s(x) \frac{dx}{\sqrt{1-x^2}} \qquad \text{and} \qquad \hat{s}_n = \frac{2}{\pi(1+\delta_{0n})} \sum_{i=0}^{4} w_i T_n(x_i) s(x_i)$$

the x_i's being the 5 Gauss-Lobatto quadrature points for the weight $w(x) = (1-x^2)^{-1/2}$: $\{x_i\} = \{-\cos(i\pi/4), 0 \le i \le 4\} = \left\{-1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1\right\}$

The source and its Chebyshev interpolant



$$\hat{s}_0 = -0.03004,\ \hat{s}_1 = 1.130,\ \hat{s}_2 = 0.2715,\ \hat{s}_3 = 0.04488,\ \hat{s}_4 = 0.005474$$



$$\hat{s}_n - \tilde{s}_n = 2.0\,10^{-7},\ 3.2\,10^{-6},\ 4.5\,10^{-5},\ 5.4\,10^{-4},\ 1.0\,10^{-12}$$

Matrix of the differential operator

The matrices of derivative operators with respect to the Chebyshev basis $(T_0, T_1, T_2, T_3, T_4)$ are

$$\frac{d}{dx} = \begin{pmatrix} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \frac{d^2}{dx^2} = \begin{pmatrix} 0 & 0 & 4 & 0 & 32 \\ 0 & 0 & 0 & 24 & 0 \\ 0 & 0 & 0 & 0 & 48 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

so that the matrix of the differential operator $\frac{d^2}{dx^2} - 4\frac{d}{dx} + 4\,\mathrm{Id}$ on the r.h.s. of Eq. (8) is

$$A_{kl} = \begin{pmatrix} 4 & -4 & 4 & -12 & 32 \\ 0 & 4 & -16 & 24 & -32 \\ 0 & 0 & 4 & -24 & 48 \\ 0 & 0 & 0 & 4 & -32 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

Resolution by means of a Galerkin method

Galerkin basis :

$$\begin{aligned}\phi_0(x) &:= T_2(x) - T_0(x) = 2x^2 - 2\\ \phi_1(x) &:= T_3(x) - T_1(x) = 4x^3 - 4x\\ \phi_2(x) &:= T_4(x) - T_0(x) = 8x^4 - 8x^2\end{aligned}$$

Each of the ϕ_i satisfies the boundary conditions: $\phi_i(-1) = \phi_i(1) = 0$. Note that the ϕ_i's are not orthogonal.

Transformation matrix Chebyshev $\longrightarrow$ Galerkin: $\tilde{\phi}_{ki} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

such that $\phi_i(x) = \sum_{k=0}^{4} \tilde{\phi}_{ki}\, T_k(x)$.

Chebyshev coefficients and Galerkin coefficients: $u(x) = \sum_{k=0}^{4} \tilde{u}_k\, T_k(x) = \sum_{i=0}^{2} \tilde{\tilde{u}}_i\, \phi_i(x)$

The matrix $\tilde{\phi}_{ki}$ relates the two sets of coefficients via the matrix product $\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{\phi}} \times \tilde{\tilde{\boldsymbol{u}}}$

Galerkin system

For Galerkin method, the test functions are equal to the trial functions, so that the condition of small residual writes

$$(\phi_i, Lu - s) = 0 \Longleftrightarrow \sum_{j=0}^{3} (\phi_i, L\phi_j)\, \tilde{\tilde{u}}_j = (\phi_i, s)$$

with

$$\begin{aligned}
(\phi_i, L\phi_j) &= \sum_{k=0}^{4}\sum_{l=0}^{4} (\tilde{\phi}_{ki} T_k, L\tilde{\phi}_{lj} T_l) = \sum_{k=0}^{4}\sum_{l=0}^{4} \tilde{\phi}_{ki}\tilde{\phi}_{lj} (T_k, LT_l) \\
&= \sum_{k=0}^{4}\sum_{l=0}^{4} \tilde{\phi}_{ki}\tilde{\phi}_{lj} (T_k, \sum_{m=0}^{4} A_{ml} T_m) = \sum_{k=0}^{4}\sum_{l=0}^{4} \tilde{\phi}_{ki}\tilde{\phi}_{lj} \sum_{m=0}^{4} A_{ml} (T_k, T_m) \\
&= \sum_{k=0}^{4}\sum_{l=0}^{4} \tilde{\phi}_{ki}\tilde{\phi}_{lj} \frac{\pi}{2} (1 + \delta_{0k}) A_{kl} = \frac{\pi}{2} \sum_{k=0}^{4}\sum_{l=0}^{4} (1 + \delta_{0k}) \tilde{\phi}_{ki} A_{kl} \tilde{\phi}_{lj}
\end{aligned}$$

Resolution of the Galerkin system

In the above expression appears the transpose matrix

$$Q_{ik} := {}^{\mathrm{t}}\left[(1+\delta_{0k})\tilde{\phi}_{ki}\right] = \begin{pmatrix} -2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The small residual condition amounts then to solve the following linear system in $\tilde{\tilde{\boldsymbol{u}}} = (\tilde{\tilde{u}}_0, \tilde{\tilde{u}}_1, \tilde{\tilde{u}}_2)$:

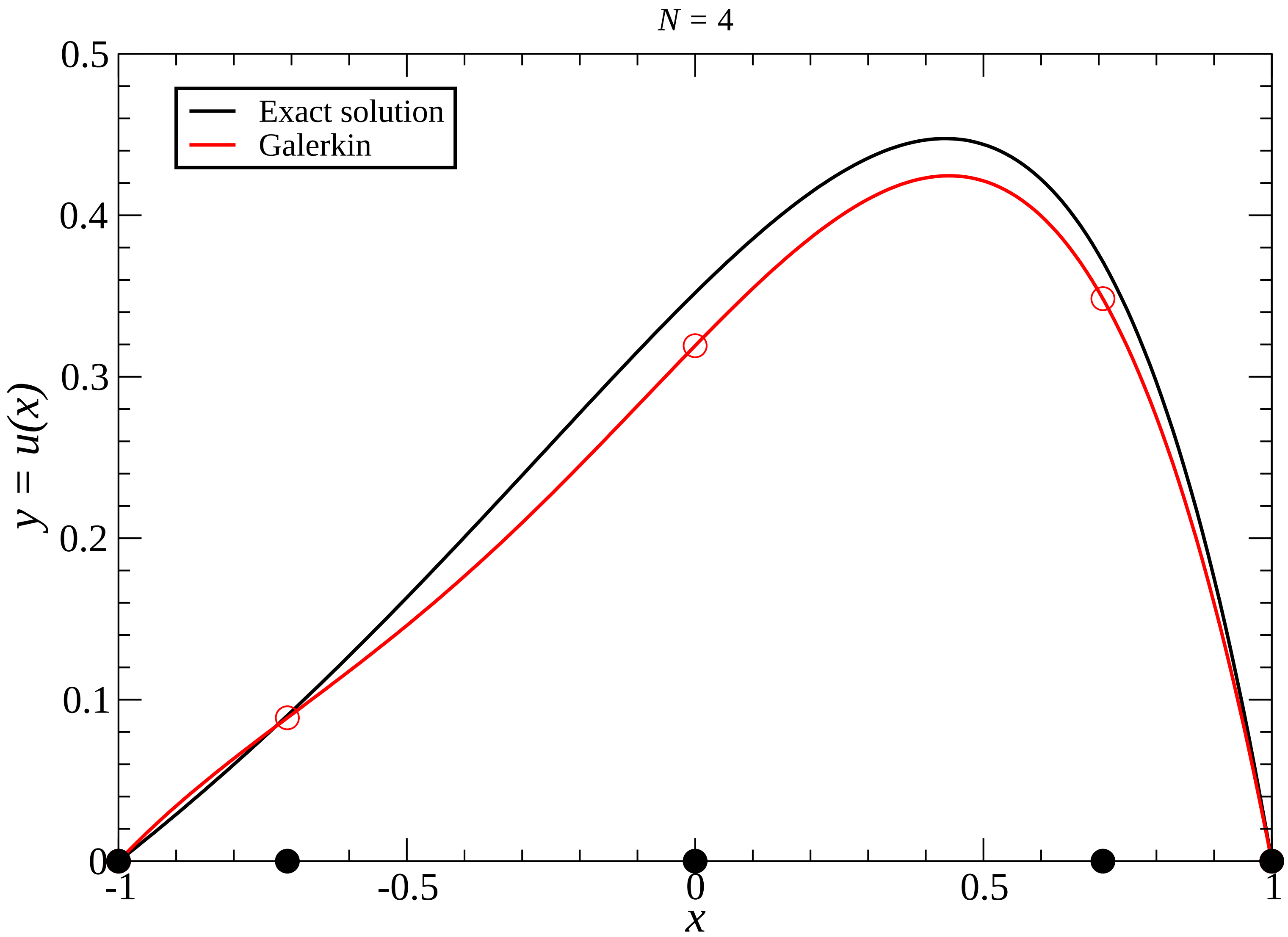
$$\boldsymbol{Q} \times \boldsymbol{A} \times \tilde{\boldsymbol{\phi}} \times \tilde{\tilde{\boldsymbol{u}}} = \boldsymbol{Q} \times \tilde{\boldsymbol{s}}$$

with $\boldsymbol{Q} \times \boldsymbol{A} \times \tilde{\boldsymbol{\phi}} = \begin{pmatrix} 4 & -8 & -8 \\ 16 & -16 & 0 \\ 0 & 16 & -52 \end{pmatrix}$ and $\boldsymbol{Q} \times \tilde{\boldsymbol{s}} = \begin{pmatrix} 0.331625 \\ -1.08544 \\ 0.0655592 \end{pmatrix}$

The solution is found to be $\tilde{\tilde{u}}_0 = -0.1596,\ \tilde{\tilde{u}}_1 = -0.09176,\ \tilde{\tilde{u}}_2 = -0.02949$.

The Chebyshev coefficients are obtained by taking the matrix product by $\tilde{\boldsymbol{\phi}}$:
$\tilde{u}_0 = 0.1891,\ \tilde{u}_1 = 0.09176,\ \tilde{u}_2 = -0.1596,\ \tilde{u}_3 = -0.09176,\ \tilde{u}_4 = -0.02949$

Comparison with the exact solution



Exact solution: $u(x) = e^x - \dfrac{\sinh 1}{\sinh 2}\, e^{2x} - \dfrac{e}{1+e^2}$

Resolution by means of a tau method

Tau method : trial functions = test functions = Chebyshev polynomials $T_0, \ldots, T_4$.
Enforce the boundary conditions by additional equations.
Since $T_n(-1) = (-1)^n$ and $T_n(1) = 1$, the boundary condition operator has the matrix

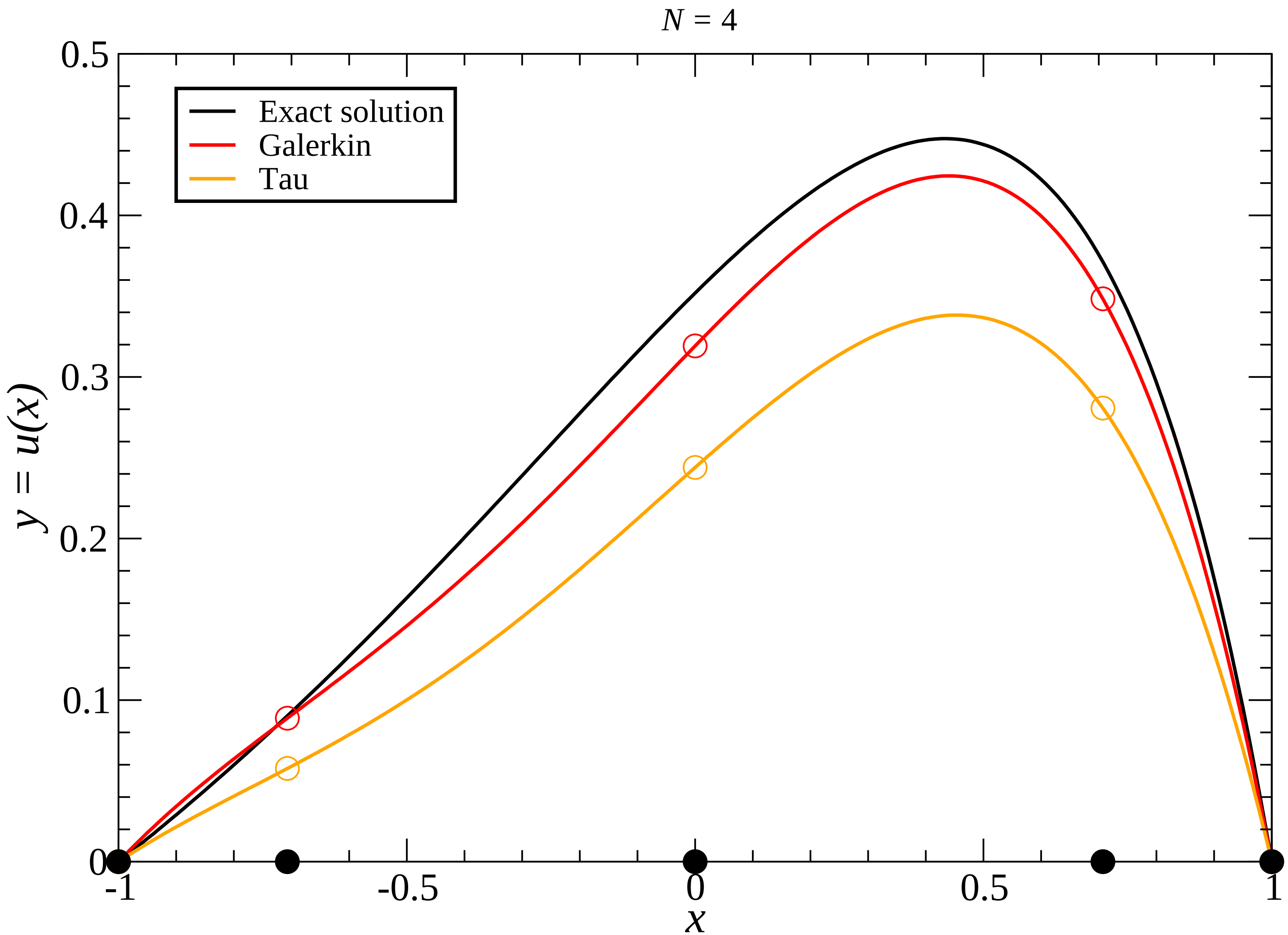
$$b_{pk} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{10}$$

The T_n's being an orthogonal basis, the tau system is obtained by replacing the last two rows of the matrix $\boldsymbol{A}$ by (10):

$$\begin{pmatrix} 4 & -4 & 4 & -12 & 32 \\ 0 & 4 & -16 & 24 & -32 \\ 0 & 0 & 4 & -24 & 48 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{u}_0 \\ \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{u}_4 \end{pmatrix} = \begin{pmatrix} \hat{s}_0 \\ \hat{s}_1 \\ \hat{s}_2 \\ 0 \\ 0 \end{pmatrix}$$

The solution is found to be
$\tilde{u}_0 = 0.1456, \ \tilde{u}_1 = 0.07885, \ \tilde{u}_2 = -0.1220, \ \tilde{u}_3 = -0.07885, \ \tilde{u}_4 = -0.02360.$

Comparison with the exact solution



Exact solution: $u(x) = e^x - \dfrac{\sinh 1}{\sinh 2}\, e^{2x} - \dfrac{e}{1+e^2}$

Resolution by means of a pseudospectral method

Pseudospectral method : trial functions = Chebyshev polynomials $T_0, \ldots, T_4$ and test functions = $\delta(x - x_n)$.

The pseudospectral system is

$$\sum_{k=0}^{4} L T_k(x_n)\, \tilde{u}_k = s(x_n) \Longleftrightarrow \sum_{k=0}^{4} \sum_{l=0}^{4} A_{lk} T_l(x_n)\, \tilde{u}_k = s(x_n)$$

From a matrix point of view: $\boldsymbol{T} \times \boldsymbol{A} \times \tilde{\boldsymbol{u}} = \boldsymbol{s}$, where

$$\boldsymbol{T}_{nl} := T_l(x_n) = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & -1/\sqrt{2} & 0 & 1/\sqrt{2} & -1 \\ 1 & 0 & -1 & 0 & 1 \\ 1 & 1/\sqrt{2} & 0 & -1/\sqrt{2} & -1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

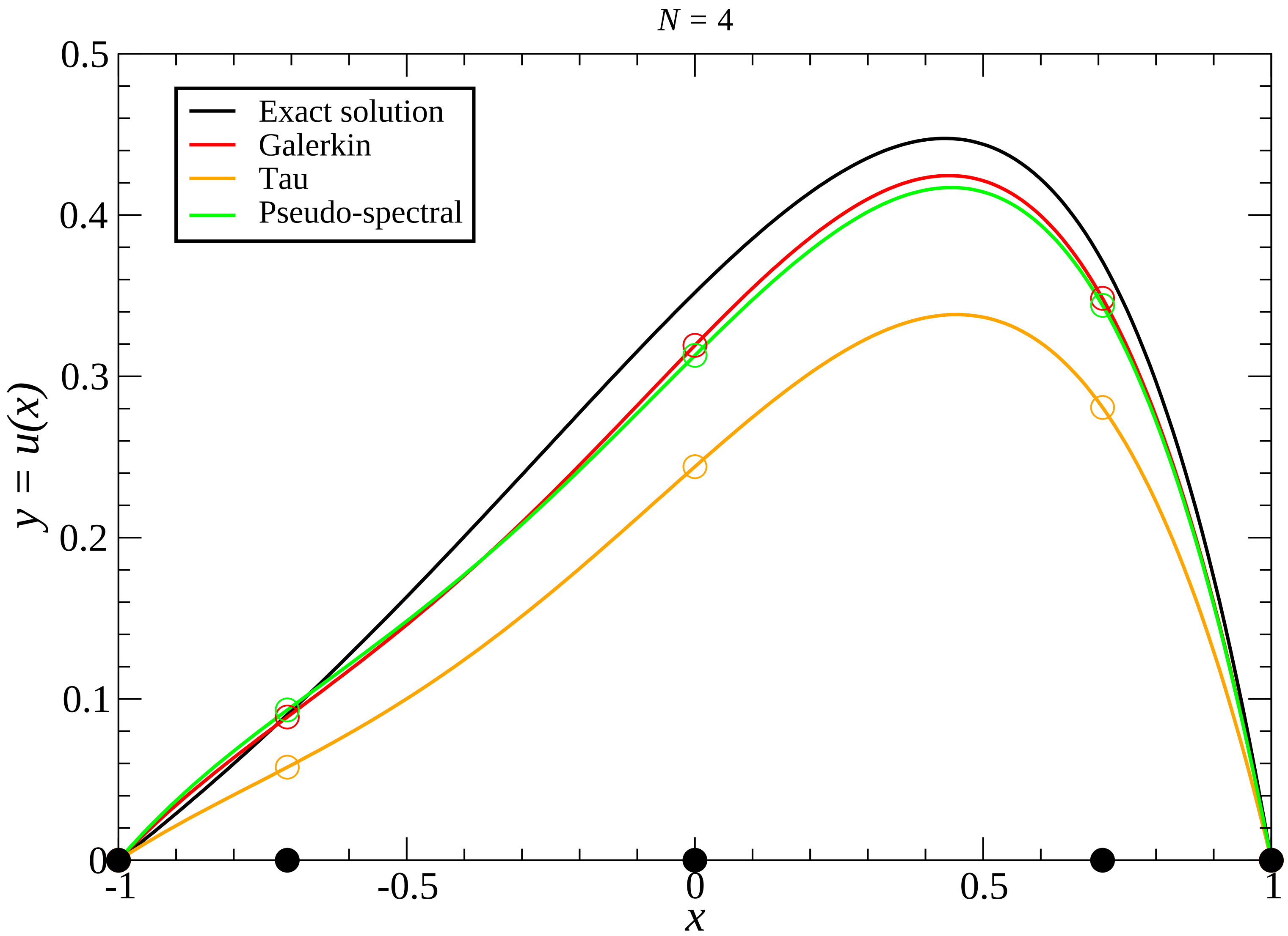
Pseudospectral system

To take into account the boundary conditions, replace the first row of the matrix $\boldsymbol{T} \times \boldsymbol{A}$ by b_{0k} and the last row by b_{1k}, and end up with the system

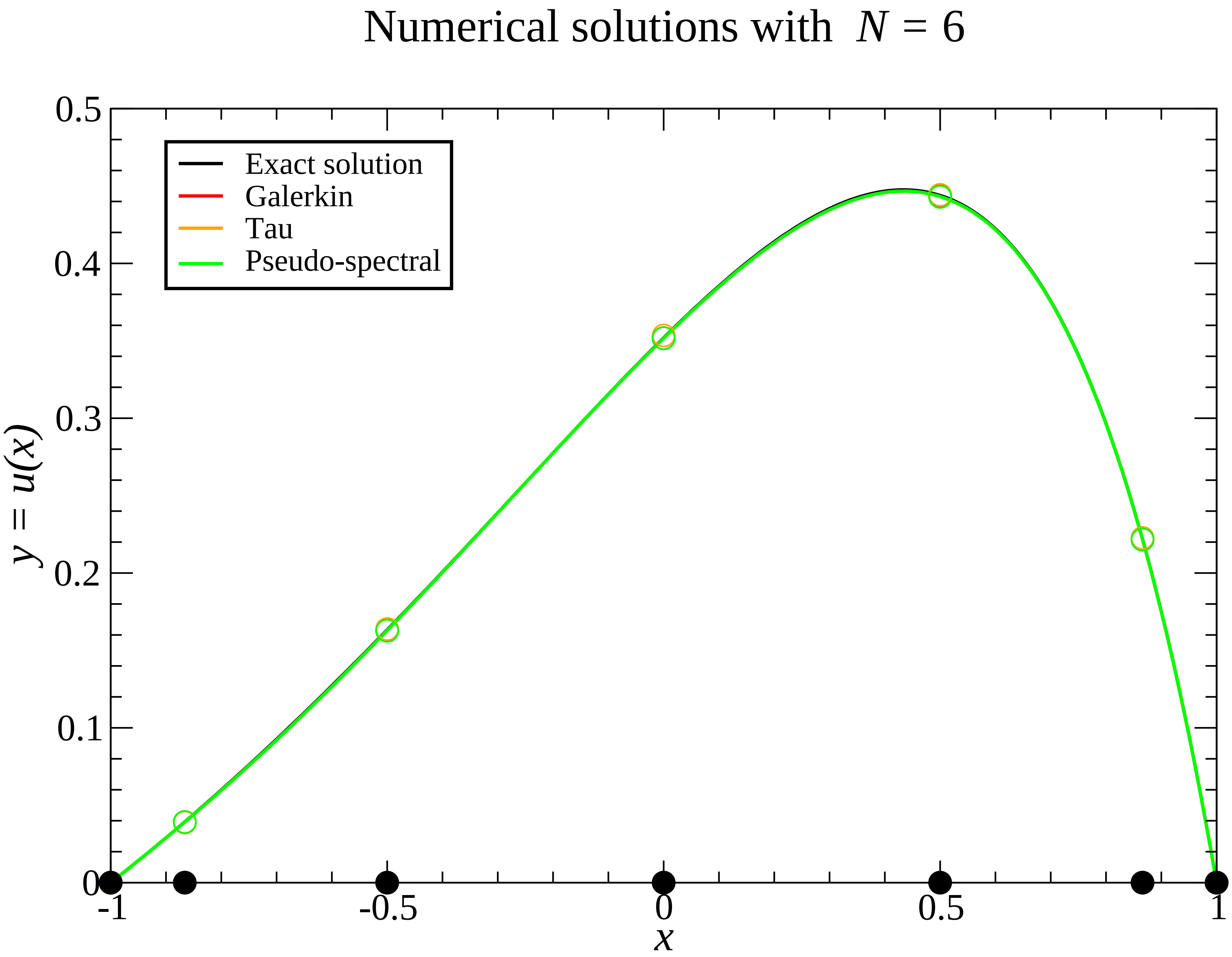
$$\begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 4 & -6.82843 & 15.3137 & -26.1421 & 28 \\ 4 & -4 & 0 & 12 & -12 \\ 4 & -1.17157 & -7.31371 & 2.14214 & 28 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{u}_0 \\ \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{u}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ s(x_1) \\ s(x_2) \\ s(x_3) \\ 0 \end{pmatrix}$$

The solution is found to be
$\tilde{u}_0 = 0.1875,\ \tilde{u}_1 = 0.08867,\ \tilde{u}_2 = -0.1565,\ \tilde{u}_3 = -0.08867,\ \tilde{u}_4 = -0.03104$.

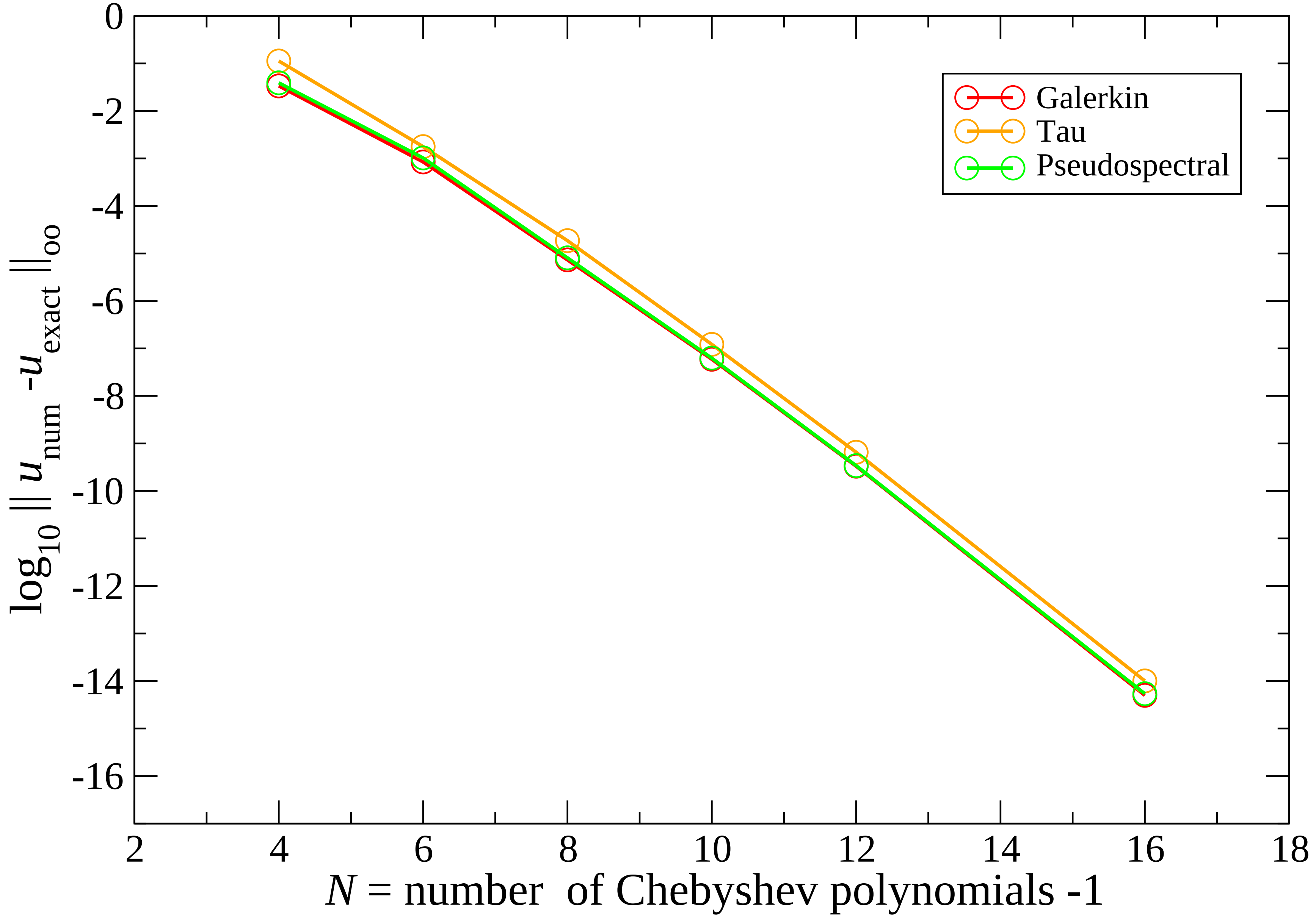
Comparison with the exact solution



Exact solution: $u(x) = e^x - \frac{\sinh 1}{\sinh 2}\, e^{2x} - \frac{e}{1+e^2}$


Numerical solutions with N = 6
Exact solution
Galerkin
Tau
Pseudo-spectral
0.5
0.4
0.3
0.2
0.1
0
-1
-0.5
0
0.5
1
x
$y = u(x)$

Exponential decay of the error with N



Not discussed here...

- Spectral methods for 3-D problems
- Time evolution
- Non-linearities
- Multi-domain spectral methods
- Weak formulation

4

Spectral methods in numerical relativity

Spectral methods developed in Meudon

Pioneered by Silvano Bonazzola & Jean-Alain Marck (1986). Spectral methods within **spherical coordinates**

- 1990 : 3-D wave equation
- 1993 : First 3-D computation of stellar collapse (Newtonian)
- 1994 : Accurate models of rotating stars in GR
- 1995 : Einstein-Maxwell solutions for magnetized stars
- 1996 : 3-D secular instability of rigidly rotating stars in GR

LORENE

Langage Objet pour la RElativite NumeriquE

A library of C++ classes devoted to multi-domain spectral methods, with adaptive spherical coordinates.

- 1997 : start of Lorene
- 1999 : Accurate models of rapidly rotating strange quark stars
- 1999 : Neutron star binaries on closed circular orbits (IWM approx. to GR)
- 2001 : Public domain (GPL), Web page: `http://www.lorene.obspm.fr`
- 2001 : Black hole binaries on closed circular orbits (IWM approx. to GR)
- 2002 : 3-D wave equation with non-reflecting boundary conditions
- 2002 : Maclaurin-Jacobi bifurcation point in GR

Code for producing the figures of the above illustrative example available from LORENE CVS server (directory `Lorene/Codes/Spectral`),

see http://www.lorene.obspm.fr

Spectral methods developed in other groups

- **Cornell group**: Black holes
- **Bartnik:** quasi-spherical slicing
- **Carsten Gundlach:** apparent horizon finder
- **Jörg Frauendiener:** conformal field equations
- **Jena group:** extremely precise models of rotating stars, cf. Marcus Ansorg's talk

Textbooks about spectral methods

- D. Gottlieb & S.A. Orszag : *Numerical analysis of spectral methods*, Society for Industrial and Applied Mathematics, Philadelphia (1977)
- C. Canuto, M.Y. Hussaini, A. Quarteroni & T.A. Zang : *Spectral methods in fluid dynamics*, Springer-Verlag, Berlin (1988)
- B. Mercier : An introduction to the numerical analysis of spectral methods, Springer-Verlag, Berlin (1989)
- C. Bernardi & Y. Maday : *Approximations spectrales de problmes aux limites elliptiques*, Springer-Verlag, Paris (1992)
- B. Fornberg : *A practical guide to pseudospectral methods*, Cambridge University Press, Cambridge (1998)
- J.P. Boyd : *Chebyshev and Fourier spectral methods*, 2nd edition, Dover, Mineola (2001) [web page]