Construction of initial data for 3+1 numerical relativity

Part 3

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Relativistic Astrophysics and Numerical Relativity
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Plan

1. Local characterization of black holes
2. Trapping horizon boundary condition for XCTS
3. Initial data for binary BH: momentarily static data

Outline

Local characterizations of black holes

Recently a **new paradigm** appeared in the theoretical approach of black holes: instead of *event horizons*, black holes are described by

- trapping horizons (Hayward 1994)
- isolated horizons (Ashtekar et al. 1999)
- dynamical horizons (Ashtekar and Krishnan 2002)

All these concepts are **local** and are based on the notion of trapped surfaces

Motivations: quantum gravity, numerical relativity

What is a trapped surface ?

1/ Expansion of a surface along a normal vector field

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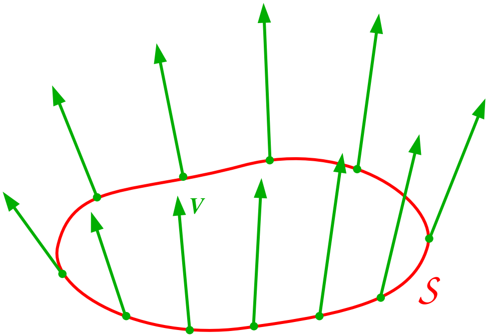

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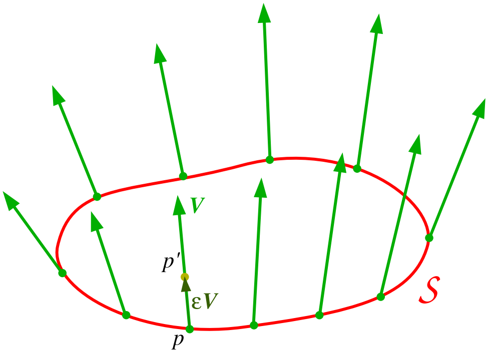

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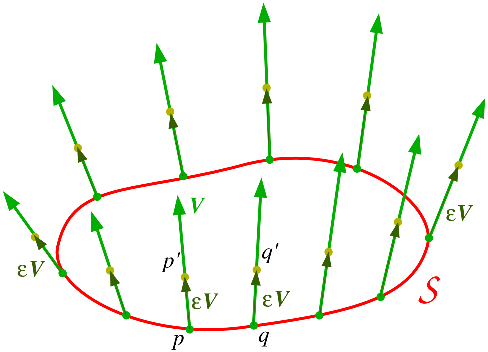

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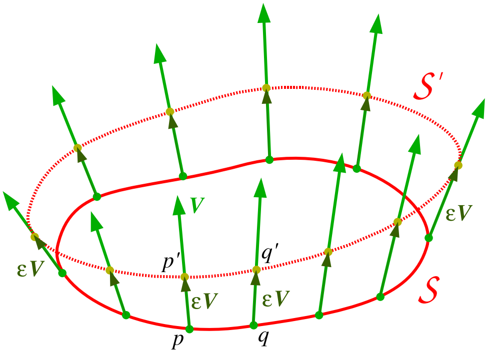

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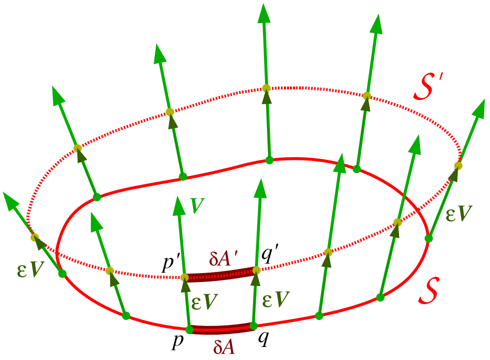

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At each point, the **expansion of** $\mathcal{S}$ **along** $\boldsymbol{v}$ is defined from the relative change in the area element δA:

$$\theta^{(\boldsymbol{v})} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \frac{\delta A' - \delta A}{\delta A} = \boldsymbol{\mathcal{L}}_{\boldsymbol{v}} \ln \sqrt{q} = q^{\mu\nu} \nabla_\mu v_\nu$$

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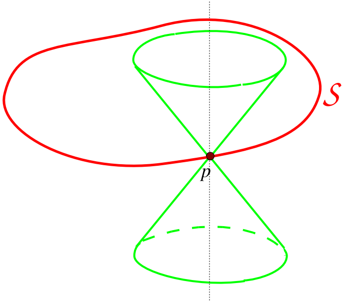

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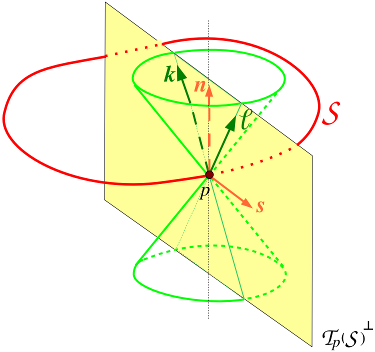

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$\exists$ two future-directed null directions orthogonal to $\mathcal{S}$:

$\boldsymbol{\ell}$ = outgoing, expansion $\theta^{(\boldsymbol{\ell})}$

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In flat space, $\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} > 0$

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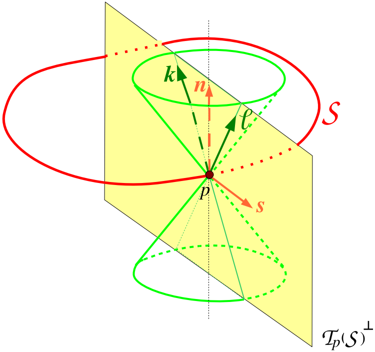

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$\mathcal{S}$ is marginally trapped $\iff$ $\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} = 0$

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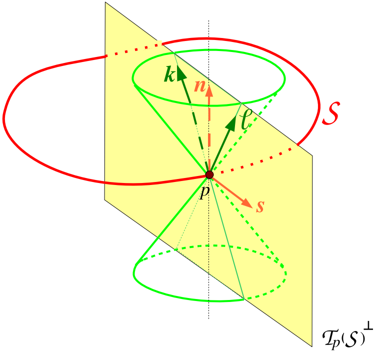

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trapped surface = **local** concept characterizing very strong gravitational fields

Link with apparent horizons

A closed spacelike 2-surface $\mathcal{S}$ is said to be outer trapped (resp. marginally outer trapped (MOTS)) iff [Hawking & Ellis 1973]

- the notions of *interior* and *exterior* of $\mathcal{S}$ can be defined (for instance spacetime asymptotically flat) $\Rightarrow$ $\boldsymbol{\ell}$ is chosen to be the *outgoing* null normal and $\boldsymbol{k}$ to be the *ingoing* one
- $\theta^{(\boldsymbol{\ell})} < 0$ (resp. $\theta^{(\boldsymbol{\ell})} = 0$)

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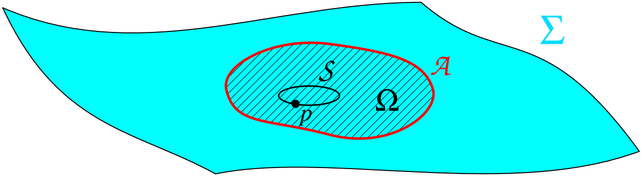


Σ: spacelike hypersurface extending to spatial infinity (Cauchy surface)

outer trapped region of Σ : Ω = set of points $p \in \Sigma$ through which there is a outer trapped surface $\mathcal{S}$ lying in Σ

apparent horizon in Σ: $\mathcal{A}$ = connected component of the boundary of Ω

Proposition [Hawking & Ellis 1973]: $\mathcal{A}$ smooth $\Longrightarrow$ $\mathcal{A}$ is a MOTS

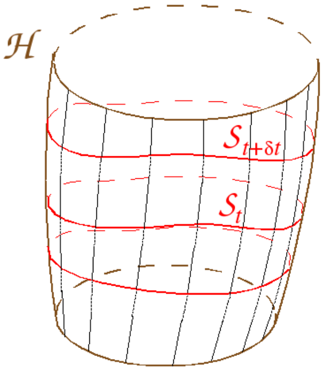
Connection with singularities and black holes

Proposition [Penrose (1965)]:
provided that the weak energy condition holds,
$\exists$ a trapped surface $\mathcal{S} \Longrightarrow \exists$ a singularity in $(\mathscr{M}, \boldsymbol{g})$ (in the form of a future inextendible null geodesic)

Proposition [Hawking & Ellis (1973)]:
provided that the cosmic censorship conjecture holds,
$\exists$ a trapped surface $\mathcal{S} \Longrightarrow \exists$ a black hole $\mathcal{B}$ and $\mathcal{S} \subset \mathcal{B}$

Local definitions of "black holes"

A hypersurface $\mathcal{H}$ of $(\mathscr{M}, \boldsymbol{g})$ is said to be



- a **future outer trapping horizon (FOTH)** iff
  (i) $\mathcal{H}$ foliated by marginally trapped 2-surfaces
  $(\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} = 0)$
  (ii) $\boldsymbol{\mathcal{L}}_{\boldsymbol{k}}\, \theta^{(\boldsymbol{\ell})} < 0$ (locally outermost trapped surf.)
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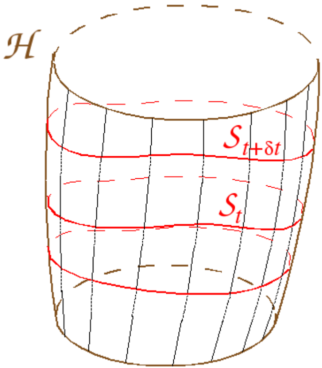

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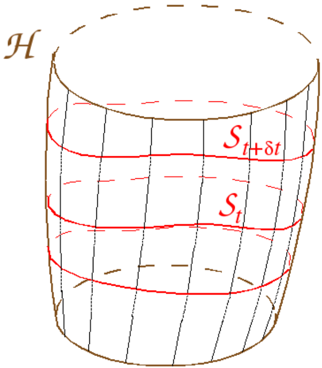

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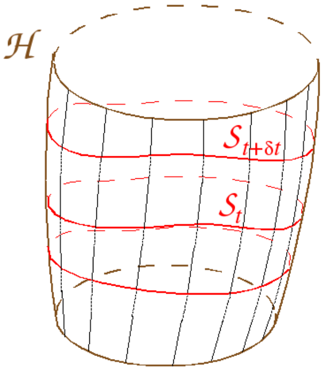

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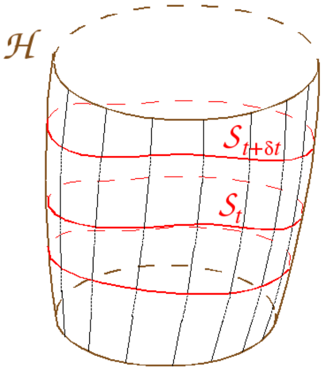

BH in equilibrium (e.g. Kerr) = IH
BH out of equilibrium = DH
generic BH = FOTH

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Dynamics of these new horizons

The *trapping horizons* and *dynamical horizons* have their **own dynamics**, ruled by Einstein equations.
In particular, one can establish for them

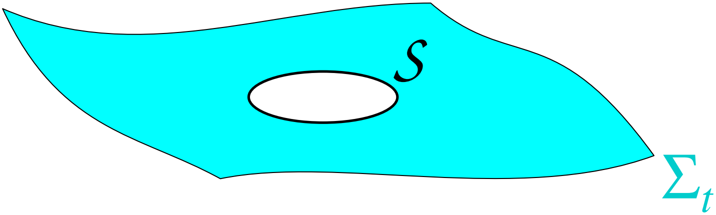
- existence and (partial) uniqueness theorems
  [Andersson, Mars & Simon, PRL **95**, 111102 (2005)],
  [Ashtekar & Galloway, Adv. Theor. Math. Phys. **9**, 1 (2005)]
- first and second laws of black hole mechanics
  [Ashtekar & Krishnan, PRD **68**, 104030 (2003)], [Hayward, PRD **70**, 104027 (2004)]
- a viscous fluid bubble analogy ("membrane paradigm" as for the event horizon), leading to a Navier-Stokes-like equation and a **positive** bulk viscosity *(event horizon = negative bulk viscosity)*
  [Gourgoulhon, PRD **72**, 104007 (2005)], [Gourgoulhon & Jaramillo, PRD **74**, 087502 (2006)]

Reviews: [Ashtekar & Krishnan, Liv. Rev. Relat. **7**, 10 (2004)], [Booth, Can. J. Phys. **83**, 1073 (2005)], [Gourgoulhon & Jaramillo, Phys. Rep. **423**, 159 (2006)]

Outline

Black hole excision

Excision method to deal with black holes: excise from the numerical domain a region containing the singularity



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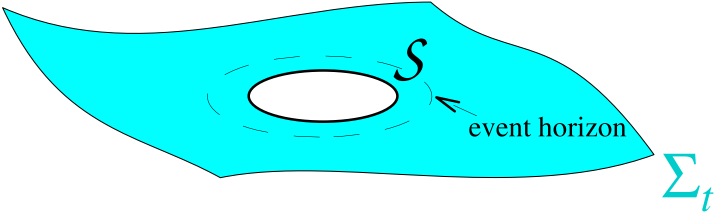


Provided that the excised region is located within the even horizon, the choice of it does not affect the exterior spacetime

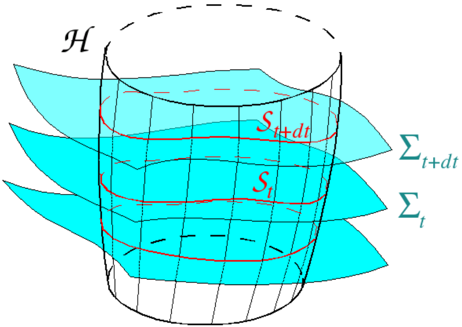
Need for boundary conditions at the excision surface

In the framework of the extended conformal thin sandwich (XCTS) method, boundary conditions on $\mathcal{S}$ are required for

- the conformal factor Ψ
- the conformal lapse $\tilde{N}$ (or equivalently the lapse N, since $N = \Psi^6 \tilde{N}$)
- the shift vector $\boldsymbol{\beta}$

Trapping horizon inner boundary

Choose the excision boundary $\mathcal{S}_t$ to be a **marginally trapped surface** for each time t



The tube $\mathcal{H} = \bigcup_{t\in\mathbb{R}} \mathcal{S}_t$

is then generically a smooth **trapping horizon**

[Andersson, Mars & Simon, PRL **95**, 111102 (2005)]

- geometrically well defined excision boundary
- ensures $\mathcal{S}_t$ is located inside the event horizon reminder
- easy to implement with spherical coordinates

Geometrical setup

Hypersurface Σ_t :

- induced metric γ (positive definite); associated connection $\boldsymbol{D}$
- future directed timelike unit normal $\boldsymbol{n}$
- extrinsic curvature $\boldsymbol{K}$: $K_{\alpha\beta} = -\nabla_\mu n_\alpha \, \gamma^\mu{}_\beta$
- lapse function N : $\underline{\boldsymbol{n}} = -N \mathbf{d}t$

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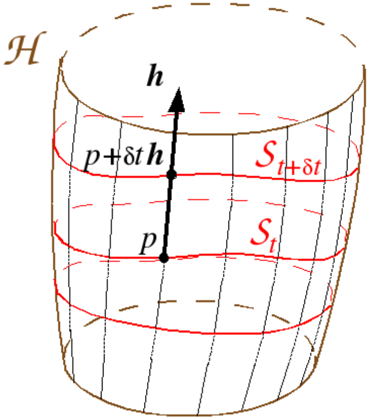
2-surface $\mathcal{S}_t$:

- induced metric $\boldsymbol{q}$ (positive definite); associated connection ${}^S\boldsymbol{D}$
- normal vector pairs (basis of $\mathcal{T}_p(\mathcal{S}_t)^\perp$): see figure
  - orthonormal basis $(\boldsymbol{n}, \boldsymbol{s})$, where $\boldsymbol{s}$ is the outgoing spacelike unit normal to $\mathcal{S}_t$ in Σ_t
  - null basis $(\boldsymbol{\ell}, \boldsymbol{k})$ (not unique: $\boldsymbol{\ell} \mapsto \boldsymbol{\ell}' = \lambda\boldsymbol{\ell}$, $\boldsymbol{k} \mapsto \boldsymbol{k}' = \mu\boldsymbol{k}$)
- extrinsic curvature, as a hypersurface of Σ_t, $\boldsymbol{H}$: $H_{\alpha\beta} = D_\mu s_\alpha \, q^\mu{}_\beta$

Privileged evolution vector on $\mathcal{H}$



Vector field $\boldsymbol{h}$ on $\mathcal{H}$ defined by

- (i) $\boldsymbol{h}$ is tangent to $\mathcal{H}$
- (ii) $\boldsymbol{h}$ is orthogonal to $\mathcal{S}_t$
- (iii) $\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, t = h^\mu \partial_\mu t = \langle \mathbf{d}t, \boldsymbol{h} \rangle = 1$

NB: (iii) $\Longrightarrow$ the 2-surfaces $\mathcal{S}_t$ are Lie-dragged by $\boldsymbol{h}$

$\boldsymbol{h} \in \mathcal{T}_p(\mathcal{S}_t)^\perp = \mathrm{Vect}(\boldsymbol{n}, \boldsymbol{s})$ and can be decomposed as $\boxed{\boldsymbol{h} = N\boldsymbol{n} + b\boldsymbol{s} = \boldsymbol{m} + b\boldsymbol{s}}$

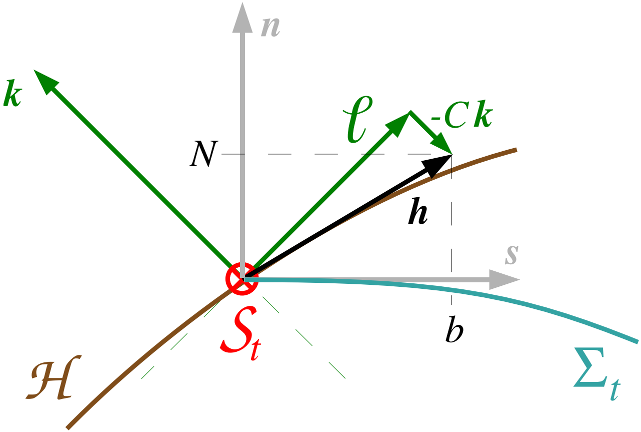
Norm of $\boldsymbol{h}$ and type of $\mathcal{H}$

Definition: $C := \frac{1}{2}\,\boldsymbol{h}\cdot\boldsymbol{h} = \frac{1}{2}\left(b^2 - N^2\right)$

$\mathcal{H}$ is spacelike (DH)	$\iff$	$\boldsymbol{h}$ is spacelike	$\iff$	$C > 0$	$\iff$	$b > N$
$\mathcal{H}$ is null (IH)	$\iff$	$\boldsymbol{h}$ is null	$\iff$	$C = 0$	$\iff$	$b = N$
$\mathcal{H}$ is timelike	$\iff$	$\boldsymbol{h}$ is timelike	$\iff$	$C < 0$	$\iff$	$b < N.$

Null basis associated with $\boldsymbol{h}$

The vectors $\boldsymbol{\ell} := \frac{1}{2}(b+N)\,(\boldsymbol{n}+\boldsymbol{s})$ and $\boldsymbol{k} := \frac{1}{b+N}\,(\boldsymbol{n}-\boldsymbol{s})$ are the unique pair of null vectors normal to $\mathcal{S}_t$ such that $\boldsymbol{\ell}\cdot\boldsymbol{k} = -1$ and $\boxed{\boldsymbol{h} = \boldsymbol{\ell} - C\boldsymbol{k}}$



Spatial coordinates

Coordinates $(x^i)_{i\in\{1,2,3\}}$ on Σ_t $\Rightarrow$ define the shift vector $\boldsymbol{\beta}$: $\boxed{\boldsymbol{\partial}_t = N\boldsymbol{n} + \boldsymbol{\beta}}$

2+1 orthogonal decomposition of the shift with respect to $\mathcal{S}_t$:

$\boxed{\boldsymbol{\beta} = \beta^\perp \boldsymbol{s} - \boldsymbol{V}}$ with $\boldsymbol{s}\cdot\boldsymbol{V} = 0$.

The coordinates (t, x^i) are comoving w.r.t. $\mathcal{H}$ iff there exists a function f not depending on t and such that

$\forall p = (t, x^1, x^2, x^3) \in \mathscr{M},\ p \in \mathcal{H} \iff f(x^1, x^2, x^3) = 0$

Special case: adapted coordinates: $f = f(x^1)$

$$\text{Coordinates } (t, x^i) \text{ comoving w.r.t. } \mathcal{H} \iff \boldsymbol{\partial}_t \text{ tangent to } \mathcal{H}$$

$$\iff \boxed{\beta^\perp = b}$$

$$\iff \boxed{\boldsymbol{h} = \boldsymbol{\partial}_t + \boldsymbol{V}}$$

Marginally trapped surface BC

One has $\boldsymbol{\ell} = \lambda(\boldsymbol{n} + \boldsymbol{s})$, with $\lambda := \frac{1}{2}(b + N)$

Hence $\theta^{(\boldsymbol{\ell})} = \lambda \left[\theta^{(\boldsymbol{n})} + \theta^{(\boldsymbol{s})}\right]$, with

- $\theta^{(\boldsymbol{n})} = q^{\mu\nu}\nabla_\mu n_\nu = (\gamma^{\mu\nu} - s^\mu s^\nu)(-K_{\mu\nu} - D_\nu \ln N\, n_\mu) = -K + K_{ij}s^i s^j$
- $\theta^{(\boldsymbol{s})} = q^{\mu\nu}\nabla_\mu s_\nu = q^{ij}D_i s_j = (\gamma^{ij} - s^i s^j)D_i s_j = D_i s^i$ (mean curvature H of $\mathcal{S}_t$)

Therefore $\theta^{(\boldsymbol{\ell})} = \lambda \left(D_i s^i + K_{ij}s^i s^j - K\right)$

Hence the marginally trapped surface condition:

$$\theta^{(\boldsymbol{\ell})} = 0 \iff \boxed{D_i s^i + K_{ij}s^i s^j - K = 0}$$

Remark: minimal surface condition : $D_i s^i = 0$

Conformal decomposition : $\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij}$, $K_{ij} = \Psi^{-2}\hat{A}_{ij} + \frac{K}{3}\Psi^4\tilde{\gamma}_{ij}$, $s^i = \Psi^{-2}\tilde{s}^i$

$$\boxed{4\tilde{s}^i \tilde{D}_i \Psi + \tilde{D}_i \tilde{s}^i\, \Psi + \hat{A}_{ij}\tilde{s}^i\tilde{s}^j\, \Psi^{-3} - \frac{2}{3}K\Psi^3 = 0} \qquad (1)$$

Condition $\mathcal{L}_h \theta^{(\ell)} = 0$ on $\mathcal{S}_t$

i.e. not only $\mathcal{S}_t$ is a marginally trapped surface at time t, but remains marginally trapped at time $t + \delta t$:
Thanks to Einstein equation, the condition $\mathcal{L}_h \theta^{(\ell)} = 0$, along with $\theta^{(\ell)} = 0$, is equivalent to [Eardley, PRD **57**, 2299 (1998)]

$$-{}^{\mathcal{S}}D_a {}^{\mathcal{S}}D^a (b - N) - 2L^a \, {}^{\mathcal{S}}D_a (b - N) + A(b - N) = B(b + N) \qquad (2)$$

with ${}^{\mathcal{S}}D_a$: Levi-Civita connection associated with the metric $\boldsymbol{q}$ on $\mathcal{S}_t$

$L_a := K_{ij} s^i q^j{}_a$

$A := \frac{1}{2} {}^{\mathcal{S}}R - {}^{\mathcal{S}}D_a L^a - L_a L^a - 4\pi T_{\mu\nu}(n^\mu + s^\mu)(n^\nu - s^\nu)$

${}^{\mathcal{S}}R$: Ricci scalar of the metric $\boldsymbol{q}$ on $\mathcal{S}_t$

$B := \frac{1}{2} \hat{\sigma}_{ab} \hat{\sigma}^{ab} + 4\pi T_{\mu\nu}(n^\mu + s^\mu)(n^\nu + s^\nu)$

$\hat{\sigma}_{ab} := H_{ab} - \frac{1}{2} H q_{ab} + \sigma^{(\boldsymbol{n})}_{ab}$

Case of a non-expanding horizon

Equilibrium configuration $\iff$ $\mathcal{H}$ non-expanding horizon

Special case: isolated horizon

$\mathcal{H}$ non-expanding horizon $\iff$ $\mathcal{H}$ null hypersurface

$\iff$ $\boldsymbol{h} = N\boldsymbol{n} + b\boldsymbol{s}$ null vector ($\boldsymbol{h} = \boldsymbol{\ell}$)

$\iff$ $\boxed{b = N}$

$\iff$ $\boxed{\beta^{\perp} = N}$ (for comoving coordinates w.r.t. $\mathcal{H}$)

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Link with $\mathcal{L}_{\boldsymbol{h}}\, \theta^{(\boldsymbol{\ell})} = 0$ condition: notice $B = \lambda^{-2}\left[\frac{1}{2}\sigma^{(\boldsymbol{\ell})}_{ab}\sigma^{(\boldsymbol{\ell})\,ab} + 4\pi \boldsymbol{T}(\boldsymbol{\ell}, \boldsymbol{\ell})\right]$

Now the null Raychaudhuri equation is

$$\mathcal{L}_{\boldsymbol{h}}\, \theta^{(\boldsymbol{\ell})} + \kappa\theta^{(\boldsymbol{\ell})} + \frac{1}{2}(\theta^{(\boldsymbol{\ell})})^2 + \sigma^{(\boldsymbol{\ell})}_{ab}\sigma^{(\boldsymbol{\ell})\,ab} + 8\pi \boldsymbol{T}(\boldsymbol{\ell}, \boldsymbol{\ell}) = 0$$

In the present case ($\theta^{(\boldsymbol{\ell})} = 0$ and $\mathcal{L}_{\boldsymbol{\ell}}\, \theta^{(\boldsymbol{\ell})} = \mathcal{L}_{\boldsymbol{h}}\, \theta^{(\boldsymbol{\ell})} = 0$) it reduces to

$$\sigma^{(\boldsymbol{\ell})}_{ab}\sigma^{(\boldsymbol{\ell})\,ab} + 8\pi \boldsymbol{T}(\boldsymbol{\ell}, \boldsymbol{\ell}) = 0$$

Hence, for a non-expanding horizon, $B = 0$ and the solution to Eq. (2) is $b - N = 0$, i.e. we recover the above condition $b = N$

Non-expanding horizon: vanishing of the shear

For non-expanding horizons, we have just seen that $\sigma^{(\ell)}_{ab}\sigma^{(\ell)\,ab} + 8\pi \boldsymbol{T}(\boldsymbol{\ell},\boldsymbol{\ell}) = 0$
But, provided that matter satisfies the weak energy condition, $\boldsymbol{T}(\boldsymbol{\ell},\boldsymbol{\ell}) \geq 0$. Since $\sigma^{(\ell)}_{ab}\sigma^{(\ell)\,ab} \geq 0$, we get $\boldsymbol{T}(\boldsymbol{\ell},\boldsymbol{\ell}) = 0$ and

$$\boxed{\sigma^{(\ell)}_{ab} = 0}$$

Expressing $\sigma^{(\ell)}_{ab}$ in terms of 3+1 conformal quantities, this condition becomes

$$\frac{\partial \tilde{q}_{ab}}{\partial t} - \frac{1}{2}\left(\frac{\partial}{\partial t}\ln\tilde{q}\right)\tilde{q}_{ab} + {}^{S}\tilde{D}_a\tilde{V}_b + {}^{S}\tilde{D}_b\tilde{V}_a - {}^{S}\tilde{D}_c\tilde{V}^c\,\tilde{q}_{ab} = 0$$

where $\tilde{q}_{ab} = \Psi^{-4} q_{ab}$ (metric induced by $\tilde{\gamma}$ on $\mathcal{S}_t$)
$\tilde{V}_a := \tilde{q}_{ab}V^b$ ($-V^a$ = components of shift tangent to $\mathcal{S}_t$: $\boldsymbol{\beta} = \beta^{\perp}\boldsymbol{s} - \boldsymbol{V}$)

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$\tilde{q}_{ab}$ and $\dfrac{\partial \tilde{q}_{ab}}{\partial t}$ are part of XCTS free data: looking for quasiequilibrium configurations, it is natural to choose $\dfrac{\partial \tilde{q}_{ab}}{\partial t} = 0$

Then the vanishing shear condition becomes $\boxed{{}^{S}\tilde{D}_a \tilde{V}_b + {}^{S}\tilde{D}_b \tilde{V}_a - {}^{S}\tilde{D}_c \tilde{V}^c\, \tilde{q}_{ab} = 0}$
i.e. $\boldsymbol{V}$ must be a **conformal Killing vector** of $\tilde{\boldsymbol{q}}$ (or $\boldsymbol{q}$ as well) [Cook & Pfeiffer, PRD **70**, 104016 (2004)]

Non-expanding horizon: vanishing of the shear

Conformal Killing vector condition :

$$ {}^{S}\tilde{D}_a \tilde{V}_b + {}^{S}\tilde{D}_b \tilde{V}_a - {}^{S}\tilde{D}_c \tilde{V}^c \, \tilde{q}_{ab} = 0 \qquad (3) $$

But on a (2-dimensional) sphere, all metrics are conformally related
Let $\breve{\boldsymbol{q}}$ be the *round metric* associated with some coordinates $(x^a) = (\theta, \varphi)$ on $\mathcal{S}_t$:
$\breve{q}_{ab} dx^a \, dx^b = r^2(d\theta^2 + \sin^2 \theta d\varphi^2)$
Any Killing vector $\boldsymbol{\xi}$ of $\breve{\boldsymbol{q}}$ is a conformal Killing vector of $\tilde{\boldsymbol{q}}$; hence

$$ \boldsymbol{V} = \omega \, \boldsymbol{\xi}, \quad \omega = \mathrm{const} $$

is a solution of Eq. (3)
For instance $\boldsymbol{\xi} = \boldsymbol{\partial}_\varphi$

Summary: BC for XCTS scheme

XCTS: needs BC on $\mathcal{S}_0$ for Ψ, $\tilde{N} = \Psi^{-6} N$ and $\beta^i \leftrightarrow (\beta^\perp, V^a)$

We assume here comoving coordinates ($b = \beta^\perp$)

General trapping horizon:

Conditions $\theta^{(\ell)} = 0$ and $\mathcal{L}_h \theta^{(\ell)} = 0$ yield respectively

- $4\tilde{s}^i \tilde{D}_i \Psi + \tilde{D}_i \tilde{s}^i \, \Psi + \hat{A}_{ij} \tilde{s}^i \tilde{s}^j \, \Psi^{-3} - \frac{2}{3} K \Psi^3 \overset{\mathcal{S}_0}{=} 0, \quad \hat{A}^{ij} = \frac{1}{2\tilde{N}} \left[\dot{\tilde{\gamma}}^{ij} + (\tilde{L}\beta)^{ij} \right]$
- $-{}^{\mathcal{S}}D_a {}^{\mathcal{S}}D^a (\beta^\perp - N) - 2L^a \, {}^{\mathcal{S}}D_a (\beta^\perp - N) + A(\beta^\perp - N) \overset{\mathcal{S}_0}{=} B(\beta^\perp + N)$

Non-expanding horizon ($+ \ \dot{\tilde{\gamma}}_{ij} = 0$ as part of free data) :

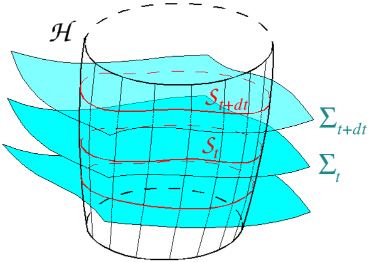
Conditions $\theta^{(\ell)} = 0$, $\mathcal{L}_h \theta^{(\ell)} = 0$ and $\sigma^{(\ell)} = 0$ yield respectively

- $4\tilde{s}^i \tilde{D}_i \Psi + \tilde{D}_i \tilde{s}^i \, \Psi + \frac{1}{2\tilde{N}} (\tilde{L}\beta)_{ij} \tilde{s}^i \tilde{s}^j \, \Psi^{-3} - \frac{2}{3} K \Psi^3 \overset{\mathcal{S}_0}{=} 0$
- $\beta^\perp \overset{\mathcal{S}_0}{=} N$
- V conformal Killing vector of $\tilde{q}$, e.g. $V \overset{\mathcal{S}_0}{=} \omega \, \partial_\varphi$

Quasi-equilibrium BC

Non-expanding horizon BC = 4 conditions,
whereas we have 5 equations in the XCTS scheme, for the 5 quantities Ψ, $\tilde{N} = \Psi^{-6} N$ and $\beta^i \leftrightarrow (\beta^\perp, V^a)$
$\Longrightarrow$ a 5th BC is required !

No priviledged geometrical choice for this 5th BC: this reflects the freedom in the choice of the slicing: the 5th BC can be a BC on the lapse N and can be freely chosen



Remark: the condition $K = 0$ (part of free data) implies maximal slicing but is not sufficient to fully determine the foliation $(\Sigma_t)_{t\in\mathbb{R}}$: the value of N at the inner boundary $\mathcal{S}_t$ must be provided

Choice of lapse function

Suggested choices of lapse:

- arbitrary simple Dirichlet/Newmann BC [Cook & Pfeiffer, PRD **70**, 104016 (2004)] :
  $$\frac{\partial}{\partial r}(N\Psi) \stackrel{\mathcal{S}_0}{=} 0, \quad N\Psi \stackrel{\mathcal{S}_0}{=} 0 \quad \text{or} \quad \frac{\partial}{\partial r}(N\Psi) \stackrel{\mathcal{S}_0}{=} \frac{N\Psi}{2r}$$
- constant "surface gravity" [Jaramillo, Ansorg & Limousin, gr-qc/0610006] :
  $$s^i D_i N - N K_{ij} s^i s^j = \kappa_0 = \text{const}$$
  $$\iff \tilde{s}^i \tilde{D}_i N - \left(\Psi^{-4} \hat{A}_{ij} \tilde{s}^i \tilde{s}^j + \frac{1}{3} K \Psi^2 \right) N = \kappa_0$$

Outline

Brill-Lindquist initial data

Framework: find initial data for **two black holes momentarily at rest**

Use the conformal transverse traceless method (CTT)
Again, do something simple:

- choice of the initial 3-dimensional manifold: twice-punctured $\mathbb{R}^3$: $\Sigma_0 = \mathbb{R}^3 \backslash \{O_1, O_2\}$
- choice of the CTT free data: $\tilde{\gamma}_{ij} = f_{ij}$, $\hat{A}^{\mathrm{TT}}_{ij} = 0$, $K = 0$, $\tilde{E} = 0$ and $\tilde{p}^i = 0$

The constraint equations are then

$$\Delta \Psi + \frac{1}{8} (LX)_{ij} (LX)^{ij} \, \Psi^{-7} = 0 \qquad (4)$$

$$\Delta X^i + \frac{1}{3} \mathcal{D}^i \mathcal{D}_j X^j = 0 \qquad (5)$$

$\Delta := \mathcal{D}_i \mathcal{D}^i$ flat Laplacian, $\mathcal{D}_i$ flat connection ($\mathcal{D}_i = \partial_i$ in Cartesian coord.)

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Momentarily static solution of (5) : $\boldsymbol{X} = 0$
Then (4) reduces to Laplace equation

$$\boxed{\Delta\Psi = 0}$$

Brill-Lindquist initial data

Brill-Lindquist solution to Laplace equation [Brill & Lindquist, Phys. Rev. **131**, 471 (1963)] :

$$\Psi = 1 + \frac{\alpha_1}{\|\boldsymbol{r} - \boldsymbol{r}_1\|} + \frac{\alpha_2}{\|\boldsymbol{r} - \boldsymbol{r}_2\|}$$

α_1, α_2 : constants, $\|.\|$ norm with respect to $\boldsymbol{f}$,
$\boldsymbol{r}_1$ (resp. $\boldsymbol{r}_2$) location of puncture O_1 (resp. O_2)

The physical initial data are $\begin{cases} \gamma_{ij} = \Psi^4 f_{ij} \\ K_{ij} = 0 \quad \text{(momentarily static)} \end{cases}$

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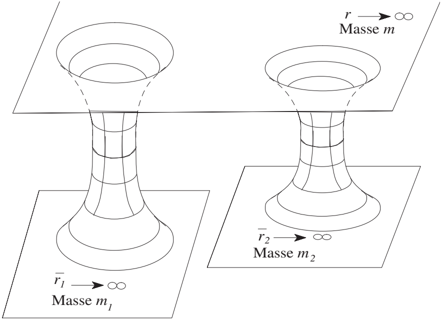


Riemannian manifold with three asymptotically flat ends

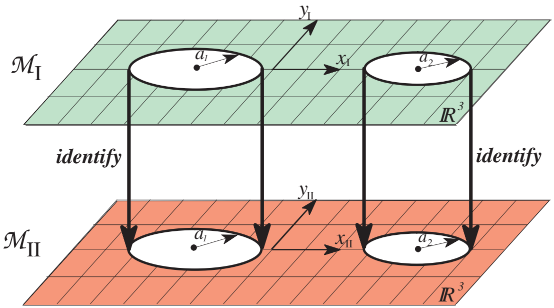
ADM mass of "upper" sheet:
$m = 2(\alpha_1 + \alpha_2)$
ADM mass of "lower" sheets:

$$m_1 = 2\alpha_1 + 2\frac{\alpha_1\alpha_2}{\|\boldsymbol{r}_1 - \boldsymbol{r}_2\|}$$

$$m_2 = 2\alpha_2 + 2\frac{\alpha_1\alpha_2}{\|\boldsymbol{r}_1 - \boldsymbol{r}_2\|}$$

Misner-Lindquist initial data

Same free data as before: $\tilde{\gamma}_{ij} = f_{ij}$, $\hat{A}^{\rm TT}_{ij} = 0$, $K = 0$, $\tilde{E} = 0$ and $\tilde{p}^i = 0$
but change the manifold Σ_0 :



excise two balls $\mathcal{B}_1$ and $\mathcal{B}_2$ from $\mathbb{R}^3$, make a copy and glue it at the surface of the balls

Again select the momentarily static solution to the momentum constraint: $\boldsymbol{X} = 0$, so that the Hamiltonian constraint reduces to Laplace equation:

$$\Delta\Psi = 0$$

Misner-Lindquist initial data

Misner-Lindquist solution to Laplace equation

[Misner, Ann. Phys. **24**, 102 (1963)], [Lindquist, J. Math. Phys. **4**, 938 (1963)] :

$$\Psi = 1 + \sum_{n=1}^{\infty} \left(\frac{\alpha_{1n}}{\|\boldsymbol{r} - \boldsymbol{r}_{1n}\|} + \frac{\alpha_{2n}}{\|\boldsymbol{r} - \boldsymbol{r}_{2n}\|} \right)$$

3-parameter solutions : d = distance between the center of the excised balls
a_1 = radius of ball 1
a_2 = radius of ball 2

$\alpha_{1n} = F_1(d, a_1, a_2, n)$, $\alpha_{2n} = F_2(d, a_1, a_2, n)$,
$\boldsymbol{r}_{1n} = \boldsymbol{G}_1(d, a_1, a_2, n)$, $\boldsymbol{r}_{2n} = \boldsymbol{G}_2(d, a_1, a_2, n)$

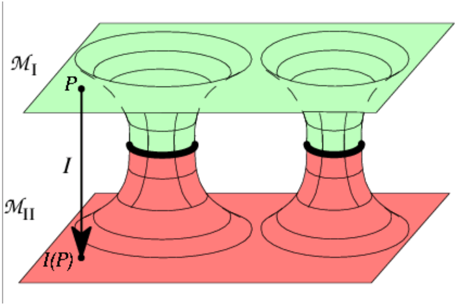
(see e.g. [Andrade & Price, PRD **56**, 6336 (1997)] for the explicit form of F_1, F_2, $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$)

In the special case $a_1 = a_2 =: a$ these expressions are

$$\alpha_{1n} = \alpha_{2n} = \frac{c}{\sinh(n\mu)}, \quad \boldsymbol{r}_{1n} = -c \coth(n\mu)\, \boldsymbol{e}_x, \quad \boldsymbol{r}_{2n} = c \coth(n\mu)\, \boldsymbol{e}_x,$$

with (c, μ) related to (d, a) by $d = c \coth \mu$ and $a = \dfrac{c}{\sinh \mu}$

Misner-Lindquist initial data



Riemannian manifold with two asymptotically flat ends

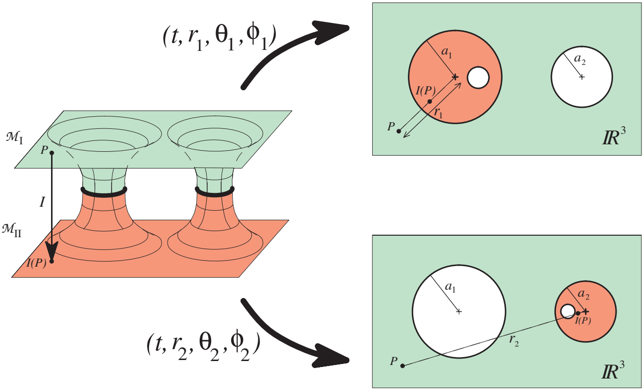
"upper" sheet and "lower" sheet are isometric

ADM mass of each sheet ($a_1 = a_2$ case):

$$m = 4c \sum_{n=1}^{\infty} \frac{1}{\sinh(n\mu)}$$

Misner-Lindquist initial data

Isometry between the two sheets



Two explicit forms of the isometry, as *inversions* w.r.t. each throat:

- spherical coordinates centered on O_1: $I\ :\ (r_1, \theta_1, \varphi_1) \mapsto (a_1^2/r_1, \theta_1, \varphi_1)$
- spherical coordinates centered on O_2: $I\ :\ (r_2, \theta_2, \varphi_2) \mapsto (a_2^2/r_2, \theta_2, \varphi_2)$