Black holes: from event horizons to trapping horizons

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Laboratoire de Physique Théorique
Orsay, France
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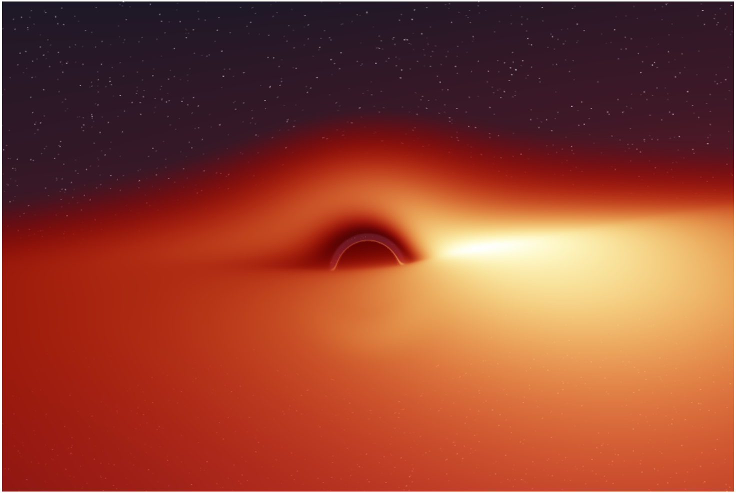
Plan

Outline

What is a black hole ?

... for the astrophysicist: a very deep gravitational potential well



[J.A. Marck, CQG **13**, 393 (1996)]

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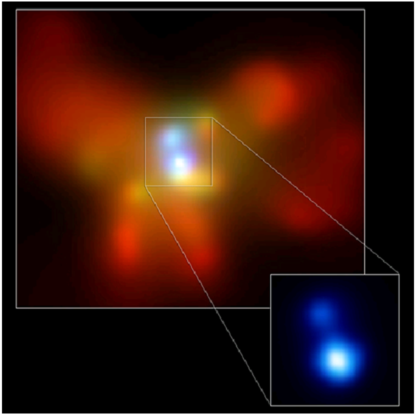
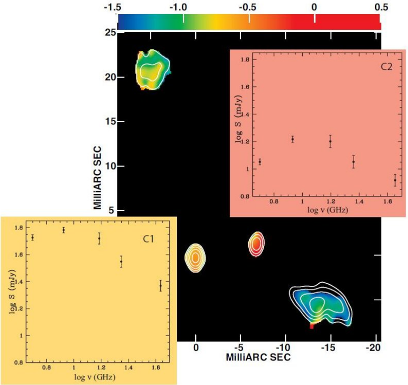


Binary BH in galaxy NGC 6240

$d = 1.4$ kpc

[Komossa et al., ApJ **582**, L15 (2003)]



Binary BH in radio galaxy 0402+379

$d = 7.3$ pc

[Rodriguez et al., ApJ **646**, 49 (2006)]

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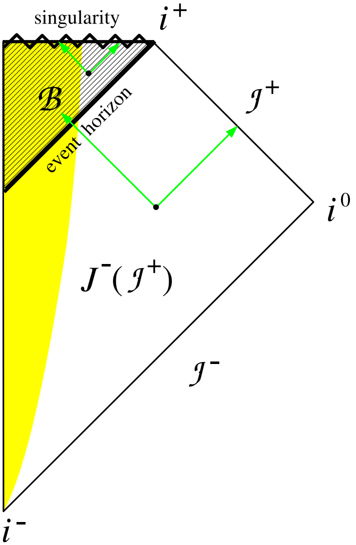

... for the mathematical physicist:

$$\boxed{\mathcal{B} := \mathscr{M} - J^-(\mathscr{I}^+)}$$

i.e. the region of spacetime where light rays cannot escape to infinity

- $(\mathscr{M}, \boldsymbol{g})$ = asymptotically flat manifold
- $\mathscr{I}^+$ = future null infinity
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event horizon: $\mathcal{H} := \partial J^-(\mathscr{I}^+)$
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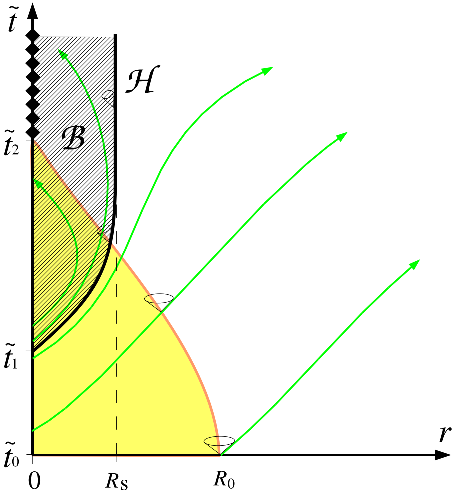

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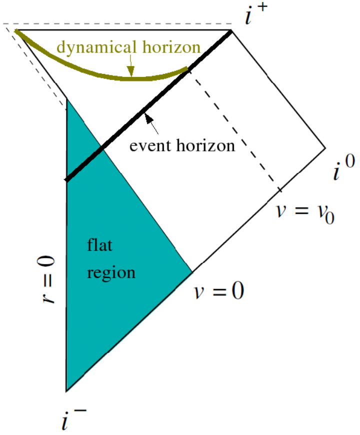
Determination of $\dot{J}^-(\mathscr{I}^+)$ requires the knowledge of the entire future null infinity. Moreover this is *not locally linked with the notion of strong gravitational field:*



Example of event horizon in a **flat** region of spacetime:

Vaidya metric, describing incoming radiation from infinity:

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right) dv^2 + 2dv\,dr + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

with

$$\begin{array}{ll} m(v) = 0 & \text{for } v < 0 \\ dm/dv > 0 & \text{for } 0 \le v \le v_0 \\ m(v) = M_0 & \text{for } v > v_0 \end{array}$$

[Ashtekar & Krishnan, LRR **7**, 10 (2004)]

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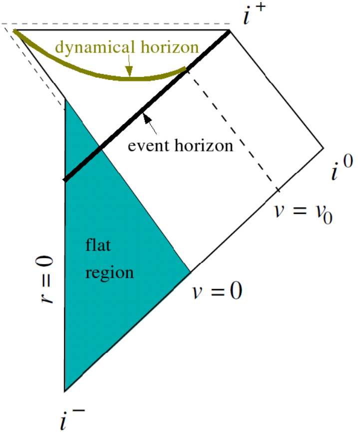

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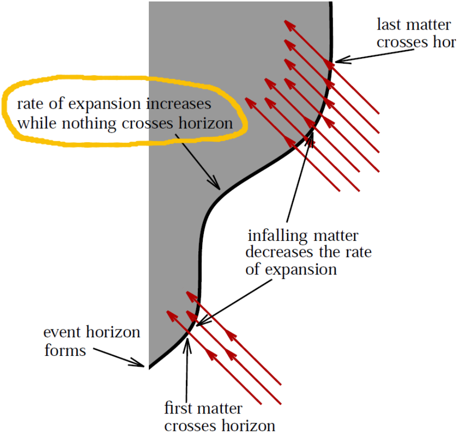
$\Rightarrow$ no local physical experiment whatsoever can locate the event horizon

Another non-local feature: teleological nature of event horizons



[Booth, Can. J. Phys. **83**, 1073 (2005)]

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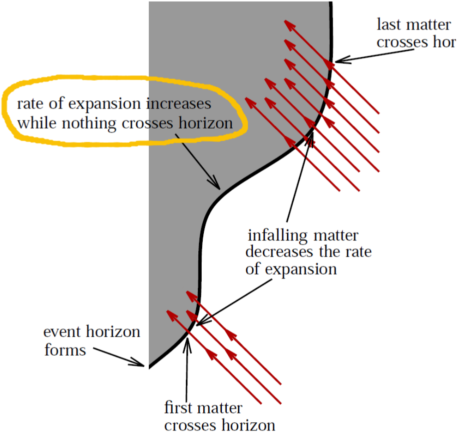

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To deal with black holes as ordinary physical objects, a **local** definition would be desirable
$\rightarrow$ quantum gravity, numerical relativity

Outline

Local characterizations of black holes

Recently a **new paradigm** has appeared in the theoretical approach of black holes: instead of *event horizons*, black holes are described by

- trapping horizons (Hayward 1994)
- isolated horizons (Ashtekar et al. 1999)
- dynamical horizons (Ashtekar and Krishnan 2002)
- slowly evolving horizons (Booth and Fairhurst 2004)

All these concepts are **local** and are based on the notion of trapped surfaces

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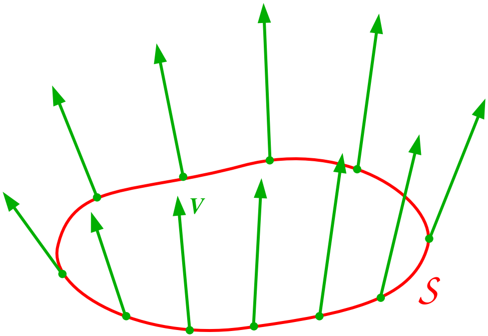

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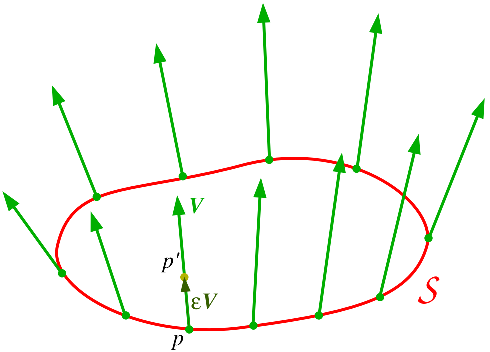

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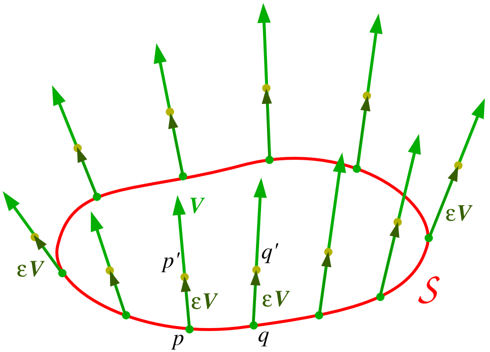

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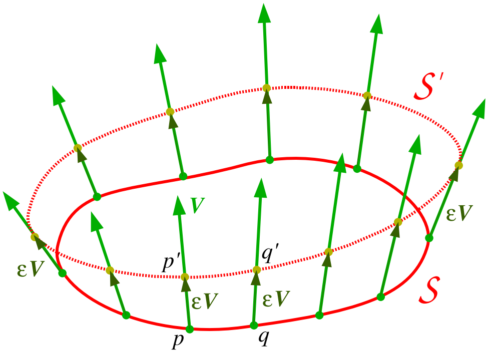

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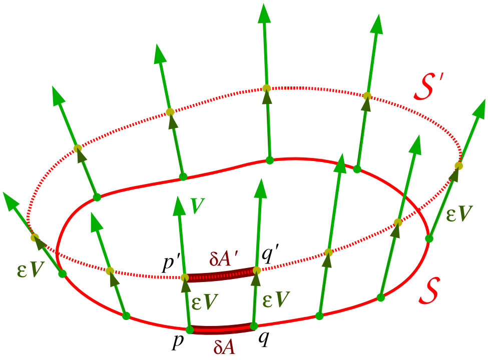

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At each point, the **expansion of $\mathcal{S}$ along $\boldsymbol{v}$** is defined from the relative change in the area element δA:

$$\boxed{\theta^{(\boldsymbol{v})} := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\frac{\delta A' - \delta A}{\delta A} = \boldsymbol{\mathcal{L}}_{\boldsymbol{v}} \ln\sqrt{q} = q^{\mu\nu}\nabla_\mu v_\nu}$$

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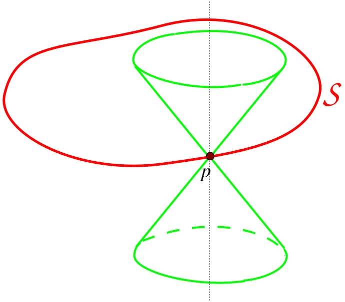

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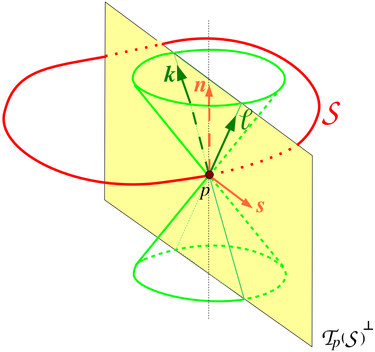

Being spacelike, $\mathcal{S}$ lies outside the light cone

$\exists$ two future-directed null directions orthogonal to $\mathcal{S}$:

$\boldsymbol{\ell}$ = outgoing, expansion $\theta^{(\boldsymbol{\ell})}$

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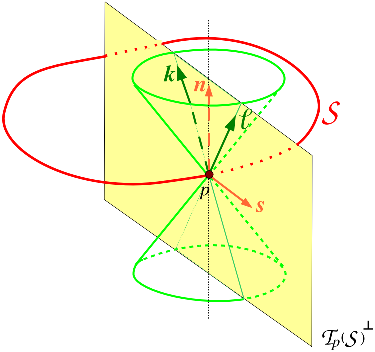

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$\mathcal{S}$ is trapped $\iff$ $\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} < 0$ [Penrose 1965]

$\mathcal{S}$ is marginally trapped $\iff$ $\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} = 0$

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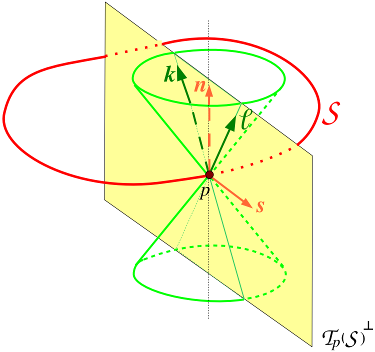

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trapped surface = **local** concept characterizing very strong gravitational fields

Link with apparent horizons

A closed spacelike 2-surface $\mathcal{S}$ is said to be outer trapped (resp. marginally outer trapped (MOTS)) iff [Hawking & Ellis 1973]

- the notions of *interior* and *exterior* of $\mathcal{S}$ can be defined (for instance spacetime asymptotically flat) $\Rightarrow$ $\boldsymbol{\ell}$ is chosen to be the *outgoing* null normal and $\boldsymbol{k}$ to be the *ingoing* one
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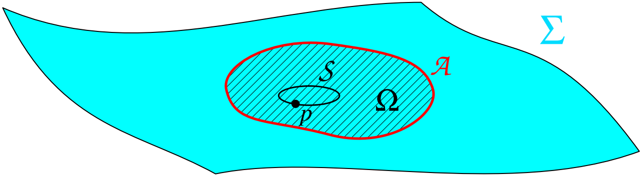


Σ: spacelike hypersurface extending to spatial infinity (Cauchy surface)

outer trapped region of Σ : Ω = set of points $p \in \Sigma$ through which there is a outer trapped surface $\mathcal{S}$ lying in Σ

apparent horizon in Σ: $\mathcal{A}$ = connected component of the boundary of Ω

Proposition [Hawking & Ellis 1973]: $\mathcal{A}$ smooth $\Longrightarrow$ $\mathcal{A}$ is a MOTS

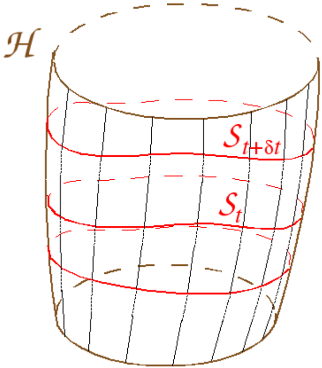
Connection with singularities and black holes

Proposition [Penrose (1965)]:
provided that the weak energy condition holds,
$\exists$ a trapped surface $\mathcal{S} \Longrightarrow \exists$ a singularity in $(\mathscr{M}, \boldsymbol{g})$ (in the form of a future inextendible null geodesic)

Proposition [Hawking & Ellis (1973)]:
provided that the cosmic censorship conjecture holds,
$\exists$ a trapped surface $\mathcal{S} \Longrightarrow \exists$ a black hole $\mathcal{B}$ and $\mathcal{S} \subset \mathcal{B}$

Local definitions of "black holes"

A hypersurface $\mathcal{H}$ of $(\mathscr{M}, \boldsymbol{g})$ is said to be



- a **future outer trapping horizon (FOTH)** iff
  (i) $\mathcal{H}$ foliated by marginally trapped 2-surfaces $(\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} = 0)$
  (ii) $\mathcal{L}_{\boldsymbol{k}}\, \theta^{(\boldsymbol{\ell})} < 0$ (locally outermost trapped surf.)
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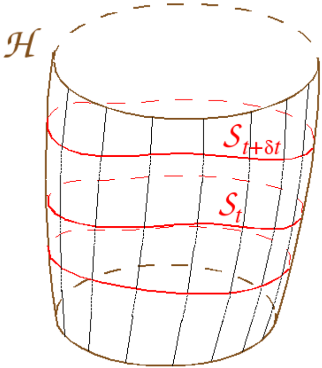

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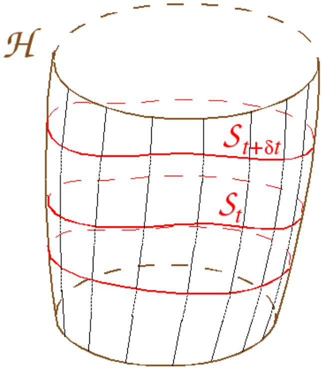

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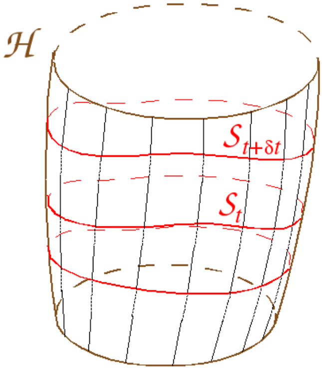

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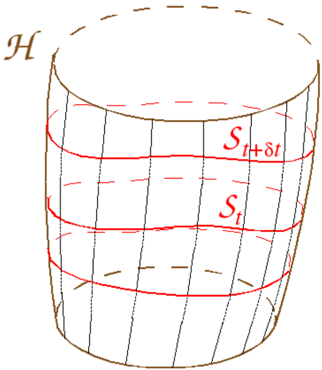

BH in equilibrium = IH
(e.g. Kerr)
BH out of equilibrium = DH
generic BH = FOTH

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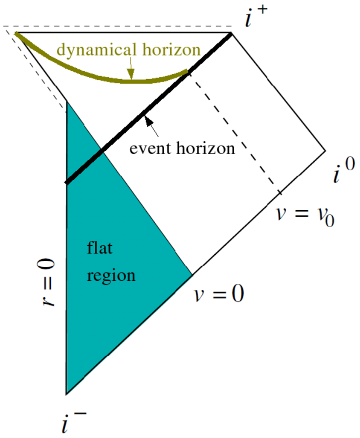
Example: Vaidya spacetime



[Ashtekar & Krishnan, LRR **7**, 10 (2004)]

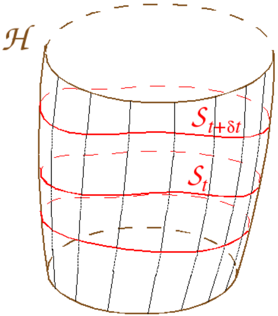
- The event horizon crosses the flat region
- The dynamical horizon lies entirely outside the flat region

Dynamics of these new horizons

The *trapping horizons* and *dynamical horizons* have their **own dynamics**, ruled by Einstein equations.
In particular, one can establish for them

- existence and (partial) uniqueness theorems
  [Andersson, Mars & Simon, PRL **95**, 111102 (2005)],
  [Ashtekar & Galloway, Adv. Theor. Math. Phys. **9**, 1 (2005)]
- first and second laws of black hole mechanics
  [Ashtekar & Krishnan, PRD **68**, 104030 (2003)], [Hayward, PRD **70**, 104027 (2004)]
- a viscous fluid bubble analogy ("membrane paradigm", as for the event horizon)
  [EG, PRD **72**, 104007 (2005)], [EG & Jaramillo, PRD **74**, 087502 (2006)]

Foliation of a hypersurface by spacelike 2-surfaces



hypersurface $\mathcal{H}$ = submanifold of spacetime $(\mathscr{M}, \boldsymbol{g})$ of codimension 1

$\mathcal{H}$ can be $\begin{cases} \text{spacelike} \\ \text{null} \\ \text{timelike} \end{cases}$

$$\mathcal{H} = \bigcup_{t \in \mathbb{R}} \mathcal{S}_t$$

$\mathcal{S}_t$ = spacelike 2-surface

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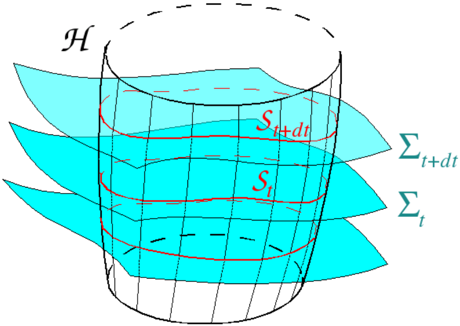

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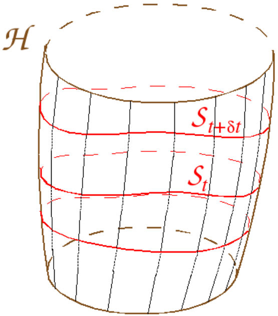
$$\mathcal{H} \text{ can be } \begin{cases} \text{spacelike} \\ \text{null} \\ \text{timelike} \end{cases}$$

$$\mathcal{H} = \bigcup_{t\in\mathbb{R}} \mathcal{S}_t$$

$\mathcal{S}_t$ = spacelike 2-surface

$\Longleftarrow$ 3+1 perspective

Foliation of a hypersurface by spacelike 2-surfaces



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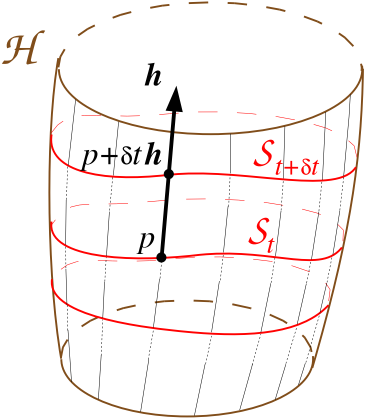
intrinsic viewpoint adopted here (i.e. not relying on extra-structure such as a 3+1 foliation)

$\boldsymbol{q}$: induced metric on $\mathcal{S}_t$ (positive definite)

$\boldsymbol{\mathcal{D}}$: connection associated with $\boldsymbol{q}$

Evolution vector on the horizon



Vector field $\boldsymbol{h}$ on $\mathcal{H}$ defined by

- (i) $\boldsymbol{h}$ is tangent to $\mathcal{H}$
- (ii) $\boldsymbol{h}$ is orthogonal to $\mathcal{S}_t$
- (iii) $\mathcal{L}_{\boldsymbol{h}}\, t = h^\mu \partial_\mu t = \langle \mathbf{d}t, \boldsymbol{h} \rangle = 1$

NB: (iii) $\Longrightarrow$ the 2-surfaces $\mathcal{S}_t$ are Lie-dragged by $\boldsymbol{h}$

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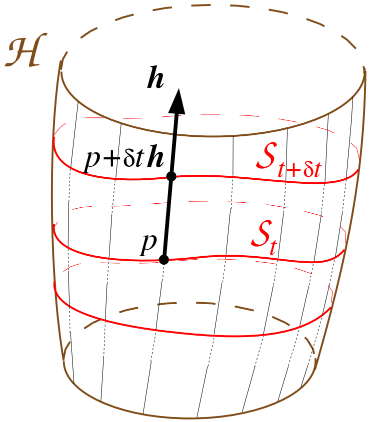

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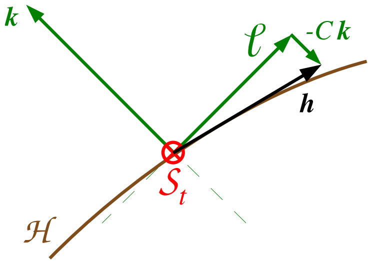
Define $C := \frac{1}{2}\boldsymbol{h}\cdot\boldsymbol{h}$

$\mathcal{H}$ is spacelike	$\iff$	$C > 0$	$\iff$	$\boldsymbol{h}$ is spacelike
$\mathcal{H}$ is null	$\iff$	$C = 0$	$\iff$	$\boldsymbol{h}$ is null
$\mathcal{H}$ is timelike	$\iff$	$C < 0$	$\iff$	$\boldsymbol{h}$ is timelike.

Normal null frame associated with the evolution vector



The foliation $(\mathcal{S}_t)_{t\in\mathbb{R}}$ entirely fixes the ambiguities in the choice of the null normal frame $(\boldsymbol{\ell}, \boldsymbol{k})$, via the evolution vector $\boldsymbol{h}$: there exists a unique normal null frame $(\boldsymbol{\ell}, \boldsymbol{k})$ such that

$$\boldsymbol{h} = \boldsymbol{\ell} - C\boldsymbol{k} \quad \text{and} \quad \boldsymbol{\ell}\cdot\boldsymbol{k} = -1$$

Normal fundamental form: $\boxed{\boldsymbol{\Omega}^{(\boldsymbol{\ell})} := -\boldsymbol{k}\cdot\boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\,\boldsymbol{\ell}}$ or $\Omega^{(\boldsymbol{\ell})}_{\alpha} := -k_\mu \nabla_\nu \ell^\mu\, q^\nu{}_\alpha$

Evolution of $\boldsymbol{h}$ along itself: $\boxed{\boldsymbol{\nabla}_{\boldsymbol{h}}\boldsymbol{h} = \kappa\,\boldsymbol{\ell} + (C\kappa - \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, C)\boldsymbol{k} - \boldsymbol{\mathcal{D}}C}$

NB: null limit : $C = 0$, $\boldsymbol{h} = \boldsymbol{\ell} \Longrightarrow \boldsymbol{\nabla}_{\boldsymbol{\ell}}\boldsymbol{\ell} = \kappa\,\boldsymbol{\ell} \Longrightarrow \kappa =$ surface gravity

Outline

Concept of black hole viscosity

- Hartle and Hawking (1972, 1973): introduced the concept of **black hole viscosity** when studying the response of the *event horizon* to external perturbations
- Damour (1979): 2-dimensional **Navier-Stokes** like equation for the event horizon $\Longrightarrow$ *shear viscosity* and *bulk viscosity*
- Thorne and Price (1986): **membrane paradigm** for black holes

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Shall we restrict the analysis to the event horizon ?

Can we extend the concept of viscosity to the local characterizations of black hole recently introduced, i.e. future outer trapping horizons and dynamical horizons ?

NB: *event horizon* = null hypersurface
future outer trapping horizon = null or spacelike hypersurface
dynamical horizon = spacelike hypersurface

Navier-Stokes equation in Newtonian fluid dynamics

$$\rho\left(\frac{\partial v^i}{\partial t} + v^j \nabla_j v^i\right) = -\nabla^i P + \mu \Delta v^i + \left(\zeta + \frac{\mu}{3}\right) \nabla^i (\nabla_j v^j) + f^i$$

or, in terms of fluid momentum density $\pi_i := \rho v_i$,

$$\boxed{\frac{\partial \pi_i}{\partial t} + v^j \nabla_j \pi_i + \theta \pi_i = -\nabla_i P + 2\mu \nabla^j \sigma_{ij} + \zeta \nabla_i \theta + f_i}$$

where θ is the fluid expansion:

$$\theta := \nabla_j v^j$$

and σ_{ij} the velocity shear tensor:

$$\sigma_{ij} := \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right) - \frac{1}{3} \theta\, \delta_{ij}$$

P is the pressure, μ the shear viscosity, ζ the bulk viscosity and f_i the density of external forces

Original Damour-Navier-Stokes equation

Hyp: $\mathcal{H}$ = null hypersurface (particular case: black hole **event horizon**)
Then $\boldsymbol{h} = \boldsymbol{\ell}$ ($C = 0$)

Damour (1979) has derived from Einstein equation the relation

$$\boxed{{}^{\mathcal{S}}\!\mathcal{L}_{\boldsymbol{\ell}}\,\boldsymbol{\Omega}^{(\boldsymbol{\ell})} + \theta^{(\boldsymbol{\ell})}\boldsymbol{\Omega}^{(\boldsymbol{\ell})} = \boldsymbol{\mathcal{D}}\kappa - \boldsymbol{\mathcal{D}}\cdot\boldsymbol{\sigma}^{(\boldsymbol{\ell})} + \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{\ell})} + 8\pi\vec{\boldsymbol{q}}^{*}\boldsymbol{T}\cdot\boldsymbol{\ell}}$$

or equivalently

$$\boxed{{}^{\mathcal{S}}\!\mathcal{L}_{\boldsymbol{\ell}}\,\boldsymbol{\pi} + \theta^{(\boldsymbol{\ell})}\boldsymbol{\pi} = -\boldsymbol{\mathcal{D}}P + 2\mu\boldsymbol{\mathcal{D}}\cdot\boldsymbol{\sigma}^{(\boldsymbol{\ell})} + \zeta\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{\ell})} + \boldsymbol{f}} \quad (*)$$

with $\boldsymbol{\pi} := -\frac{1}{8\pi}\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$ momentum surface density

$P := \frac{\kappa}{8\pi}$ pressure

$\mu := \frac{1}{16\pi}$ shear viscosity

$\zeta := -\frac{1}{16\pi}$ bulk viscosity

$\boldsymbol{f} := -\vec{\boldsymbol{q}}^{*}\boldsymbol{T}\cdot\boldsymbol{\ell}$ external force surface density ($\boldsymbol{T}$ = stress-energy tensor)

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$(*)$ is identical to a 2-dimensional Navier-Stokes equation

Original Damour-Navier-Stokes equation (con't)

Introducing a coordinate system (t, x^1, x^2, x^3) such that

- t is compatible with $\boldsymbol{\ell}$: $\mathcal{L}_{\boldsymbol{\ell}}\, t = 1$
- $\mathcal{H}$ is defined by $x^1 = \mathrm{const}$, so that $x^a = (x^2, x^3)$ are coordinates spanning $\mathcal{S}_t$

then

$$\boldsymbol{\ell} = \frac{\partial}{\partial t} + \boldsymbol{V}$$

with $\boldsymbol{V}$ tangent to $\mathcal{S}_t$: velocity of $\mathcal{H}$'s null generators with respect to the coordinates x^a [Damour, PRD **18**, 3598 (1978)].

Then

$$\theta^{(\boldsymbol{\ell})} = \mathcal{D}_a V^a + \frac{\partial}{\partial t} \ln \sqrt{q} \qquad q := \det q_{ab}$$

$$\sigma^{(\boldsymbol{\ell})}_{ab} = \frac{1}{2}\left(\mathcal{D}_a V_b + \mathcal{D}_b V_a\right) - \frac{1}{2}\theta^{(\boldsymbol{\ell})}\, q_{ab} + \frac{1}{2}\frac{\partial q_{ab}}{\partial t}$$

compare

Negative bulk viscosity of event horizons

From the Damour-Navier-Stokes equation, $\boxed{\zeta = -\frac{1}{16\pi} < 0}$

This negative value would yield to a *dilation or contraction instability* in an ordinary fluid

It is in agreement with the tendency of a null hypersurface to continually contract or expand

The event horizon is stabilized by the teleological condition imposing its expansion to vanish in the far future (equilibrium state reached)

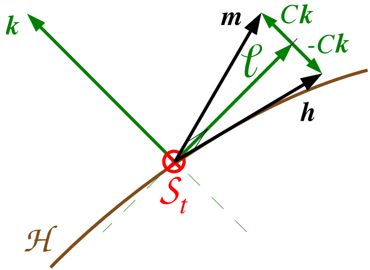
Generalization to the non-null case

Starting remark: in the null case (event horizon), ℓ plays two different roles:

- evolution vector along $\mathcal{H}$ (e.g. term ${}^{\mathcal{S}}\!\mathcal{L}_{\ell}$)
- normal to $\mathcal{H}$ (e.g. term $\vec{q}^{*}\boldsymbol{T}\cdot\boldsymbol{\ell}$)

When $\mathcal{H}$ is no longer null, these two roles have to be taken by two different vectors:

- evolution vector: obviously $\boldsymbol{h}$
- vector normal to $\mathcal{H}$: a natural choice is $\boxed{\boldsymbol{m} := \boldsymbol{\ell} + C\boldsymbol{k}}$



Generalized Damour-Navier-Stokes equation

From the contracted Ricci identity applied to the vector $\boldsymbol{m}$ and projected onto $\mathcal{S}_t$: $(\nabla_\mu \nabla_\nu m^\mu - \nabla_\nu \nabla_\mu m^\mu)\, q^\nu{}_\alpha = R_{\mu\nu} m^\mu q^\nu{}_\alpha$ and using Einstein equation to express $R_{\mu\nu}$, one gets an evolution equation for $\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$ along $\mathcal{H}$:

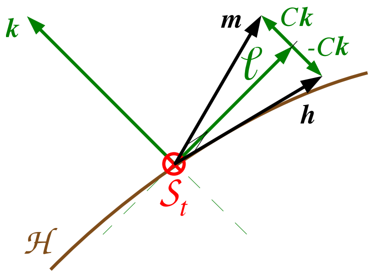
$$\boxed{{}^{\mathcal{S}}\!\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{\Omega}^{(\boldsymbol{\ell})} + \theta^{(\boldsymbol{h})}\, \boldsymbol{\Omega}^{(\boldsymbol{\ell})} = \boldsymbol{\mathcal{D}}\kappa - \boldsymbol{\mathcal{D}} \cdot \boldsymbol{\sigma}^{(\boldsymbol{m})} + \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{m})} - \theta^{(\boldsymbol{k})}\boldsymbol{\mathcal{D}}C + 8\pi \vec{\boldsymbol{q}}^{*}\boldsymbol{T} \cdot \boldsymbol{m}}$$

- $\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$: normal fundamental form of $\mathcal{S}_t$ associated with null normal $\boldsymbol{\ell}$
- $\theta^{(\boldsymbol{h})}$, $\theta^{(\boldsymbol{m})}$ and $\theta^{(\boldsymbol{k})}$: expansion scalars of $\mathcal{S}_t$ along the vectors $\boldsymbol{h}$, $\boldsymbol{m}$ and $\boldsymbol{k}$ respectively
- $\boldsymbol{\mathcal{D}}$: covariant derivative within $(\mathcal{S}_t, \boldsymbol{q})$
- κ : component of $\boldsymbol{\nabla}_{\boldsymbol{h}}\boldsymbol{h}$ along $\boldsymbol{\ell}$
- $\boldsymbol{\sigma}^{(\boldsymbol{m})}$: shear tensor of $\mathcal{S}_t$ along the vector $\boldsymbol{m}$
- C : half the scalar square of $\boldsymbol{h}$

Null limit (event horizon)

If $\mathcal{H}$ is a null hypersurface,

$$\boldsymbol{h} = \boldsymbol{m} = \boldsymbol{\ell} \qquad \text{and} \qquad C = 0$$



and we recover the original Damour-Navier-Stokes equation:

$$\boxed{{}^{\mathcal{S}}\!\boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\,\boldsymbol{\Omega}^{(\boldsymbol{\ell})} + \theta^{(\boldsymbol{\ell})}\boldsymbol{\Omega}^{(\boldsymbol{\ell})} = \boldsymbol{\mathcal{D}}\kappa - \boldsymbol{\mathcal{D}}\cdot\boldsymbol{\sigma}^{(\boldsymbol{\ell})} + \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{\ell})} + 8\pi\vec{\boldsymbol{q}}^{*}\boldsymbol{T}\cdot\boldsymbol{\ell}}$$

Case of future trapping horizons

Definition [Hayward, PRD **49**, 6467 (1994)] :
$\mathcal{H}$ is a **future trapping horizon** iff $\theta^{(\boldsymbol{\ell})} = 0$ and $\theta^{(\boldsymbol{k})} < 0$.
The generalized Damour-Navier-Stokes equation reduces then to

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[EG, PRD **72**, 104007 (2005)]

NB: Notice the change of sign in the $-\frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{h})}$ term with respect to the original Damour-Navier-Stokes equation

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The explanation: it is $\theta^{(\boldsymbol{m})}$ which appears in the general equation and

$$\theta^{(\boldsymbol{m})} + \theta^{(\boldsymbol{h})} = 2\theta^{(\boldsymbol{\ell})} \Longrightarrow \begin{cases} \text{event horizon } (\boldsymbol{m} = \boldsymbol{h}) & : \quad \theta^{(\boldsymbol{m})} = \theta^{(\boldsymbol{\ell})} \\ \text{trapping horizon } (\theta^{(\boldsymbol{\ell})} = 0) & : \quad \theta^{(\boldsymbol{m})} = -\theta^{(\boldsymbol{h})} \end{cases}$$

Viscous fluid form

$$\boxed{{}^{\mathcal{S}}\!\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\,\boldsymbol{\pi} + \theta^{(\boldsymbol{h})}\boldsymbol{\pi} = -\boldsymbol{\mathcal{D}}P + \frac{1}{8\pi}\boldsymbol{\mathcal{D}}\cdot\boldsymbol{\sigma}^{(\boldsymbol{m})} + \zeta\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{h})} + \boldsymbol{f}}$$

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Similar to the Damour-Navier-Stokes equation for an event horizon, except

- the **Newtonian-fluid** relation between *stress* and *strain* does not hold: $\boldsymbol{\sigma}^{(\boldsymbol{m})}/8\pi \neq 2\mu\boldsymbol{\sigma}^{(\boldsymbol{h})}$, rather $\boldsymbol{\sigma}^{(\boldsymbol{m})}/8\pi = [\boldsymbol{\sigma}^{(\boldsymbol{h})} + 2C\boldsymbol{\sigma}^{(\boldsymbol{k})}]/8\pi$

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- positive bulk viscosity

This positive value of bulk viscosity shows that FOTHs and DHs behave as "ordinary" physical objects, in perfect agreement with their local nature

Outline

Angular momentum of trapping horizons

Definition [Booth & Fairhurst, CQG **22**, 4545 (2005)]: Let φ be a vector field on $\mathcal{H}$ which

- is tangent to $\mathcal{S}_t$
- has closed orbits
- has vanishing divergence with respect to the induced metric: $\boldsymbol{\mathcal{D}} \cdot \boldsymbol{\varphi} = 0$ (weaker than being a Killing vector of $(\mathcal{S}_t, \boldsymbol{q})$!)

For dynamical horizons, $\theta^{(\boldsymbol{h})} \neq 0$ and there is a unique choice of φ as the generator (conveniently normalized) of the curves of constant $\theta^{(\boldsymbol{h})}$
[Hayward, PRD **74**, 104013 (2006)]

The *generalized angular momentum associated with* φ is then defined by

$$\boxed{J(\boldsymbol{\varphi}) := -\frac{1}{8\pi} \oint_{\mathcal{S}_t} \langle \boldsymbol{\Omega}^{(\boldsymbol{\ell})}, \boldsymbol{\varphi} \rangle \, {}^{\mathcal{S}}\boldsymbol{\epsilon},}$$

Remark 1: does not depend upon the choice of null vector $\boldsymbol{\ell}$, thanks to the divergence-free property of $\boldsymbol{\varphi}$

Remark 2:

- coincides with Ashtekar & Krishnan's definition for a dynamical horizon
- coincides with Brown-York angular momentum if $\mathcal{H}$ is timelike and $\boldsymbol{\varphi}$ a Killing vector

Angular momentum flux law

Under the supplementary hypothesis that φ is transported along the evolution vector $\boldsymbol{h}$: $\mathcal{L}_{\boldsymbol{h}}\,\varphi = 0$, the generalized Damour-Navier-Stokes equation leads to

$$\boxed{\frac{d}{dt}J(\varphi) = -\oint_{\mathcal{S}_t} \boldsymbol{T}(\boldsymbol{m},\varphi)\,{}^{\mathcal{S}}\boldsymbol{\epsilon} - \frac{1}{16\pi}\oint_{\mathcal{S}_t}\left[\boldsymbol{\sigma}^{(\boldsymbol{m})} : \mathcal{L}_{\varphi}\,\boldsymbol{q} - 2\theta^{(\boldsymbol{k})}\varphi\cdot\boldsymbol{\mathcal{D}}C\right]{}^{\mathcal{S}}\boldsymbol{\epsilon}}$$

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Two interesting limiting cases:

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i.e. Eq. (6.134) of the *Membrane Paradigm* book (Thorne, Price & MacDonald 1986)

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- $\mathcal{H} =$ future trapping horizon :

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Area evolution law for an event horizon

$A(t)$: area of the 2-surface $\mathcal{S}_t$; ${}^{\mathcal{S}}\!\boldsymbol{\epsilon}$: volume element of $\mathcal{S}_t$; $\bar{\kappa}(t) := \frac{1}{A(t)} \int_{\mathcal{S}_t} \kappa \, {}^{\mathcal{S}}\!\boldsymbol{\epsilon}$

Integrating the null Raychaudhuri equation on $\mathcal{S}_t$, one gets

$$\boxed{\frac{d^2 A}{dt^2} - \bar{\kappa}\frac{dA}{dt} = -\int_{\mathcal{S}_t} \left[8\pi \boldsymbol{T}(\boldsymbol{\ell},\boldsymbol{\ell}) + \boldsymbol{\sigma}^{(\boldsymbol{\ell})} : \boldsymbol{\sigma}^{(\boldsymbol{\ell})} - \frac{(\theta^{(\boldsymbol{\ell})})^2}{2} + (\bar{\kappa} - \kappa)\theta^{(\boldsymbol{\ell})} \right] {}^{\mathcal{S}}\!\boldsymbol{\epsilon}} \quad (1)$$

[Damour, 1979]

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[Damour, 1979]

Simplified analysis : assume $\bar{\kappa} = \mathrm{const} > 0$:

- Cauchy problem $\Longrightarrow$ diverging solution of the homogeneous equation: $\dfrac{dA}{dt} = \alpha \exp(\bar{\kappa} t)$!

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$A(t)$: area of the 2-surface $\mathcal{S}_t$; ${}^{\mathcal{S}}\boldsymbol{\epsilon}$: volume element of $\mathcal{S}_t$; $\bar{\kappa}(t) := \frac{1}{A(t)} \int_{\mathcal{S}_t} \kappa \, {}^{\mathcal{S}}\boldsymbol{\epsilon}$

Integrating the null Raychaudhuri equation on $\mathcal{S}_t$, one gets

$$\boxed{\frac{d^2 A}{dt^2} - \bar{\kappa}\frac{dA}{dt} = -\int_{\mathcal{S}_t} \left[8\pi \boldsymbol{T}(\boldsymbol{\ell},\boldsymbol{\ell}) + \boldsymbol{\sigma}^{(\boldsymbol{\ell})} : \boldsymbol{\sigma}^{(\boldsymbol{\ell})} - \frac{(\theta^{(\boldsymbol{\ell})})^2}{2} + (\bar{\kappa} - \kappa)\theta^{(\boldsymbol{\ell})}\right] {}^{\mathcal{S}}\boldsymbol{\epsilon}} \quad (1)$$

[Damour, 1979]

Simplified analysis : assume $\bar{\kappa} = \mathrm{const} > 0$:

- Cauchy problem $\Longrightarrow$ diverging solution of the homogeneous equation: $\frac{dA}{dt} = \alpha \exp(\bar{\kappa} t)$!
- correct treatment: impose $\frac{dA}{dt} = 0$ at $t = +\infty$ (teleological !)

$$\frac{dA}{dt} = \int_t^{+\infty} D(u) e^{\bar{\kappa}(t-u)} \, du \qquad D(t) \text{ : r.h.s. of Eq. (1)}$$

Non causal evolution

Area evolution law for a dynamical horizon

Dynamical horizon : $C > 0$; $\kappa' := \kappa - \mathcal{L}_{\boldsymbol{h}} \ln C$; $\bar{\kappa}'(t) := \frac{1}{A(t)} \int_{\mathcal{S}_t} \kappa' \, {}^{\mathcal{S}}\boldsymbol{\epsilon}$

From the $(\boldsymbol{m}, \boldsymbol{h})$ component of Einstein equation, one gets

$$\boxed{\frac{d^2 A}{dt^2} + \bar{\kappa}' \frac{dA}{dt} = \int_{\mathcal{S}_t} \left[8\pi \boldsymbol{T}(\boldsymbol{m}, \boldsymbol{h}) + \boldsymbol{\sigma}^{(\boldsymbol{h})} : \boldsymbol{\sigma}^{(\boldsymbol{m})} + \frac{(\theta^{(\boldsymbol{h})})^2}{2} + (\bar{\kappa}' - \kappa')\theta^{(\boldsymbol{h})} \right] {}^{\mathcal{S}}\boldsymbol{\epsilon}} \quad (2)$$

[EG & Jaramillo, PRD **74**, 087502 (2006)]

Area evolution law for a dynamical horizon

Dynamical horizon : $C > 0$; $\kappa' := \kappa - \mathcal{L}_{\boldsymbol{h}} \ln C$; $\bar{\kappa}'(t) := \frac{1}{A(t)} \int_{\mathcal{S}_t} \kappa' \, {}^{\mathcal{S}}\boldsymbol{\epsilon}$

From the $(\boldsymbol{m}, \boldsymbol{h})$ component of Einstein equation, one gets

$$\boxed{\frac{d^2 A}{dt^2} + \bar{\kappa}' \frac{dA}{dt} = \int_{\mathcal{S}_t} \left[8\pi \boldsymbol{T}(\boldsymbol{m}, \boldsymbol{h}) + \boldsymbol{\sigma}^{(\boldsymbol{h})} : \boldsymbol{\sigma}^{(\boldsymbol{m})} + \frac{(\theta^{(\boldsymbol{h})})^2}{2} + (\bar{\kappa}' - \kappa')\theta^{(\boldsymbol{h})} \right] {}^{\mathcal{S}}\boldsymbol{\epsilon}} \quad (2)$$

[EG & Jaramillo, PRD **74**, 087502 (2006)]

Simplified analysis : assume $\bar{\kappa}' = \mathrm{const} > 0$
(OK for small departure from equilibrium [Booth & Fairhurst, PRL **92**, 011102 (2004)]):
Standard Cauchy problem :

$$\frac{dA}{dt} = \left. \frac{dA}{dt} \right|_{t=0} + \int_0^t D(u) e^{\bar{\kappa}'(u-t)} \, du \qquad D(t) : \text{ r.h.s. of Eq. (2)}$$

Causal evolution, in agreement with local nature of dynamical horizons

Outline

Applications to numerical relativity

- **Initial data**: isolated horizons (helical symmetry)
  [EG, Grandclément & Bonazzola, PRD **65**, 044020 (2002)]
  [Grandclément, EG & Bonazzola, PRD **65**, 044021 (2002)]
  [Cook & Pfeiffer, PRD **70**, 104016 (2004)]

- **A posteriori analysis**: estimating mass, linear and angular momentum of formed black holes
  [Schnetter, Krishnan & Beyer, PRD **74**, 024028 (2006)]
  [Cook & Whiting, PRD **76**, 041501 (2007)]
  [Krishnan, Lousto & Zlochower, PRD **76**, 081501(R) (2007)]

- **Numerical construction of spacetime**: inner boundary conditions for a constrained scheme with "black hole excision"
  [Jaramillo, EG, Cordero-Carrión, & J.M. Ibáñez, PRD **77**, 047501 (2008)]
  [Vasset, Novak & Jaramillo, PRD **79**, 124010 (2009)]

Outline

Review articles

- A. Ashtekar and B. Krishnan : *Isolated and dynamical horizons and their applications*, Living Rev. Relativity **7**, 10 (2004) ; http://www.livingreviews.org/lrr-2004-10
- I. Booth : *Black hole boundaries*, Canadian J. Phys. **83**, 1073 (2005) ; http://arxiv.org/abs/gr-qc/0508107
- E. Gourgoulhon and J.L. Jaramillo : *A 3+1 perspective on null hypersurfaces and isolated horizons*, Phys. Rep. **423**, 159 (2006) ; http://arxiv.org/abs/gr-qc/0503113
- E. Gourgoulhon and J. L. Jaramillo : *New theoretical approaches to black holes*, New Astron. Rev. **51**, 791 (2008) ; http://arxiv.org/abs/0803.2944
- S.A. Hayward : *Dynamics of black holes*, Adv. Sci. Lett. **2**, 205 (2009); http://arxiv.org/abs/0810.0923
- B. Krishnan : *Fundamental properties and applications of quasi-local black hole horizons*, Class. Quantum Grav. **25**, 114005 (2008); http://arxiv.org/abs/0712.1575