General relativity computations with SageManifolds

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Outline

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- In 1969, during his PhD under Pirani supervision, Ray d'Inverno wrote `ALAM (Atlas Lisp Algebraic Manipulator)` and used it to compute the Riemann tensor of Bondi metric. The original calculations took Bondi and his collaborators 6 months to go. The computation with ALAM took 4 minutes and yielded to the discovery of 6 errors in the original paper [J.E.F. Skea, *Applications of SHEEP* (1994)]

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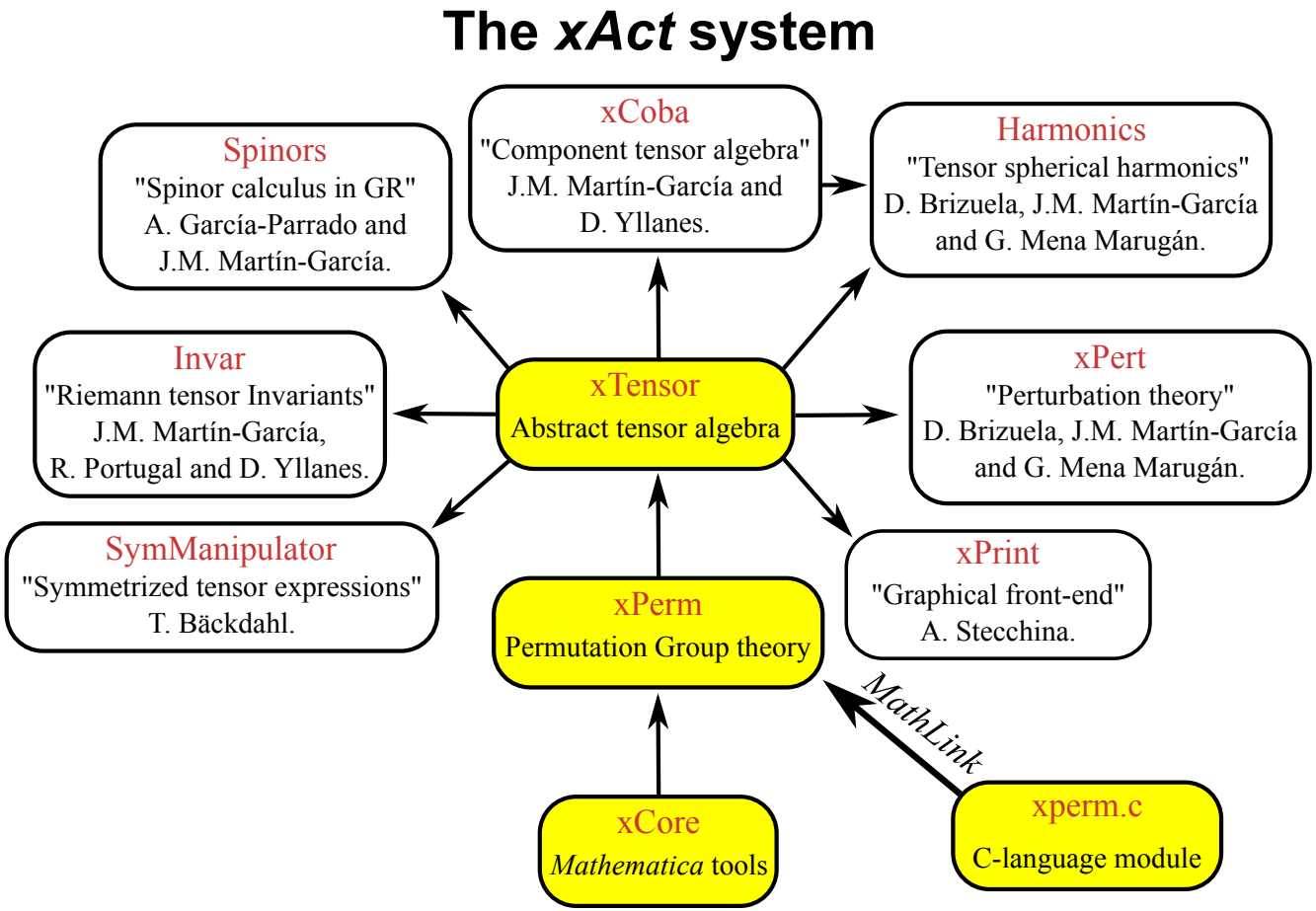
- Since then, many softwares for tensor calculus have been developed...

An example of modern software: The xAct suite

Free packages for tensor computer algebra in Mathematica, developed by José Martín-García et al. http://www.xact.es/



[García-Parrado Gómez-Lobo & Martín-García, Comp. Phys. Comm. **183**, 2214 (2012)]

Software for differential geometry

Packages for general purpose computer algebra systems:

- xAct free package for Mathematica [J.-M. Martin-Garcia]
- Ricci free package for Mathematica [J. L. Lee]
- MathTensor package for Mathematica [S. M. Christensen & L. Parker]
- GRTensor package for Maple [P. Musgrave, D. Pollney & K. Lake]
- DifferentialGeometry included in Maple [I. M. Anderson & E. S. Cheb-Terrab]
- Atlas 2 for Maple and Mathematica
- ...

Standalone applications:

- SHEEP, Classi, STensor, based on Lisp, developed in 1970's and 1980's (free) [R. d'Inverno, I. Frick, J. Åman, J. Skea, et al.]
- Cadabra field theory (free) [K. Peeters]
- SnapPy topology and geometry of 3-manifolds, based on Python (free) [M. Culler, N. M. Dunfield & J. R. Weeks]
- ...

cf. the complete list at `http://www.xact.es/links.html`

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The mission

Create a viable free open source alternative to Magma, Maple, Mathematica and Matlab.

Some advantages of SageMath

SageMath is free

Freedom means

1. everybody can use it, by downloading the software from http://sagemath.org
2. everybody can examine the source code and improve it

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- no need to learn any specific syntax to use it
- easy access for students
- Python is a very powerful *object oriented language*, with a neat syntax

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SageMath is developing and spreading fast

...sustained by an enthousiast community of developers

Object-oriented notation in Python

As an object-oriented language, Python (and hence SageMath) makes use of the following **postfix notation** (same in C++, Java, etc.):

```
result = object.function(arguments)
```

In a procedural language, this would be written as

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result = function(object,arguments)
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Examples

1. `riem = g.riemann()`
2. `lie_t_v = t.lie_der(v)`

NB: no argument in example 1

SageMath approach to computer mathematics

SageMath relies on a `Parent` / `Element` scheme: each object x on which some calculus is performed has a "parent", which is another SageMath object X representing the set to which x belongs.
The calculus rules on x are determined by the *algebraic structure* of X.
Conversion rules prior to an operation, e.g. $x + y$ with x and y having different parents, are defined at the level of the parents

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Example

```
sage: x = 4 ; x.parent()
Integer Ring
sage: y = 4/3 ; y.parent()
Rational Field
sage: s = x + y ; s.parent()
Rational Field
sage: y.parent().has_coerce_map_from(x.parent())
True
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This approach is similar to that of Magma and is different from that of Mathematica, in which everything is a tree of symbols

Outline

The SageManifolds project

http://sagemanifolds.obspm.fr/

Aim

Implement real smooth manifolds of arbitrary dimension in Sage and tensor calculus on them

In particular:

- one should be able to introduce an arbitrary number of coordinate charts on a given manifold, with the relevant transition maps
- tensor fields must be manipulated as such and not through their components with respect to a specific (possibly coordinate) vector frame

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Concretely, the project amounts to creating new Python classes, such as `Manifold`, `Chart`, `TensorField` or `Metric`, within SageMath's **Parent/Element framework**.

Implementing coordinate charts

Given a (topological) manifold M of dimension $n \geq 1$, a **coordinate chart** is a homeomorphism $\varphi : U \to V$, where U is an open subset of M and V is an open subset of $\mathbb{R}^n$.

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Examples:

- at least 2 charts are necessary to cover the n-dimensional sphere $\mathbb{S}^n$ $(n \geq 1)$ and the torus $\mathbb{T}^2$
- at least 3 charts are necessary to cover the real projective plane $\mathbb{RP}^2$

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In SageManifolds, an arbitrary number of charts can be introduced

To fully specify the manifold, one shall also provide the *transition maps* on overlapping chart domains (SageManifolds class `CoordChange`)

Implementing scalar fields

A **scalar field** on manifold M is a smooth mapping

$$\begin{array}{rccc} f: & U \subset M & \longrightarrow & \mathbb{R} \\ & p & \longmapsto & f(p) \end{array}$$

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The various coordinate representations F, $\hat{F}$, ... of f are stored as a *Python dictionary* whose keys are the charts C, $\hat{C}$, ...:

$$f.\texttt{\_express} = \left\{C : F,\ \hat{C} : \hat{F}, \ldots\right\}$$

with $f(\underbrace{p}_{\text{point}}) = F(\underbrace{x^1, \ldots, x^n}_{\substack{\text{coord. of } p \\ \text{in chart } C}}) = \hat{F}(\underbrace{\hat{x}^1, \ldots, \hat{x}^n}_{\substack{\text{coord. of } p \\ \text{in chart } \hat{C}}}) = \ldots$

The scalar field algebra

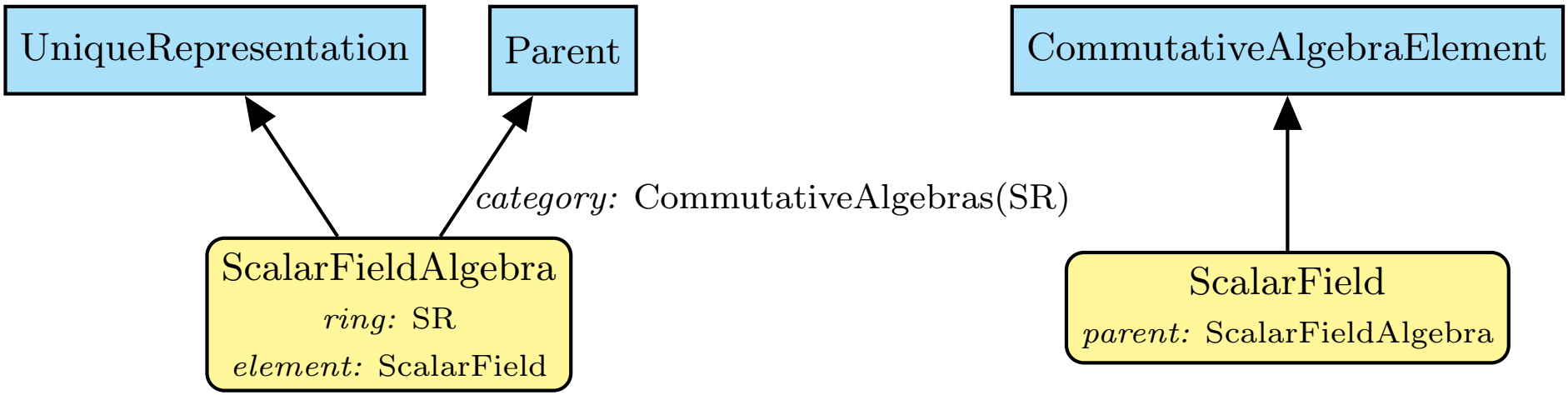
Given an open subset $U \subset M$, the set $C^\infty(U)$ of scalar fields defined on U has naturally the structure of a **commutative algebra over** $\mathbb{R}$:

1. it is clearly a vector space over $\mathbb{R}$
2. it is endowed with a commutative ring structure by pointwise multiplication:

$$\forall f, g \in C^\infty(U), \quad \forall p \in U, \quad (f.g)(p) := f(p)g(p)$$

The algebra $C^\infty(U)$ is implemented in SageManifolds via the class `ScalarFieldAlgebra`.

Classes for scalar fields



Native Sage class

SageManifolds class
(differential part)

Outline

Various ways to install/access SageMath

- Install on your computer:
  3 options:
  - compile from source (Linux, MacOS X):

    ```
    git clone git://github.com/sagemath/sage.git
    cd sage
    MAKE='make -j8' make
    ```

  - install a compiled binary version (Linux, MacOS X)
  - run in virtual machine (Windows)
- Sage Debian Live USB key:
  http://sagedebianlive.metelu.net/
  comes along with SageMath (boosted with octave, scilab), Geogebra, LaTex, gimp, vlc, LibreOffice,...
- SageMathCloud:
  https://cloud.sagemath.com/
- SageMathCell:
  Single cell mode: http://sagecell.sagemath.org/

Various ways to run SageMath

- Console mode:
  run the command `sage`
- Standard Sage Notebook:
  run the command `sage -n`
  $\Longrightarrow$ worksheet file format: `sws`
- Jupyter Notebook[1]:
  run the command `sage -n jupyter`
  $\Longrightarrow$ worksheet file format: `ipynb`
- SageMathCloud:
  in your browser, open `https://cloud.sagemath.com/`
  $\Longrightarrow$ worksheet file format: `sagews`, `ipynb`

[1] the future standard notebook

A full example: deriving and solving the TOV equations

Let us use SageManifolds to (i) derive the Tolman-Oppenheimer-Volkoff (TOV) equations from the Einstein equation and (ii) to solve the TOV system to get some numerical models of relativistic stars

See the worksheet at
http://nbviewer.jupyter.org/github/egourgoulhon/NewCompStarSchool/blob/master/WorkSheets/TOV.ipynb

The source is stored at GitHub, from which it can be downloaded:
https://github.com/egourgoulhon/NewCompStarSchool

A copy of the worksheet is also publicly available on the SageMathCloud (click on the icon "Files"):
https://cloud.sagemath.com/projects/8f20b8d0-aac0-4454-95d5-dc929acae1e5/files/TOV.ipynb

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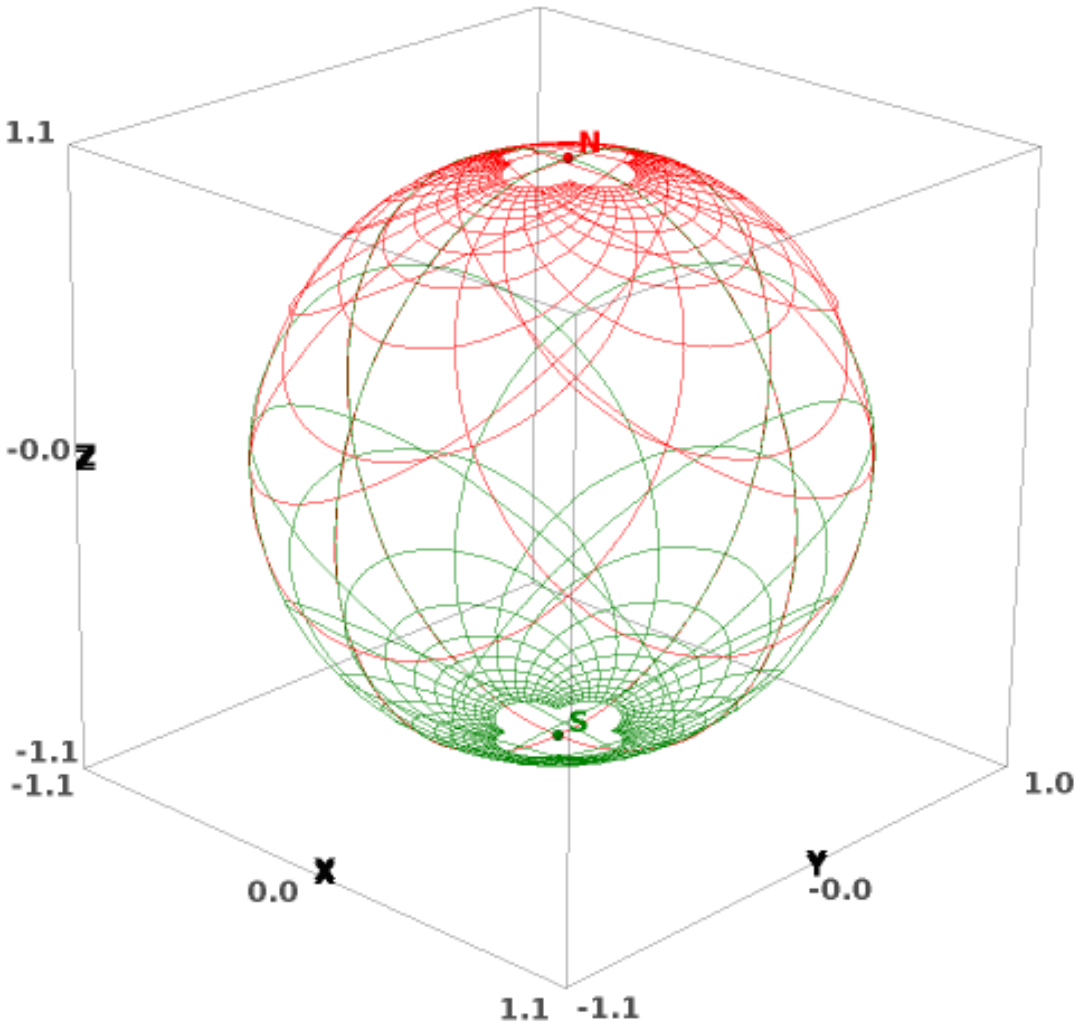
The 2-sphere example



Stereographic coordinates on the 2-sphere

Two charts:

- X_1: $\mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$
- X_2: $\mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^2$

← picture obtained via function `Chart.plot()`

See the worksheet at
http://sagemanifolds.obspm.fr/examples/html/SM_sphere_S2.html

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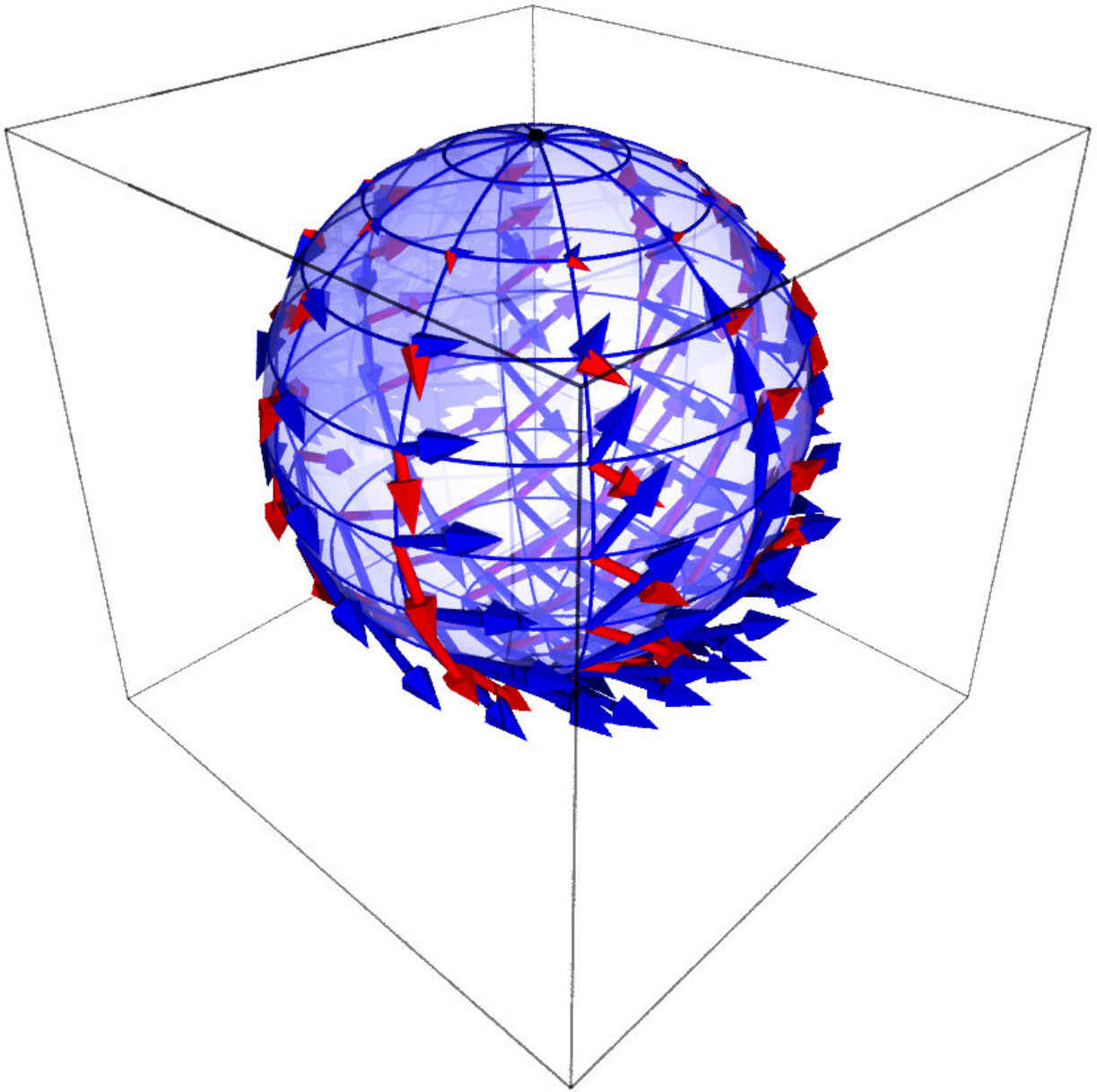


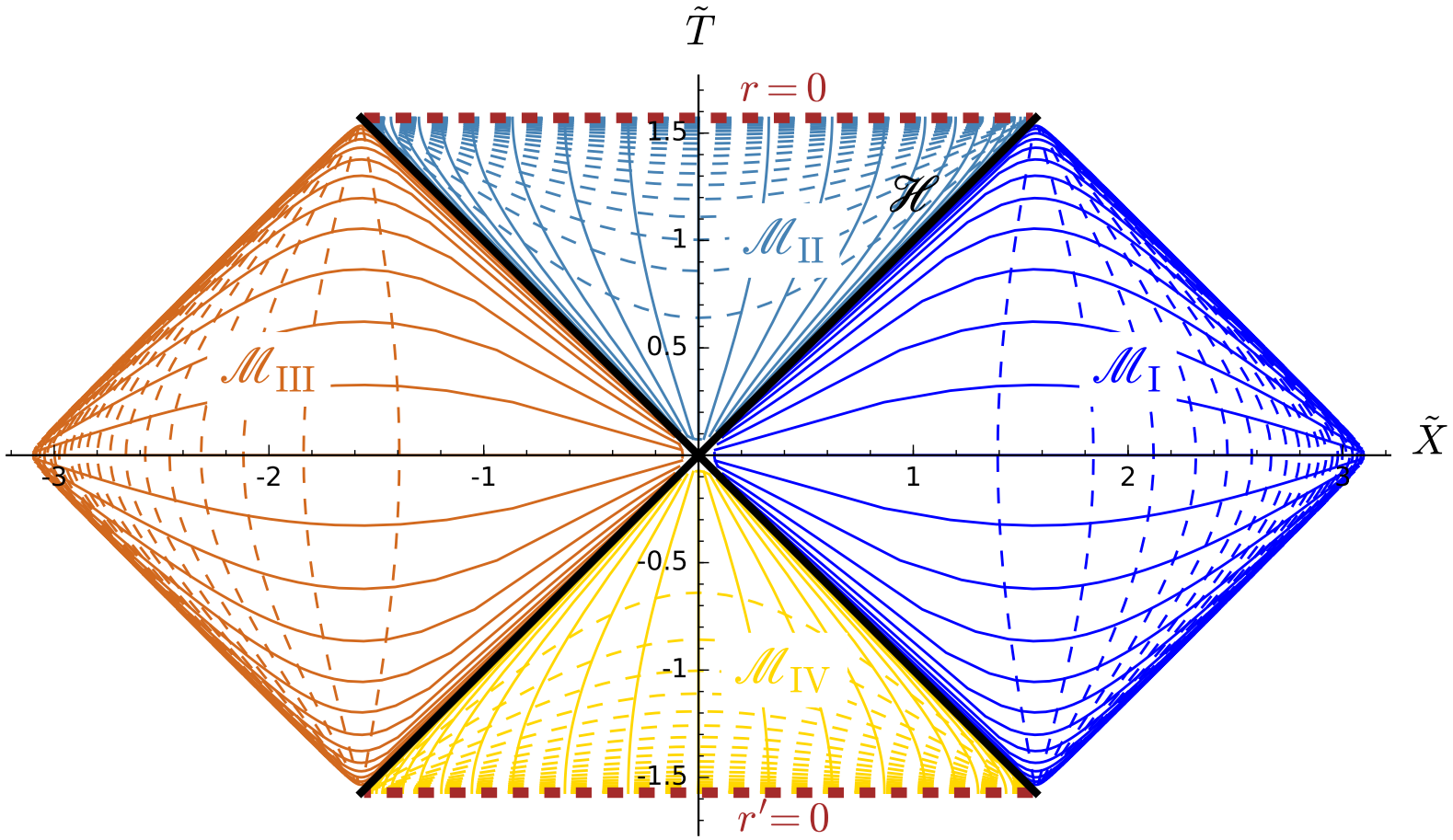
Vector frame associated with the stereographic coordinates (x, y) from the North pole

- $\frac{\partial}{\partial x}$
- $\frac{\partial}{\partial y}$

← picture obtained via function `VectorField.plot()`

Charts on Schwarzschild spacetime

The Carter-Penrose diagram



Two charts of standard Schwarzschild-Droste coordinates (t, r, θ, φ) plotted in terms of compactified coordinates $(\tilde{T}, \tilde{X}, \theta, \varphi)$; see the worksheet at
http://luth.obspm.fr/~luthier/gourgoulhon/bh16/sage.html

Outline

Conclusion and perspectives

- **SageManifolds** is a work in progress
  $\sim$ 64,000 lines of Python code up to now (including comments and doctests)
- A preliminary version (v0.9) is freely available (GPL) at
  http://sagemanifolds.obspm.fr/

Current status

Already present (v0.9):

- maps between manifolds, pullback operator
- submanifolds, pushforward operator
- curves in manifolds
- standard tensor calculus (tensor product, contraction, symmetrization, etc.), even on non-parallelizable manifolds
- all monoterm tensor symmetries
- exterior calculus (wedge product, exterior derivative, Hodge duality)
- Lie derivatives of tensor fields
- affine connections, curvature, torsion
- pseudo-Riemannian metrics, Weyl tensor
- some plotting capabilities (charts, points, curves, vector fields)
- parallelization (on tensor components) of CPU demanding computations, via the Python library `multiprocessing`

Current status

- *Not implemented yet (but should be soon):*
  - extrinsic geometry of pseudo-Riemannian submanifolds
  - computation of geodesics (numerical integration via SageMath/GSL or `Gyoto`)
  - integrals on submanifolds

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  - integrals on submanifolds
- *Future prospects:*
  - add more graphical outputs
  - add more functionalities: symplectic forms, fibre bundles, spinors, variational calculus, etc.
  - connection with numerical relativity: using SageMath to explore numerically-generated spacetimes

Integration into SageMath

SageManifolds is aimed to be fully integrated into SageMath

- The algebraic part (tensors on free modules of finite rank) has been submitted to SageMath Trac as ticket #15916 and got a positive review $\Longrightarrow$ integrated in SageMath 6.6
- The differential part has been split in various tickets for submission to SageMath Trac (cf. the metaticket #18528); 4 tickets have been already accepted and integrated in SageMath 7.3
- Until complete integration, the full SageManifold has to be downloaded from `http://sagemanifolds.obspm.fr/`
- SageManifolds v0.9 is installed in the SageMathCloud $\Longrightarrow$ open a free account and use it online: `https://cloud.sagemath.com/`

Want to join the project or simply to stay tuned?

visit `http://sagemanifolds.obspm.fr/`
(download page, documentation, example worksheets, mailing list)