The initial data problem for 3+1 numerical relativity

Part 1

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Plan

1. The initial data problem
2. Conformal transverse-traceless method
3. Conformal thin sandwich method

Outline

Initial data for the Cauchy problem

In lecture 1, we have seen

3+1 decomposition $\Longrightarrow$ Einstein equation $=$ Cauchy problem with constraints

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$$R + K^2 - K_{ij}K^{ij} = 16\pi E$$
$$D_j K^j_{\ \ i} - D_i K = 8\pi p_i$$

NB: the matter distribution $(E, \boldsymbol{p})$ may have some additional constraints from its own.

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- **The astrophysical problem:** make sure that the obtained solution to the constraint equations have something to do with the physical system that one wish to study.

A first naive approach

Notice that the constraints involve a single hypersurface Σ_0, not a foliation $(\Sigma_t)_{t\in\mathbb{R}}$. In particular, neither the lapse function N nor the shift vector appear β in these equations

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However, this approach is not satisfactory:
only 4 equations for 6 unknowns K_{ij} and there is no natural prescription for choosing arbitrarily two among the six components K_{ij}

Various approaches to the initial data problem

- **Conformal methods:** initiated by Lichnerowicz (1944) and extended by
  - Choquet-Bruhat (1956, 1971)
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In this lecture we focus on conformal methods

Outline

Starting point

Conformal decomposition introduced in Lecture 1:

$$\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij} \quad \text{and} \quad A^{ij} = \Psi^{-10} \hat{A}^{ij}$$

The Hamiltonian and momentum constraints become respectively

$$\boxed{\tilde{D}_i \tilde{D}^i \Psi - \frac{1}{8} \tilde{R} \Psi + \frac{1}{8} \hat{A}_{ij} \hat{A}^{ij} \, \Psi^{-7} + \left(2\pi E - \frac{1}{12} K^2 \right) \Psi^5 = 0}$$

$$\boxed{\tilde{D}_j \hat{A}^{ij} - \frac{2}{3} \Psi^6 \tilde{D}^i K = 8\pi \Psi^{10} p^i}$$

Longitudinal/transverse decomposition of $\hat{A}^{ij}$

York (1973,1979) splitting of $\hat{A}^{ij}$:

$$\boxed{\hat{A}^{ij} = (\tilde{L}X)^{ij} + \hat{A}^{ij}_{\mathrm{TT}}}$$

with

- $(\tilde{L}X)^{ij}$ = **conformal Killing operator** associated with the metric $\tilde{\gamma}$ and acting on the vector field $\boldsymbol{X}$:

$$(\tilde{L}X)^{ij} := \tilde{D}^i X^j + \tilde{D}^j X^i - \frac{2}{3}\tilde{D}_k X^k \,\tilde{\gamma}^{ij}$$

- $\hat{A}^{ij}_{\mathrm{TT}}$ traceless and transverse (i.e. divergence-free) with respect to the metric $\tilde{\gamma}$: $\tilde{\gamma}_{ij}\hat{A}^{ij}_{\mathrm{TT}} = 0$ and $\tilde{D}_j\hat{A}^{ij}_{\mathrm{TT}} = 0$

NB: both the longitudinal part and the TT part are traceless: $\tilde{\gamma}_{ij}(\tilde{L}X)^{ij} = 0$ and $\tilde{\gamma}_{ij}\hat{A}^{ij}_{\mathrm{TT}} = 0$

Longitudinal/transverse decomposition of $\hat{A}^{ij}$

Determining $\boldsymbol{X}$ and $\hat{A}^{ij}_{\rm TT}$:

Considering the divergence of $\hat{A}^{ij}$, we see that $\boldsymbol{X}$ must be a solution of the vector differential equation

$$\boxed{\tilde{\Delta}_L\, X^i = \tilde{D}_j \hat{A}^{ij}}$$

where $\tilde{\boldsymbol{\Delta}}_L$ is the **conformal vector Laplacian**:

$$\boxed{\tilde{\Delta}_L\, X^i := \tilde{D}_j (\tilde{L}X)^{ij} = \tilde{D}_j \tilde{D}^j X^i + \frac{1}{3} \tilde{D}^i \tilde{D}_j X^j + \tilde{R}^i{}_j X^j}$$

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The operator $\tilde{\boldsymbol{\Delta}}_L$ is elliptic and its kernel is reduced to **conformal Killing vectors**, i.e. vectors $\boldsymbol{C}$ that satisfy $(\tilde{L}C)^{ij} = 0$ (generators of conformal isometries, if any)

- if Σ_0 is a *closed manifold* (i.e. compact without boundary): the solution $\boldsymbol{X}$ exists; it may be not unique, but $(\tilde{L}X)^{ij}$ is unique;
- if (Σ_0, γ) is an *asymptotically flat manifold*: there exists a unique solution $\boldsymbol{X}$ which vanishes at spatial infinity

Conclusion: the longitudinal/transverse decomposition exists and is unique

Conformal transverse-traceless form of the constraints

Defining $\tilde{E} := \Psi^8 E$ and $\tilde{p}^i := \Psi^{10} p^i$, the Hamiltonian constraint (Lichnerowicz equation) and the momentum constraint become respectively

$$\boxed{\tilde{D}_i \tilde{D}^i \Psi - \frac{\tilde{R}}{8}\Psi + \frac{1}{8}\left[(\tilde{L}X)_{ij} + \hat{A}^{\mathrm{TT}}_{ij}\right]\left[(\tilde{L}X)^{ij} + \hat{A}^{ij}_{\mathrm{TT}}\right]\Psi^{-7} + 2\pi\tilde{E}\Psi^{-3} - \frac{K^2}{12}\Psi^5 = 0} \quad (1)$$

$$\boxed{\tilde{\Delta}_L\, X^i - \frac{2}{3}\Psi^6 \tilde{D}^i K = 8\pi\tilde{p}^i} \quad (2)$$

where $(\tilde{L}X)_{ij} := \tilde{\gamma}_{ik}\tilde{\gamma}_{jl}(\tilde{L}X)^{kl}$ and $\hat{A}^{\mathrm{TT}}_{ij} := \tilde{\gamma}_{ik}\tilde{\gamma}_{jl}\hat{A}^{kl}_{\mathrm{TT}}$

Free data and constrained data

In view of the above system, we see clearly which part of the initial data on Σ_0 can be freely chosen and which part is "constrained":

- **free data**:
  - conformal metric $\tilde{\gamma}$
  - symmetric traceless and transverse[1] tensor $\hat{A}^{\mathrm{TT}}_{ij}$
  - scalar field K
  - conformal matter variables: $(\tilde{E}, \tilde{p}^i)$
- **constrained data** (or "determined data"):
  - conformal factor Ψ, obeying the *non-linear* elliptic equation (1)
  - vector $\boldsymbol{X}$, obeying the *linear* elliptic equation (2)

[1] traceless and transverse are meant with respect to $\tilde{\gamma}$

Strategy for construction initial data

York (1979) CTT method:

1. choose $(\tilde{\gamma}_{ij}, \hat{A}_{ij}^{\mathrm{TT}}, K, \tilde{E}, \tilde{p}^i)$ on Σ_0
2. solve the system (1)-(2) to get Ψ and X^i
3. construct

$$
\begin{aligned}
\gamma_{ij} &= \Psi^4 \tilde{\gamma}_{ij} \\
K^{ij} &= \Psi^{-10}\left((\tilde{L}X)^{ij} + \hat{A}^{ij}_{\mathrm{TT}}\right) + \frac{1}{3}\Psi^{-4} K \tilde{\gamma}^{ij} \\
E &= \Psi^{-8}\tilde{E} \\
p^i &= \Psi^{-10}\tilde{p}^i
\end{aligned}
$$

Then one obtains a set $(\gamma, \boldsymbol{K}, E, \boldsymbol{p})$ which satisfies the constraint equations

Decoupling on hypersurfaces of constant mean curvature

Consider the momentum constraint equation: $\tilde{\Delta}_L\, X^i - \frac{2}{3}\Psi^6 \tilde{D}^i K = 8\pi \tilde{p}^i$

If Σ_0 has a **constant mean curvature (CMC)**:

$$\boxed{K = \text{const}}$$

then $\tilde{D}^i K = 0$ and the momentum constraint equations reduces to

$$\boxed{\tilde{\Delta}_L\, X^i = 8\pi \tilde{p}^i} \qquad (3)$$

It does no longer involve Ψ

$\Longrightarrow$ decoupling of the constraint system (1)-(2)

NB: a very important case of CMC hypersurface: **maximal hypersurface**: $K = 0$

Strategy on CMC hypersurfaces

- 1st step: Solve the linear elliptic equation (3) ($\tilde{\Delta}_L X^i = 8\pi\tilde{p}^i$) to get the vector $\boldsymbol{X}$
  - if Σ_0 is a *closed manifold* (i.e. compact without boundary): the solution $\boldsymbol{X}$ exists; it may be not unique, but $(\tilde{L}X)^{ij}$ is unique;
  - if (Σ_0, γ) is an *asymptotically flat manifold*: there exists a unique solution $\boldsymbol{X}$ which vanishes at spatial infinity
- 2nd step: Inject the solution $\boldsymbol{X}$ into Lichnerowicz equation (1)

$$\tilde{D}_i\tilde{D}^i\Psi - \frac{\tilde{R}}{8}\Psi + \frac{1}{8\Psi^7}\left[(\tilde{L}X)_{ij} + \hat{A}^{\mathrm{TT}}_{ij}\right]\left[(\tilde{L}X)^{ij} + \hat{A}^{ij}_{\mathrm{TT}}\right] + \frac{2\pi\tilde{E}}{\Psi^3} - \frac{K^2}{12}\Psi^5 = 0$$

and solve the latter for Ψ (the difficult part !)

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Existence and uniqueness of solutions to Lichnerowicz equation:

- *asymptotically flat case:* (1) is solvable iff the metric $\tilde{\gamma}$ is conformal to a metric with vanishing scalar curvature (Cantor 1977)
- *closed manifold:* complete analysis carried out by Isenberg (1995) (vacuum case)

More details: see review by Bartnik and Isenberg (2004)

Conformally flat initial data on maximal slices

Simplest choice for free data $(\tilde{\gamma}_{ij}, \hat{A}^{\mathrm{TT}}_{ij}, K, \tilde{E}, \tilde{p}^i)$:

- $\tilde{\gamma}_{ij} = f_{ij}$ (flat metric)
- $\hat{A}^{\mathrm{TT}}_{ij} = 0$
- $K = 0$ ($\Sigma_0 =$ maximal hypersurface)
- $\tilde{E} = 0$ and $\tilde{p}^i = 0$ (vacuum)

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Then the constraint equations (1)-((2) reduce to

$$\boxed{\Delta\Psi + \frac{1}{8}(LX)_{ij}(LX)^{ij}\,\Psi^{-7} = 0} \tag{4}$$

$$\boxed{\Delta X^i + \frac{1}{3}\mathcal{D}^i\mathcal{D}_j X^j = 0} \tag{5}$$

where $\Delta := \mathcal{D}_i\mathcal{D}^i$ (flat Laplacian) and $(LX)^{ij} := \mathcal{D}^i X^j + \mathcal{D}^j X^i - \frac{2}{3}\mathcal{D}_k X^k\, f^{ij}$ ($\mathcal{D}_i$ flat connection: in Cartesian coordinates $\mathcal{D}_i = \partial_i$)

Asymptotic flatness $\Longrightarrow$ boundary conditions $\left\{ \begin{array}{l} \Psi|_{r\to\infty} = 1 \\ \boldsymbol{X}|_{r\to\infty} = 0 \end{array} \right.$

A (too) simple solution

Choose $\Sigma_0 \sim \mathbb{R}^3$

Then the only regular solution to $\Delta X^i + \frac{1}{3}\mathcal{D}^i\mathcal{D}_j X^j = 0$ with the boundary condition $\boldsymbol{X}|_{r\to\infty} = 0$ is

$$\boldsymbol{X} = 0$$

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Plugging this solution into the Hamiltonian constraint (4) yields Laplace equation for Ψ:

$$\Delta\Psi = 0$$

With the boundary condition $\Psi|_{r\to\infty} = 1$ the unique regular solution is

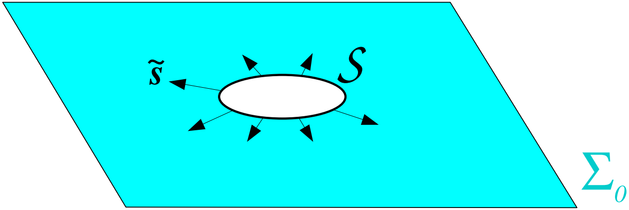
$$\Psi = 1$$

Hence the initial $(\boldsymbol{\gamma}, \boldsymbol{K})$ is $\begin{cases} \gamma_{ij} = f_{ij} \\ K_{ij} = 0 \end{cases}$ (momentarily static)

This is a standard slice $t =$ const of Minkowski spacetime

A less trivial solution

Keep the same simple free data as above, but choose for Σ_0 a less trivial topology: $\Sigma_0 \sim \mathbb{R}^3 \backslash \mathcal{B}$ ($\mathcal{B}$=ball):



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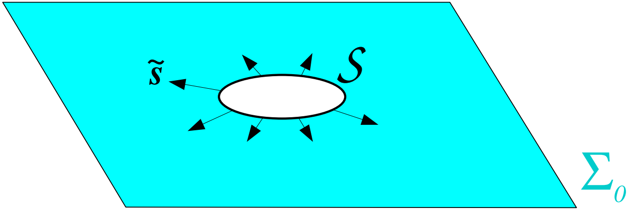

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Let us choose $\boldsymbol{X}|_{\mathcal{S}} = 0$. Altogether with the outer BC $\boldsymbol{X}|_{r\to\infty} = 0$ this yields to the following solution of momentum constraint (5)

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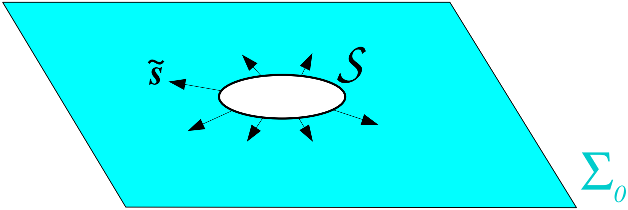

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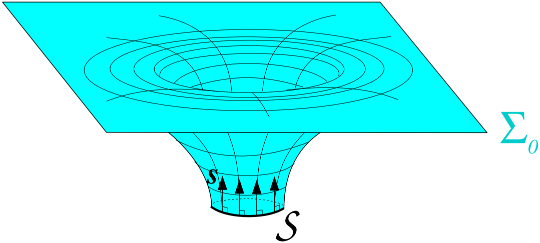
Hamiltonian constraint (4) $\Longrightarrow$ Laplace equation $\boxed{\Delta\Psi = 0}$

The choice $\Psi|_{\mathcal{S}} = 1$ would result in the same trivial solution $\Psi = 1$ as before...

A less trivial solution

In order to have something not trivial, i.e. to ensure that the metric γ will not be flat, let us demand that γ admits a **closed minimal surface**:



$\mathcal{S}$ minimal surface

$\iff$ $\mathcal{S}$'s mean curvature $= 0$

$\iff$ $D_i s^i \big|_{\mathcal{S}} = 0$

$\iff$ $\mathcal{D}_i(\Psi^6 s^i)\big|_{\mathcal{S}} = 0$

$\iff$ $\mathcal{D}_i(\Psi^4 \tilde{s}^i)\big|_{\mathcal{S}} = 0$

$\iff$

$$\boxed{\left(\frac{\partial\Psi}{\partial r} + \frac{\Psi}{2r}\right)\bigg|_{r=a} = 0} \qquad (6)$$

$\boldsymbol{s}$: unit normal to $\mathcal{S}$ for the metric γ

$\tilde{\boldsymbol{s}}$: unit normal to $\mathcal{S}$ for the metric $\tilde{\gamma}$

(r, θ, φ) : coord. sys. / $f_{ij} = \mathrm{diag}(1, r^2, r^2 \sin^2\theta)$ and $\mathcal{S}$ = sphere $\{r = a\}$

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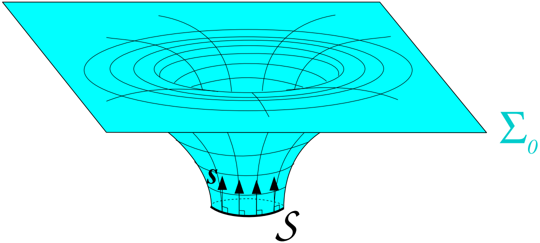

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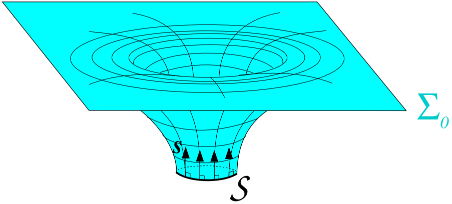
The solution to Laplace equation $\Delta\Psi = 0$ with the BC (6) and $\Psi|_{r\to\infty} = 1$ is

$$\Psi = 1 + \frac{a}{r}$$

A less trivial solution



ADM mass of that solution:

$$m = -\frac{1}{2\pi} \lim_{r\to\infty} \oint_{r=\text{const}} \frac{\partial\Psi}{\partial r}\, r^2 \sin\theta\, d\theta\, d\varphi$$

$$\Rightarrow m = 2a$$

Hence $\boxed{\Psi = 1 + \frac{m}{2r}}$

The obtained initial data is then $\begin{cases} \gamma_{ij} = \left(1 + \frac{m}{2r}\right)^4 \mathrm{diag}(1, r^2, r^2 \sin\theta) \\ K_{ij} = 0 \end{cases}$

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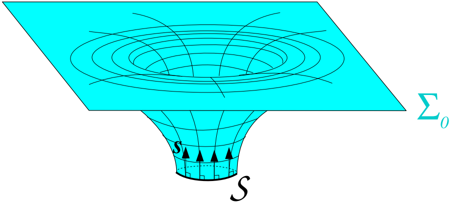

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This is a slice $t = $ const of Schwarzschild spacetime

Remember: Schwarzschild metric in isotropic coordinates (t, r, θ, φ):

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right)^2 dt^2 + \left(1+\frac{m}{2r}\right)^4 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]$$

Link with Schwarzschild coordinates (t, R, θ, φ): $R = r\left(1+\frac{m}{2r}\right)^2$

Extended solution

$\mathcal{S}$ minimal surface $\Longrightarrow$ (Σ_0, γ) can be extended *smoothly* to a larger Riemannian manifold (Σ_0', γ') by gluing a copy of Σ_0 at $\mathcal{S}$:

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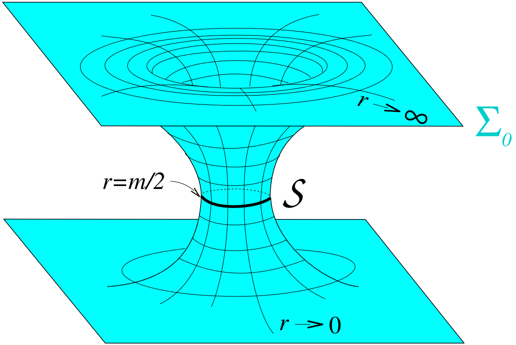
$\mathcal{S}$ minimal surface $\Longrightarrow$ (Σ_0, γ) can be extended *smoothly* to a larger Riemannian manifold (Σ'_0, γ') by gluing a copy of Σ_0 at $\mathcal{S}$:



$\mathcal{S}$ = **Einstein-Rosen bridge** between two asymptotically flat manifolds

range of r in Σ'_0 : $(0, +\infty)$

extended metric :

$$\gamma'_{ij}\, dx^i\, dx^j = \left(1 + \frac{m}{2r}\right)^4 \times \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right)$$

region $r \to 0$ = second asymptotically flat region

map $r \mapsto r' = \dfrac{m^2}{4r}$ is an isometry

This extended solution is still a slice $t =$ const of Schwarzschild spacetime
topology of $\Sigma'_0 = \mathbb{R}^3 \backslash \{O\}$ (**puncture**)

The Bowen-York solution

Same free data as before:
$\tilde{\gamma}_{ij} = f_{ij}$, $\hat{A}^{\mathrm{TT}}_{ij} = 0$, $K = 0$, $\tilde{E} = 0$ and $\tilde{p}^i = 0$
so that the constraint equations are still

$$\Delta\Psi + \frac{1}{8}(LX)_{ij}(LX)^{ij}\,\Psi^{-7} = 0 \tag{7}$$

$$\Delta X^i + \frac{1}{3}\mathcal{D}^i\mathcal{D}_j X^j = 0 \tag{8}$$

Choice of Σ_0 : $\Sigma_0 = \mathbb{R}^3 \backslash \{O\}$ (puncture topology)

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Choice of Σ_0 : $\Sigma_0 = \mathbb{R}^3 \backslash \{O\}$ (puncture topology)
Difference with previous case: $\boldsymbol{X} \neq 0$ (no longer momentarily static data)

Bowen-York (1980) solution of Eq. (8) in Cartesian coord. $(x^i) = (x, y, z)$:

$$\boxed{X^i = -\frac{1}{4r}\left(7P^i + P_j\frac{x^j x^i}{r^2}\right) - \frac{1}{r^3}\epsilon^i{}_{jk}S^j x^k}$$

Two constant vector parameters : $\begin{cases} P^i & = \text{ADM linear momentum} \\ S^i & = \text{angular momentum} \end{cases}$

The Bowen-York solution

Example: choose S^i perpendicular to P^i and choose Cartesian coordinates (x, y, z) such that $P^i = (0, P, 0)$ and $S^i = (0, 0, S)$. Then

$$
\begin{aligned}
X^x &= -\frac{P}{4}\frac{xy}{r^3} + S\frac{y}{r^3} \\
X^y &= -\frac{P}{4r}\left(7 + \frac{y^2}{r^2}\right) - S\frac{x}{r^3} \\
X^z &= -\frac{P}{4}\frac{xz}{r^3}
\end{aligned}
$$

Bowen-York extrinsic curvature: $\hat{A}^{ij} = (LX)^{ij}$:

$$
\boxed{\hat{A}^{ij} = \frac{3}{2r^3}\left[P^i x^j + P^j x^i - \left(\delta^{ij} - \frac{x^i x^j}{r^2}\right) P^k x_k\right] + \frac{3}{r^5}\left(\epsilon^i{}_{kl} S^k x^l x^j + \epsilon^j{}_{kl} S^k x^l x^i\right)}
$$

ADM linear momentum : $P_i := \frac{1}{8\pi} \lim_{\mathcal{S}_t \to \infty} \oint_{\mathcal{S}_t} (K_{jk} - K\gamma_{jk})\, (\boldsymbol{\partial}_i)^j\, s^k \sqrt{q}\, d^2y$

Angular momentum (QI) : $S_i := \frac{1}{8\pi} \lim_{\mathcal{S}_t \to \infty} \oint_{\mathcal{S}_t} (K_{jk} - K\gamma_{jk})\, (\boldsymbol{\phi}_i)^j\, s^k \sqrt{q}\, d^2y.$

The Bowen-York solution

There remains to solve (numerically !) the Hamiltonian constraint equation (7):

$$\Delta\Psi + \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\,\Psi^{-7} = 0$$

and to reconstruct $\left\{ \begin{array}{l} \gamma_{ij} = \Psi^4 f_{ij} \\ K_{ij} = \Psi^{-2}\hat{A}_{ij} \end{array} \right.$

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Remark 1: static Bowen-York solution ($P^i = 0$, $S^i = 0$) = maximal slice of Schwarzschild spacetime considered above

Remark 2: Bowen-York solution with $S^i \neq 0$ is not a slice of Kerr spacetime : it is initial data for a rotating black hole but in a non stationary state (black hole "surrounded" by gravitational radiation)

Outline

Conformal thin sandwich decomposition of extrinsic curvature

Origin: York (1999)

From Lecture 1: $\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}}\right)\tilde\gamma^{ij} = 2N\tilde A^{ij} + \frac{2}{3}\tilde D_k\beta^k\,\tilde\gamma^{ij}$

with $\tilde A^{ij} = \Psi^4 A^{ij} = \Psi^{-6}\hat A^{ij}$ and $-\mathcal{L}_{\boldsymbol{\beta}}\,\tilde\gamma^{ij} = (\tilde L\beta)^{ij} + \frac{2}{3}\tilde D_k\beta^k$

Hence

$$\hat A^{ij} = \frac{\Psi^6}{2N}\left[\dot{\tilde\gamma}^{ij} + (\tilde L\beta)^{ij}\right]$$

where $\dot{\tilde\gamma}^{ij} := \frac{\partial}{\partial t}\tilde\gamma^{ij}$

Introduce the **conformal lapse**: $\boxed{\tilde N := \Psi^{-6}N}$

then

$$\boxed{\hat A^{ij} = \frac{1}{2\tilde N}\left[\dot{\tilde\gamma}^{ij} + (\tilde L\beta)^{ij}\right]}$$

Conformal thin sandwich equations

Hamiltonian and momentum constraints become

$$\boxed{\tilde{D}_i\tilde{D}^i\Psi - \frac{\tilde{R}}{8}\Psi + \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\,\Psi^{-7} + 2\pi\tilde{E}\Psi^{-3} - \frac{K^2}{12}\Psi^5 = 0}$$

$$\boxed{\tilde{D}_j\left(\frac{1}{\tilde{N}}(\tilde{L}\beta)^{ij}\right) + \tilde{D}_j\left(\frac{1}{\tilde{N}}\dot{\tilde{\gamma}}^{ij}\right) - \frac{4}{3}\Psi^6\tilde{D}^iK = 16\pi\tilde{p}^i}$$

- **free data :** $(\tilde{\gamma}_{ij}, \dot{\tilde{\gamma}}^{ij}, K, \tilde{N}, \tilde{E}, \tilde{p}^i)$
- **constrained data:** Ψ and β^i

Extended conformal thin sandwich (XCTS)

Origin: Pfeiffer & York (2003)

Idea: instead of choosing the conformal lapse $\tilde{N}$, compute it from the Einstein equation (not a constraint !) involving the time derivative $\dot{K}$ of K:
from Lecture 1 :

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}}\right) K = -\Psi^{-4}\left(\tilde{D}_i\tilde{D}^i N + 2\tilde{D}_i \ln\Psi\, \tilde{D}^i N\right) + N\left[4\pi(E+S) + \tilde{A}_{ij}\tilde{A}^{ij} + \frac{K^2}{3}\right]$$

Combining with the Hamiltonian constraint, we get

$$\tilde{D}_i\tilde{D}^i(\tilde{N}\Psi^7) - (\tilde{N}\Psi^7)\left[\frac{1}{8}\tilde{R} + \frac{5}{12}K^2\Psi^4 + \frac{7}{8}\hat{A}_{ij}\hat{A}^{ij}\Psi^{-8} + 2\pi(\tilde{E}+2\tilde{S})\Psi^{-4}\right] + \left(\dot{K} - \beta^i\tilde{D}_i K\right)\Psi^5 = 0$$

where $\tilde{E} := \Psi^8 E$ and $\tilde{S} := \Psi^8 S$

Extended conformal thin sandwich system

PDE system of 5 equations:

$$\tilde{D}_i\tilde{D}^i\Psi - \frac{\tilde{R}}{8}\Psi + \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\,\Psi^{-7} + 2\pi\tilde{E}\Psi^{-3} - \frac{K^2}{12}\Psi^5 = 0$$

$$\tilde{D}_j\left(\frac{1}{\tilde{N}}(\tilde{L}\beta)^{ij}\right) + \tilde{D}_j\left(\frac{1}{\tilde{N}}\dot{\tilde{\gamma}}^{ij}\right) - \frac{4}{3}\Psi^6\tilde{D}^iK - 16\pi\tilde{p}^i = 0$$

$$\tilde{D}_i\tilde{D}^i(\tilde{N}\Psi^7) - (\tilde{N}\Psi^7)\left[\frac{1}{8}\tilde{R} + \frac{5}{12}K^2\Psi^4 + \frac{7}{8}\hat{A}_{ij}\hat{A}^{ij}\Psi^{-8} + 2\pi(\tilde{E}+2\tilde{S})\Psi^{-4}\right] + \left(\dot{K} - \beta^i\tilde{D}_iK\right)\Psi^5 = 0$$

- **free data :** $(\tilde{\gamma}_{ij}, \dot{\tilde{\gamma}}^{ij}, K, \dot{K}, \tilde{E}, \tilde{S}, \tilde{p}^i)$
- **constrained data:** Ψ, $\tilde{N}$ and β^i

Existence and uniqueness of solutions

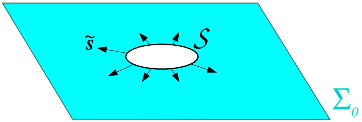
Pfeiffer & York (2005): in some cases, solutions $(\Psi, \tilde{N}, \beta^i)$ to the (non-linear !) XCTS system are not unique, even on maximal surfaces

See also analysis by Baumgarte, Ó Murchadha & Pfeiffer (2007) and Walsh (2007)

XCTS at work: a simple example

Choose the same manifold $\Sigma_0 = \mathbb{R}^3 \backslash \mathcal{B}$ ($\mathbb{R}^3$ with an excised ball) as considered previously

Choose the free data to be
$\tilde{\gamma}_{ij} = f_{ij}$, $\dot{\tilde{\gamma}}^{ij} = 0$, $K = 0$, $\dot{K} = 0$, $\tilde{E} = 0$, $\tilde{S} = 0$, $\tilde{p}^i = 0$



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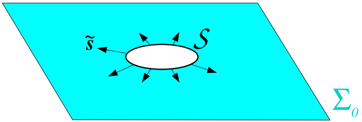

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$\Longrightarrow$ the XCTS equations reduce to

$$\Delta\Psi + \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\,\Psi^{-7} = 0 \qquad (9)$$

$$\mathcal{D}_j\left(\frac{1}{\tilde{N}}(L\beta)^{ij}\right) = 0 \qquad (10)$$

$$\Delta(\tilde{N}\Psi^7) - \frac{7}{8}\hat{A}_{ij}\hat{A}^{ij}\Psi^{-1}\tilde{N} = 0 \qquad (11)$$

with $\hat{A}^{ij} = \dfrac{1}{2\tilde{N}}(L\beta)^{ij}$

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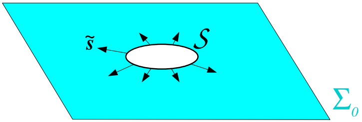

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with $\hat{A}^{ij} = \dfrac{1}{2\tilde{N}} (L\beta)^{ij}$

Choose the boundary condition $\boldsymbol{\beta}|_{\mathcal{S}} = 0$ in addition to $\boldsymbol{\beta}|_{r\to\infty} = 0$. Then, independently of the value of $\tilde{N}$, the unique solution to Eq. (10) is

$$\boxed{\boldsymbol{\beta} = 0}$$

XCTS at work: a simple example

Accordingly $\hat{A}^{ij} = 0$ and Eqs. (9) and (11) reduce to two Laplace equations:

$$\Delta\Psi = 0 \tag{12}$$

$$\Delta(\tilde{N}\Psi^7) = 0 \tag{13}$$

As previously use the minimal surface requirement for $\mathcal{S}$ to get the solution $\boxed{\Psi = 1 + \frac{m}{2r}}$ to Eq. (12).

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Regarding Eq. (13), choose the BC $\tilde{N}\big|_{\mathcal{S}} = 0$ (singular slicing). Along with the asymptotic flatness BCs $\tilde{N}\big|_{r\to\infty} = 1$ and $\Psi\big|_{r\to\infty} = 1$, this yields the solution

$$\tilde{N}\Psi^7 = 1 - \frac{m}{2r}, \text{ i.e., since } N = \Psi^6\tilde{N}, \ \boxed{N = \left(1 - \frac{m}{2r}\right)\left(1 + \frac{m}{2r}\right)^{-1}}$$

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We obtain Schwarzschild metric (in isotropic coordinates):

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right)^2 dt^2 + \left(1+\frac{m}{2r}\right)^4 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]$$

Comparing CTT and (X)CTS methods

- CTT : choose some transverse traceless part $\hat{A}^{ij}_{TT}$ of the extrinsic curvature K^{ij}, i.e. some *momentum* [2] $\Longrightarrow$ **CTT = Hamiltonian representation**
- CTS or XCTS : choose some time derivative $\dot{\tilde{\gamma}}^{ij}$ of the conformal metric $\tilde{\gamma}^{ij}$, i.e. some *velocity* $\Longrightarrow$ **(X)CTS = Lagrangian representation**

[2] recall the relation $\pi^{ij} = \sqrt{\gamma}(K\gamma^{ij} - K^{ij})$ between K^{ij} and the ADM canonical momentum

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Advantage of CTT : mathematical theory well developed; existence and uniqueness of solutions established (at least for constant mean curvature ($K = \text{const}$) slices)

Advantage of XCTS : better suited to the description of quasi-stationary spacetimes ($\rightarrow$ quasiequilibrium initial data) :

$$\frac{\partial}{\partial t} \ \text{Killing vector} \Rightarrow \dot{\tilde{\gamma}}^{ij} = 0 \ \text{and} \ \dot{K} = 0$$

[2] recall the relation $\pi^{ij} = \sqrt{\gamma}(K\gamma^{ij} - K^{ij})$ between K^{ij} and the ADM canonical momentum