Scale Relativity, Fractal Space-Time, and Quantum Mechanics†

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Abstract – This paper describes the present state of an attempt at understanding the quantum behaviour of microphysics in terms of a nondifferentiable space-time continuum having fractal (i.e., scale-dependent) properties. The fundamental principle upon which we rely is that of scale relativity, which generalizes to scale transformations Einstein's principle of relativity. After having related the fractal and renormalization group approaches, we develop a new version of stochastic quantum mechanics, in which the correspondence principle and the Schrödinger equation are demonstrated by replacing the classical time derivative by a ”quantum-covariant” derivative. Then we recall that the principle of scale relativity leads one to generalize the standard “Galilean” laws of scale transformation into a Lorentzian form, in which the Planck length-scale becomes invariant under dilations, and so plays for scale laws the same role as played by the velocity of light for motion laws. We conclude by an application of our new framework to the problem of the mass spectrum of elementary particles.

1. INTRODUCTION.

The idea that the quantum space-time of microphysics is fractal, rather than flat and Minkowskian as assumed up to now, was suggested ten years ago [1,2]. This proposal was itself based on earlier results [3-6], obtained at first by Feynman (see in particular [7] and references therein), concerning the geometrical structure of quantum paths. These studies have shown that the typical trajectories of quantum mechanical particles are continuous but nondifferentiable, and can be characterized by a fractal dimension which jumps from $D = 1$ at large length-scales to $D = 2$ at small length-scales, the transition occuring about the de Broglie scale (see refs [8,9]).

Now such a fractal dimension $D = 2$ plays a particular role in physics. It is well-known that this is the fractal dimension of Brownian motion [10], i.e. from the mathematical view-point, of a Markov-Wiener process. This remark leads us to recall a related attempt at understanding the quantum behaviour, namely, Nelson's stochastic quantum mechanics [11,12]. In this approach, it is assumed that any particle is subjected to an underlying Brownian motion of unknown origin, which is described by two (forward and backward) Wiener processes: when combined together they yield the complex nature of the wave function and they transform Newton's equation of dynamics into the Schrödinger equation.

This is precisely one of the aims of the present paper to relate the fractal and stochastic approaches: the hypothesis that the space-time is nondifferentiable and fractal implies that there are an infinity of geodesics between any couple of points [8] and provides us with a fundamental and universal origin for the double Wiener process of Nelson [9,13].

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The passage to relativistic quantum mechanics has been independently performed in the fractal approach by Ord [2] and Nottale [8]. The fractal dimension of the temporal coordinate is also found to become 2 below the de Broglie time $\tau = \hbar/<E>$ of a particle, i.e. below $\tau_o = \hbar/mc^2 = \lambda_c/c$ in rest frame, where λ_c is the Compton length of the particle. (Note that obtaining the same fractal dimension 2 for all four coordinates is expected from assuming the full trajectory *in space-time* to be characterized by fractal dimension 2). This means that for time scales $\delta t < \tau$, the trajectory is allowed to run backward in time. Such a possibility was already central in Feynman's approach to Quantum Electrodynamics (QED): Following the Wheeler-Feynman-Stückelberg interpretation of antiparticles as particles running backward in time, the open loops implied by the existence of the backward parts of the trajectory are interpreted as forming virtual particle-antiparticle pairs. This behaviour allows one to construct fractal one-particle solutions to the Dirac equation [9], as recognized long ago by Feynman (see ref. [7] and references and quotations therein).

There is also now active work on attempts at developing a relativistic version of stochastic quantum mechanics (see e.g. [14,15] and refs. therein). In this approach one introduces a four-dimensional Wiener or Bernstein process in terms of a fifth ('proper time') variable. This also implies that some parts of the trajectories are running backward in time, this behaviour being once again interpreted in terms of particle-antiparticle pairs [14].

However, as interesting as these various approaches may be, one may criticize them since they do not rely on a fundamental principle. We have suggested [8,9,16] that such a founding principle is provided to us by Einstein's postulate of relativity itself, once generalized in order to apply to scale transformations also. Let us specify the meaning of this proposal (see [9] and [16] for more details):

There are two convergent ways suggesting that the principle of relativity still needs to be generalized. The first proceeds from the above remark that the quantum paths are nondifferentiable, while the principle of relativity, in its "general" form, requires the equations of physics to be covariant under continuous and *at least two times differentiable* transformations of curvilinear coordinate systems [17]: such a general covariance leads, with the principle of equivalence, to Einstein's field equations. But one may wonder about the general form of equations which would be invariant under continuous but *nondifferentiable* transformations.

The second way to a generalized principle of relativity proceeds from an analysis of the role played by resolution in physics [9]. While in the classical domain, the resolution with which a measurement is performed does not change the physics (measuring with a better resolution only improves the precision of the measurements results), this is no longer the case in quantum mechanics. The Heisenberg relations imply a universal dependence of physical results on the resolution of the measurement apparatus. Basing ourselves on this universality and on the relative character of all scales in nature, we have proposed to incorporate resolutions into the *definition* of coordinate systems, by defining them as their 'state of scale'. In this form Einstein's far-reaching formulation of the principle of relativity, according to which "the laws of physics must apply to any system of coordinates, whatever its state" [17], can incorporate not only the effect of *motion transformations* (through the quantities which caracterize the state of motion of the reference system, such as velocity and acceleration), but also of *scale transformations*. The implementation of such a generalized principle consist in requiring both *motion -covariance* (more generally, covariance under displacements and rotations of four-dimensional coordinates systems) *and scale-covariance* [9].

Now what is the connection between these two possible extensions of Einstein's relativity ? As demonstrated in [9] and recalled in Sec.3 hereafter, one of the most straighforward manifestations of nondifferentiability is the scale dependence (more precisely: *divergence*) of *continuous* nondifferentiable physical quantities. The two approaches are then clearly convergent.

In the present paper, our goal will be mainly to develop the formalism, in particular to build connections between various mathematical tools which have been put forward in order to deal with scale transformations, namely fractals, scale-covariance, the stochastic approach and the renormalization group; then to apply it to some fundamental problems still unsolved in the standard model, in particular that of the theoretical prediction of the mass spectrum of elementary particles. We send to Refs. [8], [9] and [16] the reader interested in a more detailed description of the motivations and principles of the present approach.

The paper is organized as follows. We first briefly recall the methods and first results obtained in the fractal approach to microphysics (Sec.2). Then we address the question of the origin of the universality of fractals in nature, attributing their emergence to that of genuine continuous but *nondifferentiable* processes (Sec.3). We then develop the fractal space-time interpretation of stochastic quantum mechanics, mainly in its nonrelativistic version, then briefly address the relativistic case (Sec.4). In the subsequent section (5), the equations obtained in our first development of a theory of scale relativity are recalled. The results recently obtained by this theory concerning the theoretical prediction of several free parameters of the standard model (GUT and top quark scale, fundamental coupling constants) are briefly reviewed (Sec.5). In Sec.6, the various mathematical tools which have been put forward here (fractal dimensions, stochastic quantum mechanics, renormalization group equations) are combined together to suggest a solution to the problem of the mass spectrum of elementary particles, based on the requirements of microscopic reversibility and of scale relativity. We finally conclude by some prospects for the future development of this new field of research (Sec.7).

2. THE FRACTAL APPROACH TO QUANTUM MECHANICS.

The discovery that the typical quantum mechanical paths are continuous but nondifferentiable and may be characterized by a fractal dimension 2 may be attributed to Feynman [3,7]. Though Feynman evidently did not used the word ‘fractal’, which was coined in 1975 by Mandelbrot [10], his description of quantum mechanical paths fully corresponds to this concept. Indeed, his path integral formulation of quantum mechanics [3] allowed him to consider explicitly the geometrical structure of the various virtual paths of a quantum particle, and to demonstrate that they share common properties, in particular that, when seen at a time scale δt, the mean quadratic velocity of the particle is $<v^2> \propto \delta t^{-1}$. Assuming such a trajectory to be a fractal curve of fractal dimension D, we expect the space and time resolution to be related by the relation

$$\delta t \propto \delta x^D, \tag{2.1}$$

so that $<v^2> \approx (\delta x/\delta t)^2 \propto \delta t^{2[(1/D)-1]}$. The comparison with Feynman’s result leads $D = 2$ [8, 9]. Among the early contributions to this field, one may also quote a letter of Einstein to Pauli [18], in which he suggested that a true understanding of quantum physics could imply to give up differentiability, but certainly not the principle of general relativity.

Abbott and Wise [4] were the first to reconsider the problem of the geometrical structure of quantum paths in terms of the concept of *fractals*, introduced by Mandelbrot in 1975 [10]. They demonstrated that the length of a quantum mechanical trajectory, when observed with a space resolution δx, varies as $\mathcal{L} \propto \delta x^{-1}$ when $\delta x << \lambda$ and becomes independent of scale when $\delta x >> \lambda$, where $\lambda = \hbar/p_o$. Here $p_o = <p>$ is the average momentum of the particle, so that λ is its de Broglie length. Two informations are contained in this result. The first derives from the known expression for the scale divergence of a fractal curve [10]:

$$\mathcal{L} = \mathcal{L}_o \left(\frac{\lambda}{\delta x}\right)^{D-1} . \tag{2.2}$$

This shows that the Abbott-Wise and Feynman results are consistent and both lead to a fractal dimension $D = 2$. The additional information is that the fractal structure does not persist whatever the scale, and that there is a fast transition from fractal to nonfractal behaviour ($D = 2$ to $D = 1$) about the de Broglie scale, which Abbott and Wise identify with a quantum to classical transition (see hereafter in Sec.4 and Ref. [9] for additional details on this transition). Such a transition is indeed expected for a fractal curve whose fractal structures are developing only toward lower scales, while showing an upper cutoff at some scale λ. In such a case, neglecting the possible fluctuations during the transition ($\delta x \approx \lambda$), the scale dependence reads

$$\mathcal{L} = \mathcal{L}_o \left[1 + \left(\frac{\lambda}{\delta x}\right)^{2(D-1)}\right]^{1/2} . \tag{2.3}$$

The physical meaning and origin of such a law will be enlighted in what follows. One can, in particular, consider $\mathcal{L}$ as a curvilinear coordinate along the fractal curve. Such a curvilinear coordinate is itself scale divergent as $\mathcal{L} \propto \delta t^{(1/D)-1}$ in the fractal regime (see Eq. 2.1). But we can then introduce a renormalized coordinate $\ell = \mathcal{L}\, (\delta t/\tau_o)^{1-(1/D)}$ which will now remain finite. Each of the three coordinates can be described as a “fractal function” of ℓ and of the resolution δt:

$$x^j = x^j(\ell,\delta t) \;\Rightarrow\; \delta x^j = V^j\, \delta t + \zeta^j(\ell,\delta t)\, (\delta t/\tau_o)^{1/D} . \tag{2.4}$$

From this equation $\mathcal{L}$ can be recomputed, and this yields essentially the result of (2.3). The curvilinear coordinate ℓ is a monotonous function of time, so that the functions of $(\ell,\delta t)$ can be replaced by functions of $(t,\delta t)$.

All this reasoning still holds in space-time: the four coordinates become in this case four fractal functions depending on an invariant but scale-dependent proper time S, which can also be renormalized in order to obtain a finite invariant $\mathit{s} = S\, (\delta s/\tau_o)^{1-(1/D)}$, where s is the *classical* invariant. Note the difference between the classical invariant s and the new invariant s: the proper time s is defined along the fractal trajectory which is allowed to run backward in classical time at very small resolutions, while the standard invariant s is computed only on classical differentiable trajectories for which all time intervals remain positive.

As remarked in Refs. [2] and [8], there is a compensation between the special relativistic Lorentz contraction and the quantum scale-divergence issued from Heisenberg’s relation. Let us briefly present a new account of this effect. The proper time element δS varies as

$$\delta S \propto \delta s^{1/D} = \{c\ \delta t\ (1-v^2/c^2)^{1/2}\}^{1/D} . \tag{2.5}$$

But $(1-v^2/c^2)^{1/2} = E_0/E \approx (\delta t/\tau_0)$ from Heisenberg's relation, so that we finally obtain

$$\delta S \propto \delta t^{2/D} \tag{2.6}$$

i.e., $\delta S \propto \delta t$ for $D = 2$, while the limit $v \to c$ would have classically yielded the ligth cone result $\delta s = 0$.

The various above formula are expressed in terms of finite differences δf, identified with resolutions when concerning space and time variables. We have suggested an equivalent formulation using Non Standard Analysis [1,8,9], which allows one to replace these quantities by differentials. Then, if one jumps to a stochastic representation, the fundamental equation (2.4) becomes, for $D = 2$, nothing but the basic relation describing a Wiener process: this result will be fully used in what follows.

As we shall indeed see at length in Sec. 4, this description leads to a reformulation of Nelson's stochastic mechanics, and allows one to reach a new understanding of the origin of the complex nature of the probability amplitude of quantum mechanics and of the correspondence principle, and finally to demonstrate the Schrödinger equation (and the Klein-Gordon equation in the relativistic case). Since most of the basic quantum mechanical behavior is a mere consequence of precisely these three axioms (complex wave function, correspondence principle, Schrödinger's equation), we shall content ourselves to establish these results in the present paper, without developing any longer the fractal interpretation of quantum mechanics. Let us only sum up the additional results which may be obtained in the fractal framework:

**Geometric interpretation of classical quantities* [2,8,9]: one can show that the basic physical quantities defining a particle, such as its mass, energy, momentum, or velocity can be defined as geometric structures of its fractal trajectory. This means that we do not need any longer to consider the "particle" as a point endowed with mass which would follow some trajectory (more precisely: one of its virtual trajectories), but instead that we can *identify* the particle with the fractal structure of its trajectory.

**Fractal interpretation of quantum spin* [8,9]: we have demonstrated that a fractal trajectory of fractal dimension 2 owns a proper angular momentum ($\sigma = mr^2 d\phi/dt$ finite although $r \to 0$), while such an internal angular momentum is undefined for $D < 2$ (vanishing) and $D > 2$ (infinite). Hence the quantum spin can also be defined as a purely geometrical property of the virtual trajectories of the particle.

**Wave-particle duality* [19,8,9]: the nondifferentiability of space-time implies the existence of an *infinity* of equiprobable geodesics between any two points. Then one may admit that a quantum particle did follow one of the geodesics of this infinite family and in the same time admit that any theoretical prediction of which particular geodesical line has been followed is impossible: in other words, the theory which is to be built on the hypothesis of nondifferentiability and fractality is *not* a hidden parameter theory. Any prediction must be made in a probabilistic way using the whole family of geodesics (which defines the wave function, see below), while any position measurement will reveal the corpuscle nature of the particle. We think that this approach is able to reconcile Einstein's requirement of realism (quantum mechanics would need to be completed by the concept of a structured, non Minkowskian space-time; the fundamental laws of nature holding for individual phenomena would not be *essentially* probabilistic: the statistical nature of the theory would be a mere consequence of nondifferentiability), and Bohr's undeterminism, which becomes a properties of the infinite family of geodesics.

We send the reader interestedin a development of these and other related points to Refs. [9,8,2] and to the several recent works on the fractal approach to quantum mechanics, in particular by Sornette [20], El Naschie [21], Höfer [22] (and references quoted by these authors) and the contributors to the present volume.

3. ORIGIN OF FRACTALS : SCALE DEPENDENCE AND RENORMALIZATION GROUP.

One of the main questions that is asked concerning the emergence of fractals in natural and physical sciences is the reason for their universality [10]. While particular causes may be found for their origin by a detailed description of the various systems where they appear (chaotic dynamics, biological systems, etc...) their universality nevertheless calls for a *universal* answer.

Our suggestion, which has been developed in [9], is as follows. Since the time of Newton and Leibniz, the foundators of the integro-differentiation calculus, one basic hypothesis which is put forward in our description of physical phenomena is that of differentiability. The strength of this hypothesis has been to allow physicists to write the equations of physics in terms of differential equations. However, there is no *a priori* principle which imposes the fundamental laws of physics to be differentiable.

We shall make the opposite assumption: *the elementary laws of physics are actually nondifferentiable*. Under this conjecture, the successes of present differentiable physics are understood as applying to domains where the approximation of differentiability (or integrability) was good enough, i.e. at scales such that the effects of nondifferentiability were smoothed out; but conversely, we expect its methods to fail when confronted to truly nondifferentiable or nonintegrable phenomena, namely at very small and very large length scales, and, to a smaller extent, for chaotic systems.

The new frontier is, in our opinion, to construct a *continuous* but *nondifferentiable* physics. (We stress the fact that giving up differentiability does *not* impose giving up continuity). Set in such terms, the project may seem to be extraordinarily difficult. Fortunately, there is a fundamental key which will be a great help in this quest, namely, the concept of scale transformations.

Consider indeed a continuous but nondifferentiable function $f(x)$ between two points A_o $\{x_o, f(x_o)\}$ and A_Ω $\{x_\Omega, f(x_\Omega)\}$. Since f is non-differentiable, there exists a point A_1 of coordinates $\{x_1, f(x_1)\}$ with $x_o < x_1 < x_\Omega$, such that A_1 is not on the segment A_oA_Ω. Then the total length $\mathcal{L}_1 = \mathcal{L}(A_oA_1) + \mathcal{L}(A_1A_\Omega) > \mathcal{L}_0 = \mathcal{L}(A_oA_\Omega)$. We can now iterate the argument and find two coordinates x_{01} and x_{11} with $x_o < x_{01} < x_1$ and $x_1 < x_{11} < x_\Omega$, such that $\mathcal{L}_2 = \mathcal{L}(A_oA_{01}) + \mathcal{L}(A_{01}A_1) + \mathcal{L}(A_1A_{11}) + \mathcal{L}(A_{11}A_\Omega) > \mathcal{L}_1 > \mathcal{L}_0$. By iteration we finally construct successive approximations $f_0, f_1, ...f_n$ of $f(x)$ whose lengths $\mathcal{L}_0, \mathcal{L}_1, ...\mathcal{L}_n$ increase monotonically when the "resolution" $r \approx (x_\Omega - x_o) \times 2^{-n}$ tends to zero. In other words, continuity and nondifferentiability implies a monotonous scale dependence of f. Actually one may demonstrate that if f is continuous and everywhere nondifferentiable, then $\mathcal{L}(\varepsilon) \rightarrow \infty$ when the resolution $\varepsilon \rightarrow 0$, i.e. that f is *scale-divergent* [9].

This result is the key for a description of nondifferentiable processes in terms of differential equations. Rather than considering only the strictly nondifferentiable mathematical object $f(x)$, we shall consider its various approximations obtained from smoothing it or averaging it at various resolutions:

$$f(x,\varepsilon) = \int_{-\infty}^{+\infty} \Phi(x,y,\varepsilon)\, f(y)\, dy \tag{3.1}$$

where $\Phi(x,y,\varepsilon)$ is a smoothing function centered on x, for example a step function of width $\approx 2\varepsilon$, or a Gaussian of standard error $\approx \varepsilon$. We think that such a point of view is particularly well adapted to applications in physics: any real measurement is always performed at finite resolution (see Refs. [8,9,16] for additional comments on this point). In this framework, $f(x)$ becomes the limit when $\varepsilon \to 0$ of the family of functions $f(x,\varepsilon)$. But while $f(x,0)$ is nondifferentiable, $f(x,\varepsilon)$, which we have called a “fractal function” [9], is now differentiable for all $\varepsilon \neq 0$.

The problem of the physical description of the process where the function f intervenes is now shifted. In standard differentiable physics, it amounts to find a differential equation implying the derivatives of f, namely $\partial f/\partial x$, $\partial^2 f/\partial x^2$, ... In nondifferentiable physics, $\partial f(x)/\partial x = \partial f(x,0)/\partial x$ does not exist. But the physics of the given process will be completely described if we succeed in knowing $f(x,\varepsilon)$, which *is* differentiable, and can be solution of differential equations involving $\partial f(x,\varepsilon)/\partial x$ but also $\partial f(x,\varepsilon)/\partial \varepsilon$.

What is the meaning of the new differential $\partial f(x,\varepsilon)/\partial \varepsilon$? This is nothing but the variation of the quantity f under a *scale transformation*, i.e., a dilatation. More precisely, consider some function $\phi(\boldsymbol{x})$ and let us apply an infinitesimal dilatation $\boldsymbol{x} \to \boldsymbol{x}' = \boldsymbol{x}\,(1 + d\rho)$ to the coordinates. We obtain

$$\phi(\boldsymbol{x}') \;=\; \phi(\boldsymbol{x} + \boldsymbol{x}\,d\rho) \;=\; \phi(\boldsymbol{x}) + \frac{\partial\phi(\boldsymbol{x})}{\partial x^\mu}\,x^\mu\,d\rho \;=\; (1 + \tilde{D}\,d\rho)\,\phi(\boldsymbol{x}) \tag{3.2}$$

where $\tilde{D}$ is by definition the dilatation operator. The comparison of the two last members of this equation thus yields

$$\tilde{D} \;=\; x^\mu\,\frac{\partial}{\partial x^\mu} \;=\; r\,\frac{\partial}{\partial r} \;=\; \frac{\partial}{\partial ln r} \quad . \tag{3.3}$$

This well known form of the dilatation operator shows that the “natural” variable for resolution is $ln\varepsilon$, and that the expected new differential equations will more precisely involve quantities like $\partial f(x,\varepsilon)/\partial ln\varepsilon$. Now equations describing the scale dependence of physical beings have already been introduced in physics: these are the renormalization group equations, particularly developed in the framework of Wilson’s “multiple-scale-of-length” approach [23]. In its simplest form, a renormalization group-like equation for some essential physical quantity ϕ can be interpreted as stating that the variation of ϕ under an infinitesimal scale transformation $dln\varepsilon$ depends only on ϕ itself. This reads:

$$\frac{\partial\phi(x,\varepsilon)}{\partial ln\varepsilon} \;=\; \beta(\phi) \quad . \tag{3.4}$$

Once again looking for the simplest possible form for such an equation, we expand $\beta(\phi)$ in powers of ϕ and obtain to first order the linear equation

$$\frac{\partial\phi(x,\varepsilon)}{\partial ln\varepsilon} \;=\; a + b\,\phi\,. \tag{3.5}$$

Its solution is

$$\phi(x,\varepsilon) \;=\; \phi_0(x)\,\{1 + \zeta(x)\,(\frac{\lambda}{\varepsilon})^{-b}\} \quad , \tag{3.6}$$

where $\lambda^{-b}\zeta(x)$ is an integration “constant” and $\phi_0 = -a/b$. These notations allow us to choose $\zeta(x)$ such that

$<\zeta^2(x)> = 1$. Provided a $\neq 0$, Eq. (3.6) clearly shows two domains. Assume first $b < 0$:

(i) $\varepsilon << \lambda$: in this case $\zeta(x) \left(\frac{\lambda}{\varepsilon}\right)^{-b} >> 1$, and ϕ is given by a scale-invariant fractal-like power law with fractal dimension $D = 1{-}b$, namely $\phi(x,\varepsilon) = \phi_0(x)\ (\lambda/\varepsilon)^{-b}$.

(ii) $\varepsilon >> \lambda$: then $\zeta(x) \left(\frac{\lambda}{\varepsilon}\right)^{-b} << 1$, and ϕ becomes independent of scale.

We stress the fact that (3.6) gives us not only a fractal (scale-invariant) behaviour at small scale, but also a transition from fractal to nonfractal behaviour at scales larger than some transition scale λ. In other words, a renormalization group-like equation in its simplest (linear) form is able to provides us not only with scale-invariance, but also with the spontaneous breaking of this fundamental symmetry of nature. Only the particular case $a = 0$ yields unbroken scale-invariance, $\phi = \phi_0\ (\lambda/r)^{\delta}$, where $\delta = -b$ is a "scale dimension" [24]. Note that the corresponding equation (3.4) may be read in this case $\tilde{D}\phi = b\phi$, i.e. the scale dimension is given by the eigenvalue of the dilatation operator.

The solutions corresponding to the case $b > 0$ are symmetrical of the case $b < 0$. The scale-dependence is at large scales and is broken to yield scale-independence below the transition λ. In the present paper, we shall consider only the microphysical situation, which corresponds to $b < 0$. Note however that the case $b > 0$ is also of profound physical significance, since it is encountered in the cosmological situation [9].

In conclusion of this section, we think that the above mechanism is the clue to understanding the universality of fractals in nature. Self-similar, scale-invariant fractals with constant fractal dimension are nothing but the simplest possible behaviour of nondifferentiable, scale-dependent phenomena. They correspond to the *linear case* of scale laws, the equivalent of what are inertial frames for motion laws (this analogy will be reinforced in the following sections). The advantage of such an interpretation is that it opens several roads for generalization, the most promising being to implement the principle of scale relativity thanks to a generalization of scale invariance, namely, scale *covariance* of the equations of physics [16,9].

4. QUANTUM MECHANICS AS MECHANICS IN NONDIFFERENTIABLE SPACE.

Let us assume that space is continuous and nondifferentiable. This can be expressed by describing the position vector of a particle by a finite, continuous fractal function $\boldsymbol{x}(t,\delta t)$. Adopting the Non Standard Analysis formulation, we replace δt by the differential dt: in other words, the time variable is dissected into infinitesimal intervals dt. Our above analysis leads us to write that, between t and $t{+}dt$, the position vector varies by

$$\boldsymbol{x}(t+dt,dt) - \boldsymbol{x}(t,dt) \ = \ \boldsymbol{b}_+(\boldsymbol{x},t)\ dt \ + \ \boldsymbol{\zeta}_+(t,dt)\ (dt/\tau_0)^{\beta} \ , \tag{4.1}$$

where $\beta = 1/D$ (i.e. $\beta = 1/2$ in the quantum and Brownian motion case $D = 2$) and where $\boldsymbol{b}_+$ is an average forward velocity.

To be complete we must consider also the variation of $\boldsymbol{x}$ between $t{-}dt$ and t:

$$\boldsymbol{x}(t,dt) - \boldsymbol{x}(t-dt,dt) \ = \ \boldsymbol{b}_-(\boldsymbol{x},t)\ dt \ + \ \boldsymbol{\zeta}_-(t,dt)\ (dt/\tau_0)^{\beta} \ . \tag{4.2}$$

Equations (4.1) and (4.2) can be written in terms of instantaneous velocities

$$\boldsymbol{v}_+(\boldsymbol{x},t,dt) \ = \ \boldsymbol{b}_+(\boldsymbol{x},t) + \ \boldsymbol{\zeta}_+(t,dt)\ (dt/\tau_0)^{\beta-1} \ , \tag{4.3a}$$

$$\boldsymbol{\nu}_{-}(\boldsymbol{x},t,dt) \;=\; \boldsymbol{b}_{-}(\boldsymbol{x},t) + \;\; \boldsymbol{\zeta}_{-}(t,dt)\,(dt/\tau_{o})^{\beta-1} \quad . \tag{4.3b}$$

The nondifferentiability is evident on these expressions, since in the quantum case β–1 = –1/2, so that $dt^{\beta-1}$ is an infinite quantity. We recall that, while they would have no meaning in a standard framework, the Non Standard Analysis (NSA) framework [25,26] allows one to work explicitly with infinite and infinitesimal quantities [1,9]. In particular, (4.3) may be recovered in a very simple way thanks to the NSA method: define $\boldsymbol{V}_{+} = \boldsymbol{\nu}_{+}\,(dt/\tau_{o})^{1-\beta}$ and $\boldsymbol{V}_{-} = \boldsymbol{\nu}_{-}\,(dt/\tau_{o})^{1-\beta}$, then each components of $\boldsymbol{V}_{+}$ and $\boldsymbol{V}_{-}$ are finite numbers of the set $^{*}\mathbb{R}$ of Non Standard reals. A well-known NSA theorem states than any finite number of $^{*}\mathbb{R}$ can be decomposed in a unique way into the sum of a real (standard) number and an infinitesimal number [25]. We may then write $\boldsymbol{V}_{+} = \boldsymbol{\zeta}_{+} + \boldsymbol{b}_{+}\,(dt/\tau_{o})^{1-\beta}$, and $\boldsymbol{V}_{-} = \boldsymbol{\zeta}_{-} + \boldsymbol{b}_{-}\,(dt/\tau_{o})^{1-\beta}$, with the components of $\boldsymbol{b}_{+}$ and $\boldsymbol{b}_{-}$ being finite real numbers, *a priori* different.

As remarked by Nelson [11], while in the differentiable case only the classical part of the velocity remains (i.e., $\boldsymbol{\zeta}_{+} = \boldsymbol{\zeta}_{-} = 0$), and the forward and backward velocities are equal (i.e., $lim_{t\to 0}\{\boldsymbol{x}(t+dt,dt) - \boldsymbol{x}(t,dt)\} = lim_{t\to 0}\{\boldsymbol{x}(t,dt) - \boldsymbol{x}(t\text{-}dt,dt)\}$), there is no reason for this to remain true in the nondifferentiable case. Let us stress this point ($\boldsymbol{b}_{+} \neq \boldsymbol{b}_{-}$), since this is the essential feature which will allow classical mechanics to be transformed into quantum mechanics in our following calculations:

Because of the nondifferentiability of space-time, an infinity of geodesics will exist between any couple of points A and B, each of them having fractal (i.e. scale dependent) properties. Their ensemble will define the probability amplitude. Now at each intermediate point C, one can consider the family of incoming (backward) and outcoming (forward) geodesics and define average velocities $\boldsymbol{b}_{+}(C)$ and $\boldsymbol{b}_{-}(C)$ on these families. Once again, it is clear that, in the general nondifferentiable case, $\boldsymbol{b}_{+}$ and $\boldsymbol{b}_{-}$ are expected to be different.

As we shall see in what follows, this doubling of the velocity vector is at the origin of the complex nature of the quantum probability amplitude. We claim that it takes its origin in the very nature of the physical analysis of natural process (since Newton and Leibniz): namely, write the equations which describe the *variation* of physical quantities due to the variation of variables. This leads, in standard differentiable physics, to the integro-differential calculus and to the definition of a unique derivative, while in nondifferentiable physics this will imply a ‘doubling’ of the average velocity field.

Before proceeding further with the formalism, let us remark that, even though we are led to a reformulation of Nelson’s stochastic quantum mechanics, the interpretation is profoundly different. While Nelson assumes an underlying Brownian motion of unknown origin which acts on particles in a still Minkowskian space-time, and then introduces nondifferentiability as a by-product of this hypothesis, we assume *as a fundamental and universal principle* that space-time itself is no longer Minkowskian nor differentiable. While with Nelson’s Brownian motion hypothesis, nondifferentiability is but an approximation which is expected to break down at the scale of the underlying collisions, where a new physics should be introduced, our hypothesis of nondifferentiability is *essential* and should hold down to the smallest possible length-scales. As already remarked, the fractal hypothesis is *not* a hidden parameter theory: even though space-time could remain deterministic, its nondifferentiability implies a definitive loss of determinism of particle *trajectories*.

Let us define, following Nelson [11,12], mean forward and backward derivatives, d_{+}/dt and d_{-}/dt:

$$\frac{d_{\pm}}{dt}\,\boldsymbol{y}(t) \;=\; lim_{\Delta t\to 0\pm} \left\langle \frac{y(t+\Delta t\,) - y(t)}{\Delta t} \right\rangle \tag{4.4}$$

which, once applied to the position vector $\boldsymbol{x}$, yield the above *forward and backward mean velocities*,

$$\frac{d_+}{dt}\boldsymbol{x}(t) = \boldsymbol{b}_+ \quad ; \quad \frac{d_-}{dt}\boldsymbol{x}(t) = \boldsymbol{b}_- \quad . \tag{4.5}$$

Let us now introduce our main new method. While in every present formulations of Nelson's stochastic mechanics, one writes two systems of equations for the forward and backward processes (or for combinations of them) and eventually combine them in the end in a complex equation, we have suggested [9] to work *from the beginning* in terms of complex quantities. So we combine the forward and backward derivatives of (4.4) in a complex derivative operator

$$\frac{\widehat{d}}{dt} = \frac{(d_+ + d_-) - i\,(d_+ - d_-)}{2dt} \quad , \tag{4.6}$$

which, when applied to the position vector, yields a complex velocity [9]

$$\boldsymbol{\mathcal{V}} = \frac{\widehat{d}}{dt}\boldsymbol{x}(t) = \boldsymbol{V} - i\,\boldsymbol{U} = \frac{\boldsymbol{b}_+ + \boldsymbol{b}_-}{2} - i\,\frac{\boldsymbol{b}_+ - \boldsymbol{b}_-}{2} \quad . \tag{4.7}$$

Let us also define

$$\frac{d_v}{dt} = \frac{1}{2}\,\frac{d_+ + d_-}{dt} \quad , \quad \frac{d_u}{dt} = \frac{1}{2}\,\frac{d_+ - d_-}{dt} \quad , \tag{4.8}$$

such that $d_v\boldsymbol{x}/dt = \boldsymbol{V}$ and $d_u\boldsymbol{x}/dt = \boldsymbol{U}$. The real part $\boldsymbol{V}$ of the complex velocity $\boldsymbol{\mathcal{V}}$ generalizes the classical velocity, while its imaginary part, $\boldsymbol{U}$, is a new quantity arising from the non-differentiability. Let us now jump to a statistical representation. The position vector $\boldsymbol{x}(t)$ is now assimilated to a stochastic process which satisfies the following relations (respectively for the forward ($dt > 0$) and backward ($dt < 0$) process):

$$d\boldsymbol{x}(t) = \boldsymbol{b}_+[\boldsymbol{x}(t)]\,dt + d\boldsymbol{\xi}_+(t) = \boldsymbol{b}_-[\boldsymbol{x}(t)]\,dt + d\boldsymbol{\xi}_-(t) \quad . \tag{4.9}$$

The $d\boldsymbol{\xi}(t)$'s can be seen as originating in the above "fractal functions" $\boldsymbol{\zeta}_\pm$. One can show [9] that they amount to a Wiener process when $D = 2$ (the only case considered in the present section), i.e. that the $d\boldsymbol{\xi}(t)$'s are Gaussian with mean zero, mutually independent and such that

$$\langle d\xi_{\pm i}\; d\xi_{\pm j}\rangle = \pm\, 2\,\mathcal{D}\,\delta_{ij}\;dt \quad , \tag{4.10}$$

$\mathcal{D}$ standing for a diffusion coefficient. Its expression is easily found from the identification with the fractal approach: the transition time interval is the de Broglie time scale in rest frame, i.e. $\tau_0 = \hbar/mc^2$, so that $\mathcal{D} = \hbar/2m$, which is the value postulated by Nelson.

Equation (4.10) now allows us to get a general expression for the complex time derivative $\widehat{d}/dt$. Consider a function $f(\boldsymbol{x},t)$, and expand its total differential to second order. We get

$$df = \frac{\partial f}{\partial t}\,dt + \boldsymbol{\nabla} f\,.\,d\boldsymbol{x} + \frac{1}{2}\,\frac{\partial^2 f}{\partial x_i\,\partial x_j}\,dx_i\,dx_j \quad . \tag{4.11}$$

We may now compute the forward and backward derivatives of f. In this procedure, the mean value of $\langle dx_i\, dx_j \rangle$ reduces to $\langle d\xi_{\pm i}\; d\xi_{\pm j} \rangle$, so that the last term of (4.11) amounts to a Laplacian thanks to (4.10). We obtain

$$d_{\pm} f / dt \;=\; (\partial / \partial t \;+ \boldsymbol{b}_{\pm} \,.\, \boldsymbol{\nabla} \;\pm\; \mathcal{D}\, \Delta\;)\, f \quad . \tag{4.12}$$

Let us stop one moment on this highly meaningful result. In order to better understand it, let us assume the fractal dimension to be different from 2: in this case there is no longer a cancellation of the scale-dependent terms in (4.11), and, instead of a pure Laplacian operator in the second order term $\mathcal{D}\,\Delta f$, one would obtain an explicitly scale-dependent behaviour $\mathcal{D}\;\delta t^{\,(2/D)-1} \Delta f$. In other words, this means that the particular quantum mechanical value $D = 2$ implies that the scale symmetry becomes "hidden" in the operator formalism.

Using (4.12), we can finally give the expression for the complex time derivative operator [9]:

$$\frac{d'}{dt} \;=\; \frac{\partial}{\partial t} \;+\; \boldsymbol{\mathcal{V}} \,.\, \boldsymbol{\nabla} \;\; - i\, \mathcal{D}\, \Delta \;\; . \tag{4.13}$$

We shall now postulate that the passage from classical (differentiable) mechanics to the new nondifferentiable mechanics that is considered here can be implemented by a *unique* prescription: *Replace the standard time derivative d/dt by the new complex operator* d'/dt. In other words, d'/dt will play the role of a kind of "quantum-covariant derivative". Let us indicate the main steps by which one may generalize classical mechanics using this new correspondence principle.

We assume that any mechanical system can be characterized by a Lagrange function $\mathcal{L}(\boldsymbol{x}, \boldsymbol{\mathcal{V}}, t)$, from which an average stochastic action S is defined:

$$S \;=\; \int_{t_1}^{t_2} \langle \mathcal{L}(\boldsymbol{x},\, \boldsymbol{\mathcal{V}},\, t) \rangle \; dt \qquad . \tag{4.14}$$

The Lagrange function $\mathcal{L}$ and the action S are *a priori* complex and are obtained respectively from the classical Lagrange function $L(\boldsymbol{x}, d\boldsymbol{x}/dt, t)$ and from the classical action S precisely by applying the above prescription $d/dt \rightarrow d'/dt$. The least-action principle, applied on this new action with both ends of the above integral fixed, leads to generalized Euler-Lagrange equations [9]

$$\frac{d'}{dt}\, \frac{\partial \mathcal{L}}{\partial \mathcal{V}_i} \;=\; \frac{\partial \mathcal{L}}{\partial x_i} \quad , \tag{4.15}$$

which are exactly the equations one would have obtained from applying the correspondence ($d/dt \rightarrow d'/dt$) to the classical Euler-Lagrange equations themselves. Other fundamental results of classical mechanics are also generalized in the same way. In particular, assuming homogeneity of space *in the mean* leads to defining a complex momentum

$$\mathcal{P} \;=\; \frac{\partial \mathcal{L}}{\partial \mathcal{V}} \quad . \tag{4.16}$$

If one now considers the action as a functional of the upper limit of integration in (4.14), the variation of the action from a trajectory to another close-by trajectory, when combined with (4.15), yields a generalization of another well-known result of classical mechanics:

$$\mathcal{P} = \nabla S \quad . \tag{4.17}$$

We shall now specialize and consider Newtonian mechanics. The Lagrange function of a closed system, $L = \frac{1}{2}m\, v^2 - \mathcal{U}$, is generalized as $\mathcal{L}(\boldsymbol{x}, \mathcal{V}, t) = \frac{1}{2}\, m\mathcal{V}^{\,2} - \mathbb{U}$. Note that its real part becomes $\frac{1}{2}\, m(\boldsymbol{V}^2 - \boldsymbol{U}^2) - \mathbb{U}$, which is the Lagrangian field proposed by Guerra and Morato [27]. The Euler-Lagrange equations keep the form of Newton's fundamental equation of dynamics

$$-\nabla\mathbb{U} = m\, \frac{\mathcal{d}}{dt}\, \mathcal{V} \quad , \tag{4.18}$$

which is now written in terms of complex variables and time derivative operator.

Note that Nelson [11] arbitrarily defines the acceleration as

$$d_{\mathrm{N}}^2\boldsymbol{x}/dt^2 = \frac{1}{2}\, \frac{d_+ d_- + d_- d_+}{dt^2}\, \boldsymbol{x} \tag{4.19}$$

(it could *a priori* have been any second order combination of d_+ and d_-; however see [12]). It is easy to show that Nelson's acceleration is nothing but the real part of the complex acceleration $\mathcal{d}\mathcal{V}/dt$. Indeed, let us separate its real and imaginary parts. We find

$$\frac{\mathcal{d}\mathcal{V}}{dt} = \frac{d_V - i\, d_U}{dt}(\boldsymbol{V} - i\, \boldsymbol{U}) = \frac{d_V\, \boldsymbol{V} - d_U\, \boldsymbol{U}}{dt} - i\, \frac{d_U\, \boldsymbol{V} + d_V\, \boldsymbol{U}}{dt}$$

$$= \left(\frac{d_+ d_- + d_- d_+}{2\, dt^2} - i\, \frac{d_+^{\,2} + d_-^{\,2}}{2\, dt^2} \right) \boldsymbol{x} \quad . \tag{4.20}$$

The complex momentum $\mathcal{P}$ now reads $\mathcal{P} = m\mathcal{V}$, so that from (4.17) we arrive at the conclusion that, in this case, the *complex velocity* $\mathcal{V}$ *is a gradient*, namely the gradient of the complex action:

$$\mathcal{V} = \nabla S/m \quad . \tag{4.21}$$

This is an interesting result owing to the fact that in several derivations of Nelson's stochastic mechanics, one *assumes* that the classical velocity $\boldsymbol{V}$ (i.e. the real part of our complex velocity $\mathcal{V}$) is a gradient.

We have now at our disposal all the mathematical tools needed to derive some of the principal axioms of quantum mechanics:

Complex probability amplitude.

We introduce a complex function ψ from the complex action S,

$$\psi = e^{iS/2m\mathcal{D}} \quad , \tag{4.22}$$

which is related to the complex velocity in the following way:

$$\mathcal{V} = -2\, i\, \mathcal{D}\, \boldsymbol{\nabla}(ln\psi)\ . \tag{4.23}$$

As we shall see in what follows, ψ is solution of the Schrödinger equation and satisfies to Born's statistical interpretation of quantum mechanics, and so can be identified with the wave function (or probability amplitude) of quantum mechanics.

Principle of correspondence.

From (4.23) and the relation $\mathcal{P} = m\mathcal{V}$, we obtain:

$$\mathcal{P}\,\psi = -2\, i\, m\, \mathcal{D}\, \boldsymbol{\nabla}\psi\ , \tag{4.24}$$

with $2m\mathcal{D} = \hbar$, which is nothing but the *correspondence principle* of quantum mechanics in the case of the momentum operator, but here *demonstrated* and written in terms of an *equality* rather than a mere correspondence ($p \rightarrow -i\hbar\boldsymbol{\nabla}$), thanks to our introduction of the complex momentum $\mathcal{P}$.

Schrödinger's equation.

Let us now write the generalized Newton's equation (4.18) in terms of the new quantity ψ. It takes the form

$$\boldsymbol{\nabla}\mathbb{U} = 2\, i\, \mathcal{D}\, m\, \frac{\hat{d}}{dt}(\boldsymbol{\nabla} ln\psi)\ . \tag{4.25}$$

Being aware that $\hat{d}$ and $\boldsymbol{\nabla}$ do not commute, we replace $\hat{d}/dt$ by its expression (4.13):

$$\boldsymbol{\nabla}\mathbb{U} = 2i\, \mathcal{D}\, m\, [\, \frac{\partial}{\partial t}\boldsymbol{\nabla} ln\psi - i\, \mathcal{D}\, \Delta(\boldsymbol{\nabla} ln\psi) - 2i\, \mathcal{D}\, (\boldsymbol{\nabla} ln\psi\, .\boldsymbol{\nabla})(\boldsymbol{\nabla} ln\psi)\,]\ . \tag{4.26}$$

This expression is simplified thanks to the three following identities, which may be established by straightforward calculation:

$$\boldsymbol{\nabla}\Delta = \Delta\boldsymbol{\nabla}\ ;\ (\boldsymbol{\nabla} f\, .\boldsymbol{\nabla})(\boldsymbol{\nabla} f) = \frac{1}{2}\, \boldsymbol{\nabla}(\boldsymbol{\nabla} f)^2\ ;\ \frac{\Delta f}{f} = \Delta\, lnf + (\boldsymbol{\nabla}\, lnf)^2. \tag{4.27}$$

This implies

$$\frac{1}{2}\Delta(\boldsymbol{\nabla} ln\psi) + (\boldsymbol{\nabla} ln\psi\, .\boldsymbol{\nabla})(\boldsymbol{\nabla} ln\psi) = \frac{1}{2}\, \boldsymbol{\nabla}\, \frac{\Delta\psi}{\psi}\ , \tag{4.28}$$

and we obtain

$$\frac{\hat{d}}{dt}\mathcal{V} = -\boldsymbol{\nabla}\mathbb{U}/m = -2\, \mathcal{D}\, \boldsymbol{\nabla}\, \{\, i\, \frac{\partial}{\partial t}\, ln\psi + \mathcal{D}\, \frac{\Delta\psi}{\psi}\, \}\ . \tag{4.29}$$

Integrating this equation finally yields [9]

$$\mathcal{D}^2\, \Delta\psi + i\, \mathcal{D}\, \frac{\partial}{\partial t}\psi - \frac{\mathbb{U}}{2m}\psi = 0\ , \tag{4.30}$$

up to an arbitrary phase factor $\alpha(t)$ which may be set to zero by a suitable choice of the phase. Replacing $\mathcal{D}$ by $\hbar/2m$, we get Schrödinger's equation

$$\frac{\hbar^2}{2m}\ \Delta\psi\ +\ i\hbar\ \frac{\partial}{\partial t}\psi\ =\ \mathbb{U}\,\psi\ . \tag{4.31}$$

Born's statistical interpretation.

Let us set $\psi\psi^\dagger = \rho$. Then, as already well-known, the imaginary part of the Schrödinger equation reads

$$\partial\rho/\partial t + div(\rho\boldsymbol{V}) = 0, \tag{4.32}$$

which is recognized as an equation of continuity. Since $\boldsymbol{V}$ generalizes the classical velocity, ρ is straighforwardly interpreted as a probability density. This could have been also obtained directly from the Markov-Wiener initial process which, as such, satisfies two (forward and backward) Fokker-Planck equations [11,12]. These Fokker-Planck equations can also be combined into a unique complex equation:

$$\partial\rho/\partial t + div(\rho\boldsymbol{\mathcal{V}})\ =\ -i\ \mathcal{D}\,\Delta\rho\ . \tag{4.33}$$

the real part of which is (4.32).

Quantum-classical transition.

We may now come back, as promised in Sec. 2, on the problem of the quantum-classical transition. This question has recently known a renewal of interest (see [28] and references therein) and is certainly not trivial. Indeed the existence of macroscopic quantum systems shows that it cannot be reduced to a microscopic to macroscopic transition. While for a plane wave describing some beam of free particles, the transition is clearly given by the associated de Broglie wavelength (which defines the transition to geometric optics, and also the resolution of the “microscope” which would use this beam as “illuminating” source), this is no longer the case for more complicated quantum systems. The solution proposed by Zurek and others [28] consists in remarking that a quantum system is rarely isolated, but interacts with its environment. The effect of this interaction amounts to a Brownian motion which implies a very fast transition to classical behaviour around the *thermal* de Broglie length, $\lambda_{th} = \hbar/\sqrt{(2mkT)}$. Are we able to recover these two transitions in the fractal/stochastic approach ?

In order to answer this question, let us write the elementary process (4.3) in terms of the average velocities $\boldsymbol{U}$ and $\boldsymbol{V}$ rather than in terms of $\boldsymbol{b}_+$ and $\boldsymbol{b}_-$:

$$\boldsymbol{v}(\boldsymbol{x},t,\delta t)\ =\ \boldsymbol{V}(\boldsymbol{x},t) +\ c\ \boldsymbol{\zeta}_v(t,\delta t)\ (\delta t/\tau_o)^{-1/2}\ , \tag{4.34a}$$

$$\boldsymbol{u}(\boldsymbol{x},t,\delta t)\ =\ \boldsymbol{U}(\boldsymbol{x},t) +\ c\ \boldsymbol{\zeta}_u(t,\delta t)\ (\delta t/\tau_o)^{-1/2}\ , \tag{4.34b}$$

where the $\boldsymbol{\zeta}$'s are dimensionless ($<\boldsymbol{\zeta}^2> = 1$). This form has the advantage to provide us with an explicit appearance of the classical velocity $\boldsymbol{V}$. Knowing that $\tau_o = \hbar/mc^2$, (4.34a) reads

$$\boldsymbol{v}(\boldsymbol{x},t,\delta t)\ =\ \boldsymbol{V}(\boldsymbol{x},t)\ \{\ 1 +\ \boldsymbol{\zeta}_v(t,\delta t)\left(\frac{\hbar}{m\boldsymbol{V}^2\delta t}\right)^{1/2}\ \}\ . \tag{4.35}$$

We thus find the temporal transition to occur about the nonrelativistic de Broglie time, $\tau_{\rm dB} = \hbar/E = \hbar/(\frac{1}{2}m\boldsymbol{V}^2)$. If we look for the spatial transition, then using the basic relation $\delta x^2 = \hbar\delta t/m$ (2.1 and 4.10), Eq. (4.35) reads

$$\boldsymbol{v}(\boldsymbol{x},t,\delta t) \;=\; \boldsymbol{V}(\boldsymbol{x},t)\ \{\ 1 + \ \boldsymbol{\zeta}_v \left(\frac{\hbar}{m\boldsymbol{V}\delta x}\right)\}\ . \tag{4.36}$$

This result confirms that the spatial transition occurs at the de Broglie length $\lambda_{\rm dB} = \hbar/m\boldsymbol{V}$. But a similar reasoning will allow us to also obtain a similar relation for the nonclassical velocity $\boldsymbol{U}$:

$$\boldsymbol{u}(\boldsymbol{x},t,\delta t) \;=\; \boldsymbol{U}(\boldsymbol{x},t)\ \{\ 1 + \ \boldsymbol{\zeta}_u \left(\frac{\hbar}{m\boldsymbol{U}\delta x}\right)\}\ . \tag{4.36}$$

The velocity $\boldsymbol{U}$ is related to the probability density:

$$\boldsymbol{U} = \frac{\hbar}{2m}\ \boldsymbol{\nabla} ln\rho\ \ , \tag{4.37}$$

so that we expect another transition at a scale $\lambda_{\rm u} = 1/\left|\boldsymbol{\nabla} ln\rho^{1/2}\right|$. For example, in the case of the fundamental state of the hydrogen atom, we get $\rho^{1/2} = 2\ e^{-r/r_{\rm B}}$, where $r_{\rm B}$ is the Bohr radius, so that we find $\lambda_{\rm u} = r_{\rm B}$, as expected. For a Gaussian distribution of probability density with dispersion σ_x, we get $\boldsymbol{\nabla} ln\rho^{1/2} = x/\sigma_x^{\ 2}$, and since the variable x is constrained to values $|x| \approx \sigma_x$, we finally obtain $\lambda_{\rm u} \approx \sigma_x$. Then, using Heisenberg's relation, this becomes $\lambda_{\rm u} \approx \hbar/m{<}v^2{>}^{1/2}$, which corresponds to the thermal de Broglie length when expressed in terms of temperature. This second transition applies in particular to systems which have no classical counterpart.

In conclusion, we have demonstrated that the fractal-nonfractal transition is indeed generally coinciding with the quantum-classical transition: but here the transition is intrinsic to the description rather than due to an additional interaction with the environment.

Relativistic case.

Let us close this section by a very brief account of the manner the previous nonrelativistic theory can be generalized in the relativistic case. The problem of the stochastic approach to relativistic quantum mechanics has be recently considered by several authors (see Refs. [14,15,29] and refs. therein). As in the non-relativistic case, a fractal interpretation may be given to these attempts.

Starting from the hypothesis that *space-time* is nondifferentiable, we have seen in Sec. 2 that this implies the various virtual space-time trajectories of particles to be fractal, that one can define a scale-dependent invariant (i.e. proper time) and its finite renormalized counterpart s on these trajectories, and that the four coordinates can be described as four fractal functions, i.e. four finite functions of proper time and of resolution, which are nondifferentiable (i.e., $\partial x^\mu(s,\delta s{=}0)/\partial s = \infty$). We expect, as in the non-relativistic case, the average four-velocities not to be equal in the forward and the backward (time reversal) process, so that we may write:

$$(dx^\mu)_\pm \;=\; b_\pm^{\ \mu}\ ds \ +\ d\xi_\pm^{\ \mu} \tag{4.38}$$

with

$$<d\xi_\pm^{\ \mu}\ d\xi_\pm^{\ \nu}> \;=\; \pm\, 2\ \mathcal{D}\ \delta^{\,\mu\nu} ds\ \ . \tag{4.39}$$

The "quantum-covariant derivative" then writes:

$$\frac{d̸}{ds} = \frac{\partial}{\partial s} + \boldsymbol{\mathcal{V}}^{\mu}\,\partial_{\mu} - i\,\mathcal{D}\,\partial^{\mu}\,\partial_{\mu} \quad , \tag{4.40}$$

where $\boldsymbol{\mathcal{V}}^{\mu}$ is the four-dimensional analogue of the complex velocity $\boldsymbol{\mathcal{V}}$. The difficulty of the relativistic case is that ξ is a Wiener process in $\boldsymbol{R}^4$, while the x^{μ}'s correspond to a (+,–,–,–) signature. Dohrn and Guerra [29] have shown that a possible solution to this problem was to introduce two metrics, a Brownian metric and a Riemannian metric, related by a compatibility condition. Such a method allowed Zastawniak [15] to derive the Klein-Gordon equation from an *s*-stationary Markov diffusion in $\boldsymbol{R}^4$, while Serva [14] obtains it in an equivalent way, by using a Bernstein process. Concerning the Dirac equation, Gaveau *et al*. [30] have shown that it may be obtained from a Poisson process whose real version gives rise to the telegrapher's equation. Anyway these approaches can easily be reformulated in terms of a complex formalism which generalizes that described hereabove for the nonrelativistic case. An account of this generalization will be given elsewhere [13].

5. SCALE RELATIVITY

In the previous section, we have shown how one could recover standard quantum mechanics as the simplest theory that one may construct on the fractal and nondifferentiability hypothesis. We shall now see that the principle of scale relativity and the subsequent requirement of scale covariance is able, not only to provide us with standard quantum mechanics, but also to generalize it in its frontier domain, i.e. at very small length scales and very high energy.

The question that we shall now address is that of *finding the laws of scale transformations which meets the principle of scale relativity*. We shall sum up in this section the reasoning and first results obtained in this framework (see Refs [9,16] for more details), while Sec. 6 will be dedicated to its application to the problem of the mass spectrum of elementary particles.

The principle of scale relativity may be implemented by requiring that the equations of physics be written in a *covariant* way under *scale transformations*. Are the standard scale laws (those described by renormalization group equations, or by a fractal or power-law behaviour) scale-covariant ? As seen in Sec.3, they are usually described (far from the transition to scale-independence) by laws such as $\varphi = \varphi_0\,(\lambda/r)^{\delta}$, with δ a *constant* scale-dimension (which may differ from the standard value $\delta = 1$ by an *anomalous dimension* term [24]). This means that a scale transformation $r \rightarrow r'$ writes:

$$ln\,\frac{\varphi(r')}{\varphi_0} = ln\,\frac{\varphi(r)}{\varphi_0} + \mathbb{V}\;\delta(r) \quad , \tag{5.1a}$$

$$\delta(r') = \delta(r) \quad , \tag{5.1b}$$

where we have set:

$$\mathbb{V} = ln(r/r') \quad . \tag{5.2}$$

The choice of a logarithmic form for the writing of the scale transformation and the definition of the fundamental resolution parameter $\mathbb{V}$ is justified by the expression of the dilatation operator $\tilde{D} = \partial/\partial ln r$ (Eq. 3.3). The *relative* character of $\mathbb{V}$ is evident: in the same way as only *velocity differences* have a physical meaning (Galilean relativity of motion), only $\mathbb{V}$ *differences* have a physical meaning (relativity of scales). We have then suggested [16] to caracterize this relative resolution parameter $\mathbb{V}$ as a “state of scale” of the coordinate system, in analogy with Einstein’s formulation of the principle of relativity [17], in which the relative velocity characterizes the state of motion of the reference system.

Now in such a frame of thought, the problem of finding the laws of linear transformation of fields in a scale transformation $r \to r'$ amounts to finding four quantities, $A(\mathbb{V})$, $B(\mathbb{V})$, $C(\mathbb{V})$, and $D(\mathbb{V})$, where $\mathbb{V} = ln\,(r/r')$, such that

$$ln \frac{\varphi(r')}{\varphi_o} = A(\mathbb{V}) \;\; ln \frac{\varphi(r)}{\varphi_o} + B(\mathbb{V}) \;\; \delta(r) \quad , \tag{5.3a}$$

$$\delta(r') = C(\mathbb{V}) \;\; ln \frac{\varphi(r)}{\varphi_o} + D(\mathbb{V}) \;\; \delta(r) \quad . \tag{5.3b}$$

Set in this way, it immediately appears that the current “scale-invariant” scale transformation law of the standard renormalization group (5.1), given by $A = 1$, $B = \mathbb{V}$, $C = 0$ and $D = 1$, corresponds to the Galileo group. This is also clear from the law of composition of dilatations, $r \to r' \to r''$, which has a simple additive form,

$$\mathbb{V}'' = \mathbb{V} + \mathbb{V}' \, . \tag{5.4}$$

However the general solution to the “special relativity problem” (namely, find A, B, C and D from the principle of relativity) is the Lorentz group [31,16]. Then we have suggested [16] to replace the standard law of dilatation, $r \to r' = \rho r$ by a new Lorentzian relation. However, while the relativistic symmetry is universal in the case of the laws of motion, this is not true for the laws of scale. Indeed, physical laws are no longer dependent on resolution for scales larger than the classical/quantum transition (identified with the fractal/nonfractal transition in our approach) that has been analysed above. This implies that the dilatation law must remain Galilean above this transition scale.

For simplicity, we shall consider in what follows only the one-dimensional case. We define the resolution as $r = \delta x = c\ \delta t$, and we set $\lambda_o = c\ \tau_{dB} = \hbar c/E$. In its rest frame, λ_o is thus the Compton length of the system or particle considered, i.e. in the first place the Compton length of the electron (this will be better justified in Sec. 6). Our new law of dilatation reads, for $r < \lambda_o$ and $r' < \lambda_o$

$$ln \frac{r'}{\lambda_o} = \frac{ln(r/\lambda_o) + ln\rho}{1 + \dfrac{ln\rho \;\; ln(r/\lambda_o)}{ln^2(\lambda_o/\mathbb{\Lambda})}} \quad . \tag{5.5}$$

This relation introduces a fundamental length scale $\mathbb{\Lambda}$, that we have identified with the Planck length (currently 1.61605(10) x 10^{-35} m),

$$\mathbb{\Lambda} = (\hbar G/c^3)^{1/2} \quad . \tag{5.6}$$

But, as one can see from (5.5), if one starts from the scale $r = \mathbb{\Lambda}$ and apply any dilatation or contraction ρ, one gets back the scale $r' = \mathbb{\Lambda}$, whatever the initial value of λ_o (i.e., whatever the state of motion, since λ_o is Lorentz-covariant under *velocity* transformations). In other words, $\mathbb{\Lambda}$ is now interpreted as a limiting lower length-scale, impassable, invariant under dilatations and contractions. A field previously scale-dependent as $ln(\varphi/\varphi_o) = \delta_o\, ln(\lambda_o/r)$ for $r < \lambda_o$ becomes in the new framework

$$ln\,(\varphi/\varphi_o) \;=\; \frac{\delta_o\, ln\dfrac{\lambda_o}{r}}{\sqrt{1 - ln^2(\lambda_o/r)\,/\,ln^2(\lambda_o/\mathbb{\Lambda})}} \quad . \tag{5.7}$$

Note that this equation may be given the explicitly scale-covariant form $\varphi = \varphi_o\,(\lambda_o/r)^{\delta(r)}$. The main new feature of scale relativity respectively to the previous fractal or scale-invariant approaches is that the scale dimension δ and the fractal dimension $D = 1 + \delta$, which were previously constant ($D = 2$, $\delta = 1$), are now explicitly varying with scale:

$$\delta(r) \;=\; \frac{1}{\sqrt{1 - ln^2(\lambda_o/r)\,/\,ln^2(\lambda_o/\mathbb{\Lambda})}} \quad . \tag{5.8}$$

This means that the fractal dimension, which jumps from $D = 1$ to $D = 2$ at the electron Compton scale $\lambda_o = \lambda_e = \hbar/m_e c$, is now varying with scale, at first very slowly as

$$D(r) \;=\; 2\,(1 + \frac{1}{4}\,\frac{\mathbb{V}^2}{\mathbb{C}_o{}^2} \;+ \ldots) \;\;, \tag{5.9}$$

where $\mathbb{V} = ln(\lambda_o/r)$ and $\mathbb{C}_o = ln(\lambda_o/\mathbb{\Lambda})$, then tends to infinity at very small scales when $\mathbb{V} \to \mathbb{C}_o$, i.e. $r \to \mathbb{\Lambda}$. When λ_o is the Compton length of the electron, the new fundamental constant $\mathbb{C}_o$ is found to be

$$\mathbb{C}_e = ln\,(\frac{m_{\mathbb{P}}}{m_e}) \;=\; 51.52797(7) \tag{5.10}$$

from the experimental values of the electron and Planck masses [32] (the number into brackets is the uncertainty on the last digits).

Let us now consider the result which has the most direct consequences concerning the predictive power of the new theory. It is clear that the new status of the Planck length-scale as a lowest unpassable scale must be universal. In particular, it must apply also to the de Broglie and Compton scales themselves, while in their standard definition they may reach the zero length. The de Broglie and Heisenberg relations then need to be generalized. We have presented in Ref. [16] the construction of a “scale-relativistic mechanics” which allows such a generalization. But there is a very simple way to recover the result that was obtained. We have shown above and in [8] that the generalization to any fractal dimension $D = 1 + \delta$ of the de Broglie and Heisenberg relations wrote $p/p_o = (\lambda_o/\lambda)^\delta$, where p_o is the average momentum of the particle, and $\sigma_p/p_o = (\lambda_o/\sigma_x)^\delta$. Scale covariance suggests that these results are conserved, but with δ now depending on scale as given by (5.8), which is precisely the result of Ref. [16]. As a consequence the mass-energy scale and length scale are no longer inverse, but related by the scale-relativistic generalized Compton formula

$$ln\ \frac{m}{m_o} = \frac{ln\ (\lambda_o/\lambda)}{\sqrt{\left(1 - \frac{ln^2(\lambda_o/\lambda)}{ln^2(\lambda_o/\mathbb{A})}\right)}} \quad , \qquad (5.11)$$

i.e., $m/m_o=(\lambda_o/\lambda)^{\delta(\lambda)}$, with $\delta(\lambda_o) = 1$.

Concerning coupling constants, the fact that the lowest order terms of their β-function are quadratic [i.e., their renormalization group equation reads $d\alpha/d\mathbb{V} = \beta_0\alpha^2 + O(\alpha^3)$] implies that their variation with scale is unaffected by scale-relativistic corrections [16,9], *provided it is written in terms of length scale*. The passage to mass-energy scale is now performed by using (5.11).

Let us briefly recall some of the results which have been obtained in this new framework:

**Scale of Grand Unification:* Because of the new relation between length-scale and mass-scale, the theory yields a new fundamental scale, given by the *length-scale corresponding to the Planck energy*. This new scale is given to lowest order by the relation

$$ln(\lambda_Z/\lambda_{\mathbb{P}}) = \mathbb{C}_Z / \sqrt{(2)} \quad , \qquad (5.12)$$

[where $\mathbb{C}_Z \approx ln(m_{\mathbb{P}}/m_Z)$]: it is $\approx 10^{-12}$ times smaller than *W/Z* length-scale. In other words, this is but the GUT scale ($\approx 10^{14}$ GeV in the standard theory) [9].

**Unification of ChromoElectroWeak and Gravitational fields*: As a consequence, the *four* fundamental couplings, U(1), SU(2), SU(3) *and* gravitational converge in the new framework towards about the same scale, *which now corresponds to the Planck mass scale.* The GUT energy now being of the order of the Planck one ($\approx 10^{19}$GeV), the predicted lifetime of the proton ($\propto m_{\mathrm{GUT}}^4/m_{\mathrm{p}}^5 >> 10^{38}$ yrs) becomes compatible with experimental results ($> 5.5 \times 10^{32}$ yrs) [9].

**Fine structure constant*: The problem of the divergence of charges (coupling constants) and self-energy is solved. They have finite non-zero values at infinite energy in the new framework, while in the standard model they were either infinite (abelian U(1) group) either null (asymptotic freedom of nonabelian groups). Such a behaviour of the standard theory prevented one from relating the “bare” (infinite energy) values of charges to their low energy values, while this is now possible in the scale-relativistic standard model: we have found that the formal QED inverse coupling $\bar{\alpha}_0 = \frac{3}{8}\bar{\alpha}_2 + \frac{5}{8}\bar{\alpha}_1 = \frac{3}{8}\bar{\alpha}$ (where $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are respectively the U(1) and SU(2) inverse couplings), when “runned” from the electron scale down to the Planck length-scale by using its renormalization group equation, converges towards the value $4 \times (3.1411 \pm 0.0019)^2 \approx 4\pi^2$ at infinite energy [16,9,13].

Note that a value $\alpha = 1/4\pi^2$ is indeed the only possible value smaller than 1 expected from dimensional arguments for the bare coupling. Indeed, $Fr^2 = \alpha \hbar c$ has dimensional equation $\mathrm{ML^3T^{-2}}$; the ‘natural’ possible values for L and T are respectively the Compton or reduced Compton length, $\hbar/mc$ or h/mc, and the de Broglie time in rest frame, $\hbar/mc^2$ or h/mc^2. Combining all these possibilities yields $\alpha = 1/4\pi^2$, 1, 2π or $8\pi^3$. The value $1/4\pi^2$ is the only one of these possibilities which is compatible with the known strength of QED.

Conversely, the conjecture that the corresponding “bare charge” $\alpha^{1/2}$ is $1/2\pi$ allowed us to obtain a theoretical estimate of the low energy fine structure constant to better than 1‰ of its measured value [13], and to predict that the number of Higgs doublets, which contributes to $2.11\ N_H$ in the final value of $\bar{\alpha}$, is $N_H = 1$. Indeed, the running of the inverse fine structure constant from its infinite energy value to its low energy (electron scale) value reads [13,9]:

$$\bar{\alpha}\,(\lambda_e) = \bar{\alpha}\,(\mathbb{\Lambda}) + \Delta\bar{\alpha}^{(1)}_{\mathbb{\Lambda}Z} + \Delta\bar{\alpha}^{(2)}_{\mathbb{\Lambda}Z} + \Delta\bar{\alpha}^{\mathrm{L}}_{Ze} + \Delta\bar{\alpha}^{\mathrm{h}}_{Ze} + \Delta\bar{\alpha}^{\mathrm{Sc\text{-}rel}} \tag{5.13}$$

where $\bar{\alpha}\,(\mathbb{\Lambda}) = \bar{\alpha}\,(E{=}\infty) = 32\pi^2/3$; $\Delta\bar{\alpha}^{(1)}_{\mathbb{\Lambda}Z}$ is the first order variation of the inverse coupling between the Planck length-scale (i.e., infinite energy in the new framework) and the Z boson length-scale, as given by the solution to its renormalization group equation [9,13],

$$\Delta\bar{\alpha}^{(1)}_{\mathbb{\Lambda}Z} = \frac{10+N_H}{6\pi}\,ln\frac{\lambda_Z}{\mathbb{\Lambda}} = \frac{10+N_H}{6\pi}\,\mathbb{C}_Z = 23.01 + 2.11\,(N_H - 1)\ ; \tag{5.14}$$

$\Delta\bar{\alpha}^{(2)}_{\mathbb{\Lambda}Z}$ is its second order variation, which now depends on the three fundamental couplings α_1, α_2 and α_3 (which may themselves be estimated thanks to their renormalization group equations) [9,13]:

$$\Delta\bar{\alpha}^{(2)}_{\mathbb{\Lambda}Z} = -\frac{104+9N_H}{6\pi(40+N_H)}\,ln\{1-\frac{40+N_H}{20\pi}\,\alpha_1(\lambda_Z)ln\frac{\lambda_Z}{\mathbb{\Lambda}}\} + \frac{20+11N_H}{2\pi(20-N_H)}\,ln\{1+\frac{20-N_H}{12\pi}\,\alpha_2(\lambda_Z)\,ln\frac{\lambda_Z}{\mathbb{\Lambda}}\}$$

$$+\frac{20}{21\pi}\,ln\{1+\frac{7}{2\pi}\,\alpha_3(\lambda_Z)ln\frac{\lambda_Z}{\mathbb{\Lambda}}\} = 0.73 \pm 0.03\ ; \tag{5.15}$$

$\Delta\bar{\alpha}^{\mathrm{L}}_{Ze}$ is the leptonic contribution to its variation between electron and Z scales [9,13]:

$$\Delta\bar{\alpha}^{\mathrm{L}}_{Ze} = \frac{2}{3\pi}\,\{ln(\frac{m_Z}{m_e}) + ln(\frac{m_Z}{m_\mu}) + ln(\frac{m_Z}{m_\tau}) - \frac{5}{2}\} = 4.30 \pm 0.05\ ; \tag{5.16}$$

$\Delta\bar{\alpha}^{\mathrm{h}}_{Ze}$ is the hadronic contribution to its variation between electron and Z scales, which can be precisely inferred from the experimental values of the ratio R of the cross sections $\sigma(e^+e^- \to \mathrm{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ [38]

$$\Delta\bar{\alpha}^{\mathrm{h}}_{Ze} = 3.94 \pm 0.12\ ; \tag{5.17}$$

and $\Delta\bar{\alpha}^{\mathrm{Sc\text{-}rel}} = -0.18 \pm 0.01$ is the scale-relativistic correction which comes from the fact that the length-scales and mass-scales of elementary particles are no longer directly inverse in the new framework. Combining all these contributions we have obtained [13]

$$\bar{\alpha}\,(\lambda_e) = 137.08 + 2.11\,(N_H - 1) \pm 0.13\ , \tag{5.18}$$

in very good agreement with the experimental value 137.036 provided $N_H = 1$ as announced above.

**QCD coupling*: The SU(3) inverse coupling may be shown to cross the gravitational inverse coupling at also the same value $\bar{\alpha}_3 = 4\pi^2$ at the Planck mass-scale (more precisely for a mass scale $m_{\mathbb{P}}/2\pi$). This allows one to get a theoretical estimate for the value of the QCD coupling at Z scale. Indeed its renormalization group equation yields a variation of $\bar{\alpha}_3$ with scale given to second order by:

$$\bar{\alpha}_3(r) = \bar{\alpha}_3(\lambda_Z) + \frac{7}{2\pi}\,ln\frac{\lambda_Z}{r} + \frac{11}{4\pi(40+N_H)}\,ln\{1-\frac{40+N_H}{20\pi}\,\alpha_1(\lambda_Z)ln\frac{\lambda_Z}{r}\}$$

$$-\ \frac{27}{4\pi(20-N_H)}\ ln\{1+\frac{20-N_H}{12\pi}\ \alpha_2(\lambda_Z)\ ln\frac{\lambda_Z}{r}\}\ +\frac{13}{14\pi}\ ln\{1+\frac{7}{2\pi}\ \alpha_3(\lambda_Z) ln\frac{\lambda_Z}{r}\}\quad . \tag{5.19}$$

This leads to the prediction: $\alpha_3(m_Z) = 0.1155\pm0.0002$ [9,13], to be compared to the present experimental value, $\alpha_3(m_Z) = 0.112 \pm 0.003$.

**Electroweak scale*: We have argued that the electroweak / Planck scale ratio was also determined by the same number, i.e., by the bare inverse coupling $4\pi^2$ [9,13]. Namely, the relation $ln(m_{\mathbb{P}}/m) = \bar{\alpha}_0(\infty) = 4\pi^2$ yields a mass $m_{WZ} = 87.393$ GeV, closely connected to the W and Z boson masses (currently $m_Z = 91.182$ GeV, $m_W = 80.0$ GeV, so that $ln(m_{\mathbb{P}}/m_Z) = \bar{\alpha}_0(\infty)\ \{1 + O(\alpha/\pi)\}$). Moreover, we predict a new fundamental scale λ_V given by the relation $\mathbb{C}_V = \bar{\alpha}_0(\infty) = 4\pi^2$, which corresponds to a mass scale $m_V = 123.23(1)$ GeV. Candidates for particles or phenomena corresponding to such an energy are: the top quark, whose current theoretical predictions lie precisely about 120 GeV, the Higgs boson, and half the vacuum expectation value of the Higgs field (currently ≈ 246 GeV) [13].

We shall not develop any longer these predictions in the present paper, but rather focus on a new application of our methods, namely, consider the question of the origin of the mass spectrum of elementary particles from a scale-relativistic point of view.

6. THEORETICAL PREDICTION OF THE MASS SPECTRUM OF ELEMENTARY FERMIONS.

We shall now proceed further and combine together the two mathematical tools which have been developed above, namely the fractal-stochastic approach and the scale-relativistic approach. Being interested here only in the mass spectrum of elementary fermions, we shall adopt a perturbative approach (i.e., $\mathbb{V}^2/\mathbb{C}^2 << 1$). We let to future works a full development of a corresponding theory which would be valid whatever the scale.

That we can use a perturbative approach is justified by the following considerations. It is now demonstrated that that there are only three families of quarks and leptons up to the electroweak scale. The observed masses of elementary fermions vary from the electron mass [$\mathbb{C}_e = ln(m_{\mathbb{P}}/m_e) = 51.52797(7)$], which fixes the scale below which scale relativity starts (this will be demonstrated in what follows), to the top quark, whose various mass estimates are of the order of 120 GeV as recalled above. This corresponds to a ratio $\mathbb{V}/\mathbb{C} \approx 0.23$ and to a δ–factor $\delta \approx 1.028$. Then in this scale domain, the fractal dimension can clearly be written in the form $D = 2(1 + \varepsilon)$, with $\varepsilon << 1$.

In scale relativity, the elementary stochastic process (4.34) still writes

$$(\delta \boldsymbol{x})_\pm = \boldsymbol{b}_\pm\ \delta s\ +\ \delta \boldsymbol{\xi}_\pm\ , \tag{6.1}$$

but with the fluctuation $\delta\boldsymbol{\xi}$ now characterized, for resolutions $r < \hbar/m_e c$, by

$$<\delta\xi^2>\ = 2\ \mathcal{D}'\ \ \delta s^{2/D}\ , \tag{6.2}$$

where $D = D(r)$, as given by (5.8).

Let us stop on this expression. We are touching here one of the essential point of our whole approach. Having in (6.2) a fractal dimension $D \neq 2$ means that this process is no longer a Markov one, and that $\mathcal{D}'$ is no longer a diffusion coefficient in the usual sense. A fractal dimension $D < 2$ would correspond to a correlated, persistent process, while $D > 2$ corresponds to an antipersistent, *anticorrelated* process [33]. But such a behavior, for which there are correlations between the past and the future, is certainly not acceptable for the description of an *elementary* process of nature, for which we expect complete reversibility (see an account of Feynman's analysis of this point in Ref. [7]). Several reasons lead us to expect that, in the end, the effective fractal dimension must remain 2 or very close to this value.

(i) The first, most fundamental reason, is the requirement of microscopic reversibility [3,7] and of causality. Under this argument, only two values of the fractal dimension are acceptable for a description of the individual, elementary process of nature: $D = 1$ (i.e., scale dimension $\delta = 0$, meaning classical, scale-independent behaviour) and $D = 2$ (the corresponding stochastic description is then a Markov-Wiener process of mutually independent events; the scale dimension is $\delta = 1$, corresponding to a scale dependence of physical quantities). We know that these two behaviours are actually achieved in nature, in the classical and quantum realms.

(ii) The second argument is that quantum mechanics still holds (in the form of quantum field gauge theories) up to the highest energies presently reached by particle accelerators ($\approx$ 100 GeV). A theory based on a 'fractional Brownian motion' process such as given in (6.2) would be expected to be different from quantum mechanics. We have indeed seen in Sec. 4 that it was the particular value $D = 2$ of the fractal dimension which allowed the scale-dependence to be "hidden" in the operator formalism. (However this argument should be taken with caution, since the difference with a pure $D = 2$ Markov process can manifest itself through an explicit scale-dependence of physical quantities, while such a scale dependence is actually observed in relativistic QED: we shall see that it precisely provides us with a mechanism of mass generation).

(iii) The third argument is a particular, more specific, case of the second. We have demonstrated in previous works [8,9] that the quantum spin originates precisely from the fractal dimension 2 of quantum trajectories, while it would be either null or infinite if the dimension was respectively smaller or larger than 2.

We shall now see that the very existence of charged elementary particles forbids the fractal dimension to exceed 2, at least up to the *W-Z* weak bosons scale. Indeed, among the variables appearing in (6.2), the fractal dimension is not the only one which becomes scale-varying below the Compton length of the electron: this is true also of the coefficient $\mathcal{D}'$. Dimensional arguments lead us to write (6.2) in the form:

$$<\delta\xi^2> \;=\; \frac{\hbar}{m}\;\delta s\;\left(\frac{\delta s}{\tau_o}\right)^{(2/D)-1}, \tag{6.3}$$

where $\tau_o = \hbar/m_o c^2$, and where we have reintroduced the Markovian diffusion coefficient $\mathcal{D} = \hbar/2m$. We shall assume, in order to fix the ideas, that the particle considered is an electron.

Consider the last term in (6.3). We can identify $\delta s/\tau_o$ with r/λ_o, and then use the fact that we are in a perturbative regime in order to write it in the form:

$$\left(\frac{\delta s}{\tau_o}\right)^{(2/D)-1} = exp\{(\frac{2}{D(r)} - 1)\; ln(\frac{r}{\lambda_o})\} \approx 1 + (1 - \frac{2}{D(r)})\; \mathbb{V} \tag{6.4}$$

Consider now the diffusion coefficient. While it was independent of scale in nonrelativistic quantum mechanics, this is no longer the case for $r < \lambda_e = \hbar/m_e c$. Indeed the important point here is that $\mathcal{D}$ depends on the mass (i.e. self-energy) m of the particle, and that, due to Quantum Electrodynamical radiative corrections, *the self-energy of the electron is known to vary logarithmically with scale* below its Compton scale λ_e (see e.g. [34]).

The variation with scale of the self-energy in QED is obtained by writing that the mass depends on the coupling "constant" α, (itself varying with scale), and on the scale r, i.e., $m = m[\alpha(r),r]$. The resulting differential equations are the charge and mass renormalization group equations [23,34]:

$$\frac{d\alpha}{d\mathbb{V}} = \beta(\alpha) \ , \tag{6.5a}$$

$$\frac{dm}{d\mathbb{V}} = \frac{\partial m}{\partial \mathbb{V}} + \beta(\alpha) \frac{\partial m}{\partial \alpha} = \gamma(\alpha)\, m \ . \tag{6.5b}$$

Perturbative calculations yield $\gamma(\alpha) = \gamma_o\, \alpha$ and $\beta(\alpha) = \beta_o\, \alpha^2$ to lowest order, with $\gamma_o/\beta_o = 9/4$, and (6.5) is solved as [35]

$$\frac{m}{m_o} = \left(\frac{\alpha}{\alpha_o}\right)^{9/4} \ . \tag{6.6}$$

Disregarding for the moment the numerical constants coming from threshold effects, the variation of mass below λ_e is then given to lowest order by [34]

$$m(r) = m_e \ \left[1 + \frac{3\alpha_e}{2\pi} \ ln\left(\frac{\lambda_e}{r}\right) \right] \ . \tag{6.7}$$

Consider now more thoroughly (6.3) in the light of this variation of self-energy. Combining the fractal-function approach and the stochastic one, we can write the expression for the fluctuation as the product of a Markovian term $\hbar ds / m_e$ and of a resolution-dependent term:

$$<d\xi^2> = \frac{\hbar}{m_e} \ ds \ \times \ \frac{m_e}{m(r)} \ exp\left\{\left(\frac{2}{D(r)} - 1\right) ln\left(\frac{r}{\lambda_e}\right)\right\} \ . \tag{6.8}$$

Replacing $D(r) = 1+\delta(r)$ by its expression (5.8) and expanding it in terms of $\mathbb{V}/\mathbb{C}_e$, with $\mathbb{V} = ln\,(\lambda_e/r)$, we can finally write it in the form

$$<d\xi^2> = \frac{\hbar}{m_e} \ ds \ \times \ [1 + \Phi(\mathbb{V})] \ , \tag{6.9}$$

where the correction to a pure Markov-Wiener process, which combines the effects of the mass term and of the scale-dependent fractal dimension, is given to lowest order by

$$\Phi(\mathbb{V}) = \left(\frac{1}{4} \frac{\mathbb{V}^2}{\mathbb{C}_e^{\,2}} - \frac{3\alpha_e}{2\pi} \right) \mathbb{V} \ . \tag{6.10}$$

This relation means that the effect of radiative corrections amounts to defining an *effective* fractal dimension which is kept smaller than 2 below the electron scale. One can reverse the argument, and consider this behavior as a mechanism for the generation of the electron.

Assume indeed that the laws of scale become effectively Lorentzian below some universal scale λ_o, that defines a universal constant $\mathbb{C}_o = ln(\lambda_o/\Lambda)$. This means that the fractal dimension jumps from 1 to 2 at this scale, then begins to increase as $D = 2(1+\mathbb{V}^2/4\mathbb{C}_o{}^2)$ when $\mathbb{V}= ln(\lambda_o/r)$ increases. The only way to ensure that the effective dimension remains smaller than 2 is that masses become themselves scale-dependent below the scale λ_o. This is achieved provided there exists a *charged* particle of mass precisely given by $m_o= \hbar/\lambda_o c$. Indeed the effect of virtual pairs of such a charged particle will be to increase the coupling constant α, then the self-energy from Eq. (6.6), and then to decrease the fractal dimension. This process finally defines an effective fractal dimension $D_{\rm eff} < 2$ given to lowest order by

$$D_{\rm eff} \;=\; 2\,(1 \;+\; \frac{1}{4}\,\frac{\mathbb{V}^2}{\mathbb{C}_o{}^2} \;-\; \frac{3\alpha_e}{2\pi}\,)\;\;. \qquad (6.11)$$

Such a mechanism allows us to demonstrate that the transition scale to the scale-relativistic regime is actually the electron scale (the electron is simply defined as the *charged* elementary particle of smallest possible mass in nature, so that $\lambda_e = \lambda_o$). Although it does not determine the electron mass and charge themselves, we shall now see that it allows us to recover the general features of the whole mass and charge spectrum in terms of the electron parameters (m_e and α_e).

Generation of the muon.

Consider indeed Eq. (6.11). The solution brought to us by the existence of the electron is not definitive, since the scale-relativistic effect increases the effective fractal dimension in proportion to $\mathbb{V}^3$ while the QED effect is only linear in $\mathbb{V}$. If no charged particle other than the electron was to exist in nature, the effective dimension would become larger than 2 again for scales smaller than some scale λ_1 given by

$$\mathbb{V}_1 = ln\frac{\lambda_1}{\Lambda} \;=\; \sqrt{(\,\frac{6}{\pi}\,\alpha_e)}\;\;\; \mathbb{C}_e\;\;. \qquad (6.12)$$

So we expect that at least one new charged particle be created at about the scale given by (6.12). Numerically, we find $\mathbb{V}_1 = 6.08$, which corresponds to an energy of 230 MeV.

This is an encouraging result, owing to the particularly inhomogeneous distribution of elementary particles in scale. One of the features that a mechanism of generation of masses must understood is, indeed, why there is so large a gap between the electron (.511 MeV, $\mathbb{V}= 0$) and the second lightest charged particle, the muon (105.65 MeV, $\mathbb{V} = 5.3$). Moreover, several particles have masses close to the muon, such as the π (≈137 MeV, $\mathbb{V}\approx 5.5$), the strange quark s (≈240 MeV, $\mathbb{V}= 6.1$), and the u and d quarks in the proton (effective mass ≈313 MeV, $\mathbb{V}= 6.4$: note that the absolute masses of u and d quarks are much smaller, $m_u = 8.2\pm1.5$ MeV, $m_d = 14.4\pm1.5$ MeV [36], but are not effective in hadrons and mesons because of the properties of QCD).

It can then be worth to make an attempt at performing a more detailed calculation of the muon mass. Equation (6.12) does not take into account the threshhold effects and the higher order corrections in the electron self-energy. We shall only consider here the threshold effect, which is the dominant correction to (6.12). An estimate including two-loop radiative corrections will be given elsewhere [40].

Assume that the zero point of the asymptotic behaviour of mass, as described in (6.7), is not strictly λ_e, but a slightly different scale λ_0. We set $\varDelta = ln\,(\lambda_e/\lambda_0)$. The mass variation is now given, for $\mathbb{V} > \varDelta$, by

$$m(r) = m_e\ \left[\, 1 + \frac{3\alpha_e}{2\pi}\ (\mathbb{V} - \varDelta)\,\right]\ . \tag{6.13}$$

Let us call A the least order solution (6.12):

$$A\ =\ \sqrt{\left(\frac{6}{\pi}\,\alpha_e\right)}\ \ \mathbb{C}_e\ . \tag{6.14}$$

The "muon" mass is now solution of the equation:

$$\mathbb{V}^3 - A^2\,(\mathbb{V} - \varDelta)\ =\ 0\ . \tag{6.15}$$

It is clear that, provided $\varDelta > 0$, there is a first solution of this equation very close to $\varDelta$ (this solution may have no physical meaning: the transitional behaviour of $m(r)$ between $\mathbb{V} = 0$ and $\mathbb{V} = \varDelta$ should be described by a new function including a Yukawa term $exp(-2mrc/\hbar)$, which actually keeps D smaller than 2). This solution is given, to third order in $\varDelta$, by:

$$\mathbb{V}_0\ =\ \varDelta\,\left(\,1 + \frac{\varDelta^2}{A^2}\right)\ . \tag{6.16}$$

Then the physically meaningful solution we are looking for satisfies the second order equation

$$\mathbb{V}^2\ +\ \mathbb{V}_0\,\mathbb{V} +\ \mathbb{V}_0^{\,2} - A^2\ =\ 0\ . \tag{6.17}$$

The muon mass is then given by (up to the approximations considered):

$$\mathbb{V}_\mu\ =\ (A^2 - \tfrac{3}{4}\ \mathbb{V}_0^{\,2})^{1/2}\ - \tfrac{1}{2}\ \mathbb{V}_0\ . \tag{6.18}$$

Vacuum polarization being due to e^+e^- *pairs* of mass $2m_e$, it is reasonable to assume the zero point *in momentum representation* to be given by the scale $\lambda_e/2 = \hbar/2m_ec$. The passage to position representation amounts to a Laplace transform which introduces an additional numerical term equal to Euler's constant $\gamma = 0.577...$ [9]. The total threshold then amounts to $\varDelta = ln2 + \gamma \approx 1.270$. Inserting this number in (6.16) and (6.18) yields, with $A = 6.083$,

$$\mathbb{V}_\mu\ = 5.311\ . \tag{6.19}$$

The observed mass ratio of the muon and electron, $m_\mu/m_e = 206.76826(3)$ [32] corresponds to $ln(m_\mu/m_e) = 5.332$, then to $\mathbb{V}_\mu = ln(\lambda_e/\lambda_\mu) = 5.303$ when accounting for the scale-relativistic correction (5.11). Our prediction (6.19) is to 10^{-3} of this value and corresponds to $m_\mu/m_e = 208.5$. If one uses two-loop formulas, one finds an even better result [40], $\mathbb{V}_\mu = 5.3036$, or in terms of mass ratio $m_\mu/m_e = 206.84 \pm 0.38$, where the error comes from an assumed uncertainty $\pm\alpha/\pi$ on the above threshold $\varDelta$.

Anyway, even considering that the "naive" choice $2m_e$ for the threshold is not fully justified, it remains remarkable that the full range of possible values for the zero point of the asymptotic behaviour, namely $\approx \lambda_e$ to $\approx \lambda_e/4$, yields a range for $\mathbb{V}_1$ corresponding to $\approx [m_\mu\ , 2m_\mu]$.

Other particles.

Let us come back to the approximate thresholdless process for simplicity of the argument. Once the scale given by (6.12) reached, the generation of new particles having a mass given by that scale and a total charge Q_1 will allow D to remain smaller than 2. (Note that the minimal value of $D_{\rm eff}$ is given by $2 - 3\alpha_e/\pi \approx 1.993$, which remains very close to 2). The scale variation of the inverse fine structure constant due to elementary fermions of masses λ_i and charges $Q_i\, e$ (where e is the electron charge, so that the Q_i are dimensionless) is given, in terms of $\mathbb{V}_i = ln(\lambda_e/\lambda_i)$, by [37]

$$\alpha^{-1} = \alpha_e^{-1} - \frac{2}{3\pi} \sum_{i=1}^{n} \{ Q_i^{\,2} \, (\mathbb{V} - \mathbb{V}_i) \} \tag{6.20}$$

for $\mathbb{V} > \mathbb{V}_n$. Then, from (6.6), the inverse mass varies to lowest order as

$$\frac{m_e}{m} = 1 - \frac{3\alpha_e}{2\pi} \, (R\, \mathbb{V} - \sum Q_i^{2} \, \mathbb{V}_i) \ . \tag{6.21}$$

The sum of squares of charges of elementary fermions is related to the well-known ratio R intervening in the QED vacuum polarization and the e^+e^- annihilation cross section [38]:

$$R = \sum_{i=0}^{n} Q_i^{2} \ . \tag{6.22}$$

At the electron scale $\mathbb{V}_0 = 0$, $R = 1$. At the new “muon” scale $\mathbb{V}_1$ given by (6.12), there will be a minimal value for the possible ratios R_1 which will allow the mass term to compensate again the fractal dimension term. It is simply given by the slope of the scale-relativistic $\mathbb{V}^3$ increase at scale $\mathbb{V}_1$. Let us normalize the variables $\mathbb{V}$ by defining

$$X = \mathbb{V} / \mathbb{V}_1 \ . \tag{6.23}$$

In terms of this new variable, the equation for the scales of elementary fermions reads

$$X^3 - R\, X + \sum Q_i^{\,2} \, X_i = 0 \ . \tag{6.24}$$

More generally, knowing that the scale X_i is a first root of (6.24) for the next scale X_{i+1}, we find that both scales are related by the second order equation

$$X_{i+1}^{\,2} + X_i \, X_{i+1} + X_i^{2} = R_i \ , \tag{6.25}$$

whose solution is

$$X_{i+1} = (R_i - \frac{3}{4} X_i^{2})^{1/2} - \frac{1}{2} X_i \ . \tag{6.26}$$

Concerning the muon elementary particle mass scale, the searched condition is $R_1 > 3\, X^2$ at $X_2 = 1$, i.e.,

$$R_1 > 3 \ . \tag{6.27}$$

In the present paper, we shall not attempt to detail the quark contributions in terms of their fractional charges. Being mainly interested, in this first stage of our presentation of these new results, in the great lines of the mass and charge spectrum, we shall only consider integer charges, in agreement with the observed hadronic spectrum. Then the optimized solution to (6.27) is $R_1 = 4$. This means that we expect a generation of elementary fermions at about the muon mass scale with a total charge square $\Sigma Q^2 = 3$.

This result is in good agreement with the observed spectrum. Indeed the u and d quarks intervene in the scale variation of the fine structure constant through *effective* masses whose estimates vary from 0.1 GeV (i.e. the muon mass scale) [37] to their effective mass in the proton ($\approx$0.3 GeV, $\mathbb{V} \approx 6.3$) [38], while the s quark mass is $\approx$0.2 GeV ($\mathbb{V} \approx 6.0$) [37]. This yields an observed R ratio at about the scale $\mathbb{V}_1 = 6.08$ of $1_{(\mu)} + 4/3_{(u)} + 1/3_{(d)} + 1/3_{(s)} = 3$ as predicted.

The equation for the next scale of elementary particle mass is now

$$X^3 - 4X + 3 = 0 \ , \tag{6.28}$$

whose solution is $X_2 = (\sqrt{13}-1)/2 \approx 1.3028$, corresponding to $\mathbb{V}_2 = 7.92$, i.e. to a mass scale 1.52 GeV. This is another very favourable result, since the observed spectrum actually shows a new hole after the s quark, and then a new clustering including the c quark (1.27 GeV, $\mathbb{V}_c = 7.73$) and the τ lepton, (1.78 GeV, $\mathbb{V}_\tau = 8.05$). Note that for quarks, the *mass* families do not coincide with the genuine fermion families, ud,cs,tb).

Actually if one pushes further this rough model, one finds that the new charge constraint at X_2 is $R_2 > 3X_2^2 = 5.09$, implying a minimal solution $R_2 = 6$, i.e. $(\Sigma Q^2)_2 = 2$, in good agreement with the observed $(\Sigma Q^2)_{\tau+c} = 7/3$.

This leads to another mass scale given by the equation $X^3 - 6X + 2 + \sqrt{13} = 0$. The solution is $X_3 = 1.5227$ ($\mathbb{V}_3 = 9.2$) at which the total square of charges would reach the observed one ($R = 8$ for three families of leptons and quarks). This value, once again, compares well with the b quark mass (4.25 GeV, $\mathbb{V}_c = 8.9$). But it strongly disagrees with the present experimental limit of the top quark mass ($m_t \gtrsim 90$ GeV [39]), at least ten time larger.

However, such a disagreement between experiment and our rough above model is not unexpected. In particular, we have treated quark masses as unvarying with scale, while QCD effects imply a strong inverse variation with energy scale, given by $m/m_o = (\alpha_s/\alpha_{so})^{4/7}$ to lowest order [35,36]. If one improves the model and includes fractional quark charges and scale varying quark masses, one finds [40] the last $Q = 2/3$ particle (i.e. the top quark) to be pushed beyond the W and Z masses, at about $\mathbb{V} = 12$, i.e.

$$m_t \approx 120 \text{ GeV}. \tag{6.29}$$

It is remarkable that this value, which agrees with our previous prediction of a particular new scale at $\approx$123 GeV, seems to be rather insensitive on the choice of the parameters chosen for the description of the quark mass variation.

Let us conclude this section by remarking that, taken at face value, the mechanism of mass generation presented here would imply a never ending successive emergence of new mass scales for increasing energies. However, such a conclusion would disregard the fact that our mechanism is built from the combination of QED effects (based on an abelian U(1) group) and scale-relativistic effects, while beyond the electroweak symmetry breaking scale, electroweak physics becomes non-abelian (U(1) x SU(2) group). The whole question must then be set again in this case, a problem which goes beyond the goals of the present paper. A more complete account of the result which have been for the first time described here will be presented elsewhere [40].

7. CONCLUSION

Let us conclude by a summary of the content of this paper, followed by some prospect concerning some remaining open problems. The basic postulate upon which we rely is that of the continuity and nondifferentiability of space-time. The first consequence of this postulate is the dependence on scale of the various physical quantities, in the first place of position and velocity vectors. By scale dependence we mean that these quantities depend explicitly on the resolution with which measurements are performed. We have then suggested to interpret resolution as a state of scale of reference systems and, basing ourselves on the relative character of all scales in nature, to generalize Einstein's formulation of the principle of relativity by including scale transformations into its application domain. Such a principle would be implemented by the requirement of scale covariance of the equations of physics. Steps towards such a goal have been made by using three complementary mathematical tools, namely, (i) the geometrical concept of a fractal space-time continuum; (ii) renormalization group-like differential equations of scale transformations; (iii) a stochastic description based on the Markov-Wiener process. The passage from one tool to the other is performed by using the concepts of "fractal functions" (i.e. explicitly scale-dependent functions) and in eventually working in the framework of Non Standard Analysis (which allows us to transform resolutions into differentials and to account properly for infinite quantities).

Our logical advance can then be summarized as follows: the postulate of nondifferentiable continuum implies scale dependence and infinite multiplication of equiprobable geodesical lines; even though one can consider individual phenomena in our framework (and then implement Einstein's requirement of realism), we are led to a statistical approach because of the undeterminism of trajectories implied by nondifferentiability; in this statistical approach, only a fractal dimension 2 of trajectories is acceptable in order to preserve microscopic reversibility; continuous nondifferentiablity and $D = 2$ lead to a twin Markov-Wiener process which gives rise to the complex character of the formalism of quantum mechanics (the *velocities* become doublets while the *coordinates* are unchanged); we then obtain a new formulation of Nelson's stochastic quantum mechanics, in which we are led to a demonstration of the correspondence principle, of Schrödinger's equation and of Born's statistical interpretation by simply replacing the time derivative of classical differentiable mechanics by a new complex "quantum covariant derivative". Then the full strength of the principle of scale relativity is used to show that present quantum mechanics is a "Galilean" approximation of a more general theory in which the laws of scale transformations take a Lorentzian form: in the new proposed "scale-relativistic" laws, there appears an unpassable lower length-time scale, invariant under dilations and contractions, which plays for scales the same role as played by the velocity of light in motion laws. We have suggested to identify this scale with the Planck scale. After having recalled some results which have been obtained in this new framework, we finally combine our various complementary tools (fractal space-time, Markov-Wiener process, renormalization group approach, Lorentzian scale relativity) and suggest a solution to the problem of the origin of the mass spectrum of elementary fermions.

Let us finally briefly consider some prospect for the future development of this field of research. Concerning stochastic quantum mechanics, some work is still needed for a proper inclusion of spin (one can hope to see it not artificially added as in current quantum mechanics, but instead naturally emerge as a structure of the fractal virtual trajectories), then for a thorough understanding of the Dirac equation (in particular in a stochastic framework which would include trajectories running backward in classical time). Concerning scale relativity, the next step is now to include fields into the description: we shall suggest in a forthcoming paper a possible approach to this problem, which allows us to derive new relations between masses and charges of particles [40]. We shall also suggest new applications for the mechanism of mass generation that we have presented in Sec. 6: we shall reconsider in its framework the problem of the anomalous spectrum of e^+e^- pairs observed at Darmstadt in heavy-ion collisions, for which we had already suggested a fractal model [8,9], and shall apply it to new proposals concerning the electromagnetical properties of fractal media [41].

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