# Origin of complex and quaternionic wavefunctions in quantum mechanics: the scale-relativistic view

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ABSTRACT In the theory of scale relativity, one attempts to set the foundations of quantum physics on a geometric basis. The geometry of space-time is generalized to a nondifferentiable continuum, that can be characterized by its fractality. Namely, the physical quantities, among which the coordinates themselves, become in this geometry explicit functions of the scale ( i.e., these function are divergent when the scale interval tends to zero). We recall how the main tools of quantum mechanics (complex, then spinorial wave functions), and the equations they satisfy, which are obtained as geodesics equations of the fractal space-time (Schrödinger, Klein-Gordon then Pauli and Dirac equations) can be derived in such a framework. Indeed, the nondifferentiability manifests itself, because of discrete symmetry breakings of the scale variables, in terms of successive doublings of the velocity field, which are naturally accounted for by complex, then biquaternionic numbers. In this contribution, new improvements of this construction are given, including the description of the metric of a fractal space-time, a full derivation of the Born postulate, a new and more complete derivation of the Compton relation, and a generalization of the Schrödinger equation also valid for non-differentiable wave functions.
**Keywords:** scale relativity, nondifferentiable geometry, fractal space-time.

## 1 Introduction

One of the main open problems of modern physics is that of the foundation of quantum physics. It is clear that such an understanding from first principles of the postulates of quantum mechanics is impossible in the framework of the quantum theory itself (since these statements are its basic axioms), but must be looked for in an enlarged paradigm.

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In particular, if one attempts to trace back the origin of a large part of the quantum behavior, one finds it in the complex (then spinorial) nature of the probability amplitude. Feynman particularly stressed this point [1], noticing that, once the first axiom of quantum mechanics is admitted (i.e., the probability amplitude $\psi$ is complex, it is calculated as $\psi = \psi_1 + \psi_2$ for two alternative channels, and the probability density is given by $\psi\psi^\dagger$ ), quantum laws as well as classical laws can be naturally recovered. Is it possible to "understand" this axiom? In other words, is it possible to derive it from more fundamental principles, for which an intuitive comprehension would be possible?

The theory of scale relativity provides us with an extension, both of the foundation of physical theories and of the principle of relativity. So it is worth asking such questions in its frame of thought. In scale relativity, one extends the founding stones of physics by giving up the hypothesis of space-time differentiability; then one extends the principle of relativity by applying it, not only to motion laws, but also to scale laws.

One of the main and simplest consequences of the non-differentiability of space-time is that the velocity vector becomes two-valued. This two-valuedness of the velocity implies a two-valuedness of the Lagrange function, and therefore a two-valuedness of the action $S$. Finally, the wave function is defined as a re-expression for the action, so that it will also be two-valued in the simplest case.

In previous works, we have simply admitted, as a first step, to account for this two-valuedness in terms of complex numbers [2], which is indeed a natural choice. However, it is now possible to look for a physical justification of this choice. This is an important point for the understanding of quantum mechanics, but also from the viewpoint of the meaning of the scale-relativistic approach. Indeed, the equations of scale-relativity are not simply a "pasting" of real and imaginary equations, but involve the complex product from the very beginning of the calculations, as we shall see.

One of the main aims of this contribution is precisely to address this specific question: why complex numbers and why bi-quaternions? As we shall see, the answer is that complex numbers (then quaternions) achieve a particular representation of scale-relativistic physics in terms of which the fundamental equations take their simplest form: in other words, they ensure a 'covariant' representation. Other choices for the representation of the two-valuedness and for the new product are possible, but these choices would give to the Schrödinger equation a more complicated form, involving additional terms (although its physical meaning would be unchanged). Bi-quaternions follow as a further splitting, at another level, of the complex numbers.

# 2 Main steps of the derivation of the Schrödinger and Dirac equations in scale relativity

## 2.1 Foundations of scale relativity

The theory of scale relativity is based on the giving up of the hypothesis of manifold differentiability which is a key assumption of Einstein's General Relativity. In the new theory, the coordinate transformations are continuous but can be non-differentiable or differentiable (and therefore it includes General Relativity). The giving up of the assumption of differentiability implies several consequences [2], leading to subsequent steps of construction of the theory:

(1) It has been proved [2, 3, 4] that a continuous and non-differentiable curve is fractal in a general meaning, namely, its length is explicitly scale-dependent and goes to infinity when the scale interval $\varepsilon$ goes to zero, i.e., $\mathcal{L} = \mathcal{L}(\varepsilon) \to \infty$ when $\varepsilon \to 0$. This result can be readily extended to a continuous and non-differentiable manifold.

(2) The fractality of space-time [5, 6, 7, 2] involves the scale dependence of the reference frames. One therefore adds to the usual variables defining the coordinate systems (position, orientation, motion), new variables $\varepsilon$ characterizing their 'state of scale'. In particular, the coordinates become functions of these scale variables, i.e., $X = X(\varepsilon)$ (in the simplified case of only one variable). In an experimental situation, these scale variables are identified with the resolution scale of the measurement apparatus. In the case of a theoretical physics description, they are identified with the differential elements themselves, $\varepsilon = dX$, of which the coordinates become explicit functions, i.e., $X = X(dX)$.

(3) The scale variables $\varepsilon$ can never be defined in an absolute way, but only in a relative way. Namely, only their ratio $\rho = \varepsilon'/\varepsilon$ does have a physical meaning. This universal behavior leads to extend to scales the principle of relativity [7, 8, 2], in order to include in the possible changes of reference frames the new ones which are described by the transformations of these scale variables.

(4) Though the non-differentiability manifests itself at the limit $\varepsilon \to 0$, the use of differential equations is made possible by representing physical quantities by fractal functions $f[X(\varepsilon), \varepsilon]$ [2]. Even if the function $f(X, 0)$ is non-differentiable, the fractal function $f(X, \varepsilon)$ is differentiable for any $\varepsilon \neq 0$ with respect to both $X$ and $\varepsilon$. This allows one to complete the differential equations of standard physics by new differential equations of scale, which are constrained by the principle of scale relativity. In other words, we use a double differential calculus, in position space and in scale space. The study of the scale laws derived from these new scale differential equations has been developed according to various levels of relativistic transformations [8, 3, 9]. In what follows, we consider only the simplest case, namely Galilean-like scale transformations.

(5) The simplest possible scale differential equation is a first order equation, $\partial X/\partial \ln \varepsilon = \beta(X)$, which can be simplified again by Taylor-expanding the unknown function $\beta$, so that it reads $\partial X/\partial \ln \varepsilon = a + bX + \cdots$. It is solved as the sum of two terms, a scale-independent, differentiable, 'classical part' and a power-law divergent, explicitly scale-dependent, non-differentiable 'fractal part' [10],

$$X = x + \zeta \left(\frac{\lambda}{\varepsilon}\right)^{-b}, \tag{2.1}$$

where $x = -a/b$. This decomposition in a classical part and a fractal part is very general and plays a central role in the whole scale-relativity approach. When the coefficient $b$ is constant, the second term is the standard expression for the length of a fractal curve of dimension $D_F = 1 - b$ [11]. Moreover, the laws of transformation of this expression under a scale transformation $\ln(\lambda/\varepsilon) \to \ln(\lambda/\varepsilon')$ take the mathematical form of the Galileo group of transformation, and they therefore come, as required, under the principle of relativity [8].

### 2.2 Metric of a fractal space-time

In what follows, the scale variables $\varepsilon$ are identified with the differential elements themselves. In the new calculus of scale relativity, the various physical quantities can become explicit functions of these differential element, e.g., $f = f(s, ds)$, since they are now considered as variables by their own. In fractal geometry, the proper time differential $ds$ and the coordinates differentials, $dX^\mu$ are linked by the relation $(dX^\mu)^{D_F} \sim ds$. This is in accordance with equation (2.1) which reads, after differentiation,

$$dX^\mu = dx^\mu + d\xi^\mu = v^\mu ds + a^\mu \times (\lambda_c)^{1-1/D_F} \times ds^{1/D_F}, \tag{2.2}$$

where $a^\mu$ is dimensionless and where $\lambda_c$ is a length scale which must be introduced for dimensional reasons. In the case where this description holds for a quantum particle of mass $m$, it will be identified with its Compton length $\hbar/mc$ (see what follows). The elementary displacement on a fractal space-time is therefore the sum of a classical, standard differentiable element, which is leading at large scales, and of a fractal, non-standard fluctuation which is leading at small scales.

In what follows, we simplify again the description by considering only the case $D_F = 2$. For this, we base ourselves on Feynman's result [12] according to which the typical paths of quantum particles (those which contribute mainly to the path integral) are non-differentiable and (in modern words) of fractal dimension $D_F = 2$. The case $D_F \neq 2$ has been also studied in detail: it has been shown that $D_F = 2$ is a critical dimension for which the explicit scale dependence disappears in the final equations (see [3] and references therein).

Let us now show how equation (2.2) can be used to give an explicit form to the metric of a fractal space-time (disregarding at this step of the construction other consequences of nondifferentiability such as the multi-valuedness of derivatives, see next sections). The fractal fluctuations (here in four dimensions) write for fractal dimension 2

$$d\xi^\mu = \zeta^\mu \sqrt{\lambda_c \, ds}. \tag{2.3}$$

where the $\zeta^\mu$ are dimensionless highly fluctuating functions.

Due to the highly erratic character of the fractal fluctuations, we can replace them by stochastic variables such that $\langle \zeta^\mu \rangle = 0$, $\langle (\zeta^0)^2 \rangle = -1$ and $\langle (\zeta^k)^2 \rangle = 1$ ($k = 1$ to 3). We recover here (but now at the level of the metric) a description which is familiar in usual stochastic processes, which can also be separated in a regular part and a stochastic part. As we shall see in the following, we do not have to be more specific about the probability distribution of these stochastic variables. Their zero mean and unit variance is the only information needed in the subsequent calculations, which are therefore valid whatever be this distribution.

Now we can write the fractal fluctuations in terms of the coordinate differentials,

$$d\xi^\mu = \zeta^\mu \sqrt{\lambda^\mu \, dx^\mu}. \tag{2.4}$$

The identification of equations (2.3) and (2.4) leads very simply to the establishment of the expressions for the de Broglie-Einstein length-scale and time-scale from the Compton one, i.e., for two variables,

$$\lambda_x = \frac{\lambda_c}{dx/ds} = \frac{\hbar}{p_x}, \quad \tau = \frac{\lambda_c}{dt/ds} = \frac{\hbar}{E}. \tag{2.5}$$

Let us now assume that the large scale (classical) behavior is given by Riemannian metric potentials $g_{\mu\nu}(x, y, z, t)$. The invariant proper time $dS$ along a geodesic (which is therefore subjected to curvature at large scale and fractality at small scales) writes, in terms of the complete differential elements $dX^\mu = dx^\mu + d\xi^\mu$,

$$dS^2 = g_{\mu\nu} dX^\mu dX^\nu = g_{\mu\nu}(dx^\mu + d\xi^\mu)(dx^\nu + d\xi^\nu). \tag{2.6}$$

Now replacing the $d\xi$'s by their expression (2.4), one obtains a fractal metric. Let us give it in a simplified form (in order to simplify its writing), for 2 dimensions, diagonal classical metric and fractal dimension $D_F = 2$,

$$dS^2 = g_{00}(x,t)\left(1 + \zeta_0 \sqrt{\frac{\tau}{dt}}\right)^2 c^2 dt^2 - g_{11}(x,t)\left(1 + \zeta_1 \sqrt{\frac{\lambda_x}{dx}}\right)^2 dx^2. \tag{2.7}$$

Finally taking the mean on the fractal fluctuations yields

$$d\bar{S}^2 = g_{00}(x,t)\left\{1-\left(\frac{\tau}{dt}\right)\right\}c^2dt^2 - g_{11}(x,t)\left\{1+\left(\frac{\lambda_x}{dx}\right)\right\}dx^2. \tag{2.8}$$

We therefore obtain generalized fractal metric potentials which are explicitly dependent on the coordinate differential elements, in agreement with the program of refs. [7, 2]. This dependence on scale concerns also the signature, which jumps from a Minkowskian to a four dimensional Euclidean signature for time-scales smaller than $\tau = \hbar/E \leq \hbar/mc^2$ (i.e., in the highly relativistic domain where pair creations occur).

More generally the metric potentials can be written in their turn as the sum of the standard metric potentials (which describe curvature) and of divergent, highly fluctuating terms (which describe fractality), e.g. for the $g_{00}$ component,

$$\tilde{g}_{00}(x,t;dt) = g_{00}(x,t) + \gamma_{00}(x,t)\left(\frac{\tau}{dt}\right), \tag{2.9}$$

where we have kept only the leading term, owing to the fact that $<\zeta>=0$. The $\gamma_{\mu\nu}(x,t)$ are described at first approximation in terms of stochastic variables. We recover here our result [2] according to which, in the limit ($dx$, $dt \to 0$), the metric is divergent (singular) at each of its points and instants, which is the very intrinsic expression of the fractality of space-time. As a consequence, the curvature is also explicitly scale-dependent and divergent when the scale intervals tend to zero. This property ensures the fundamentally non-Riemannian character of a fractal space-time as well as the ability to characterize it in an intrinsic way.

### 2.3 Geodesics of a fractal space-time

*Infinity of fractal geodesics*

The next step in such a geometric approach consists of writing the geodesics equation. We make the conjecture that the description of quantum 'particles' can be reduced to that of these geodesics. Then their 'internal' properties are the geometrical properties of the geodesics bundle corresponding to their state, according to the various conservative quantities that define them.

Any measurement performed on the "particle" is interpreted as a selection of the geodesics bundle linked to the interaction with the measurement apparatus (that depends on its resolution) and/or to the information known about it (for example, the which-way-information in a two-slit experiment [3]).

Generalizing to space-times the definition of fractal functions, we have defined a fractal space-time as the equivalence class of a family of Riemanian manifolds, explicitly depending on the scale variables. In such space-times, the geodesics equations are also scale dependent and the number of geodesics that relate any two events (or starts from any event) is infinite. We are therefore led to adopt a generalized statistical fluid-like description where the velocity $V^\mu(s)$ is replaced by a scale-dependent velocity field $V^\mu[X^\mu(s,ds),s,ds]$.

*Discrete symmetry breaking and two-valuedness of derivative*

One of the main consequences of non-differentiability, which is the particular point that we want to emphasize in this contribution, is the breaking of the reflexion invariance of the differential element $ds$. Indeed, in terms of fractal functions $f(s,ds)$, two derivatives are defined instead of one:

$$f'_+(s,ds) = \frac{f(s+ds,ds)-f(s,ds)}{ds}, \quad f'_-(s,ds) = \frac{f(s,ds)-f(s-ds,ds)}{ds}. \tag{2.10}$$

Applied to the space-time coordinates, these two derivatives give two divergent velocity fields, $V^\mu{}_+[x(s,ds),s,ds]$ and $V^\mu{}_-[x(s,ds),s,ds]$. Each of them can be in its turn decomposed in terms of classical parts $v_+$ and $v_-$, and of fractal parts $w_+$ and $w_-$.

Then we define two "classical" derivatives $d_+/ds$ and $d_-/ds$, which, when they are applied to $x^\mu$, yield the "classical" velocity fields

$$\frac{d_+}{ds}x^\mu(s) = v_+^\mu \qquad \frac{d_-}{ds}x^\mu(s) = v_-^\mu. \tag{2.11}$$

Since there is no reason to privilegize one process rather than the other, we consider both $(+)$ and $(-)$ processes on the same footing, and we combine them in a unique twin process in terms of which the microscopic reversibility is recovered [2]. We shall in this contribution study in more detail the justification of the use of complex numbers to account for this doubling [10]. This choice, as we shall see, is, in the scale-relativity framework, at the origin of the complex nature of the wavefunction of quantum mechanics.

## 2.4 The Schrödinger equation as a geodesics equation in a fractal space

When this description is applied to the three-dimensional case of a fractal space (without fractal time), one recovers a non-relativistic situation ($v \ll c$) and a generalized Schrödinger equation is derived [2, 3, 10, 14, 15]. We shall in what follows recall the main steps of this derivation, including some improvements to previous works.

*Complex velocity*

In the non-relativistic case that is considered here, $ds$ is replaced by $dt$. One describes the elementary displacements $dX^k$, $k=1,2,3$, on the geodesics of a nondifferentiable fractal space-time in terms of the sum of two terms (omitting the indices for simplicity)

$$dX_\pm = dx_\pm + d\xi_\pm, \tag{2.12}$$

$d\xi_\pm$ representing the "fractal (differentiable) part" and $dx$, the "classical (non-differentiable) part", defined as

$$dx_\pm = v_\pm \; dt, \tag{2.13}$$

$$d\xi_\pm = \eta\sqrt{2\mathcal{D}}dt^{1/2}, \tag{2.14}$$

where $\eta$ is a stochastic variable such that $<\eta>=0$ and $<\eta^2>=1$. The two time derivatives are combined to obtain a complex derivative operator, that allows us to recover local differential time reversibility in terms of the new complex process [2]:

$$\frac{\bar{d}}{dt} = \frac{1}{2}\left(\frac{d_+}{dt} + \frac{d_-}{dt}\right) - \frac{i}{2}\left(\frac{d_+}{dt} - \frac{d_-}{dt}\right) . \tag{2.15}$$

Applying this operator to the position vector yields a complex velocity

$$\mathcal{V} = \frac{\bar{d}}{dt}x(t) = V - iU = \frac{v_+ + v_-}{2} - i\,\frac{v_+ - v_-}{2} . \tag{2.16}$$

The minus sign in front of the imaginary term is chosen here in order to obtain the Schrödinger equation in terms of $\psi$. The reverse choice would give the Schrödinger equation for the complex conjugate of the wave function $\psi^\dagger$, and would therefore be physically equivalent.

The real part, $V$, of the complex velocity, $\mathcal{V}$, represents the standard classical velocity, while its imaginary part, $-U$, is a new quantity arising from non-differentiability. At the usual classical limit, $v_+ = v_- = v$, so that $V = v$ and $U = 0$.

*Complex time-derivative operator*

In the case of a fractal dimension $D_F = 2$, as considered here, the total derivative should be written up to the second order partial derivative:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\frac{dX_i}{dt} + \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}\frac{dX_i dX_j}{dt} . \tag{2.17}$$

Let us now consider the 'classical part' of this expression. By definition, $\langle dX \rangle = dx$, so that the second term is reduced to $v.\nabla f$. Now concerning the term $dX_i dX_j/dt$, it is usually infinitesimal, but here its classical part reduces to $\langle d\xi_i \; d\xi_j \rangle/dt$. Therefore the last term of the classical part of Eq. (2.17) amounts to a Laplacian, and we obtain

$$\frac{d_\pm f}{dt} = \left(\frac{\partial}{\partial t} + v_\pm.\nabla \pm \mathcal{D}\Delta\right) f . \tag{2.18}$$

Substituting Eqs. (2.18) into Eq. (2.15), we finally obtain the expression for the complex time derivative operator [2]

$$\frac{\bar{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\mathcal{D}\Delta \; . \tag{2.19}$$

This operator $\bar{d}/dt$ plays the role of a "covariant derivative operator", namely, it will be used to write the fundamental equation of dynamics under the same form it had in the classical and differentiable case.

However, it should be remarked that the true status of the new derivative is actually different from the covariant derivative of general relativity, which amounts to subtract the effects of curvature in order to keep only the inertial part of the motion. Here, it is rather an extension of the concept of total derivative. Already in standard physics, the passage from the free Galileo-Newton's equation to its Euler form was a case of conservation of the form of equations in a more complicated situation, namely, $dv/dt = 0 \to (d/dt)v = (\partial/\partial t + v.\nabla)v = 0$. In the fractal and non-differentiable situation considered here, the three consequences of the new geometry (infinity of geodesics, fractality and two-valuedness of classical velocity) lead to add three new terms in the total derivative operator, namely $V.\nabla$, $-iU.\nabla$ and $-i\mathcal{D}\Delta$.

*Covariant mechanics induced by scale laws*

Let us now summarize the main steps by which one may generalize the standard classical mechanics using this covariance. We are now looking to motion in the standard space. In what follows, we therefore consider only the 'classical parts' of the variables, which are differentiable and independent of resolutions. The effects of the internal non-differentiable structures are now contained in the covariant derivative. We assume that the classical part' of the mechanical system under consideration can be characterized by a Lagrange function that keeps the usual form but now in terms of the complex velocity, $\mathcal{L}(x, \mathcal{V}, t)$, from which an action $\mathcal{S}$ is defined

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}(x, \mathcal{V}, t) dt. \tag{2.20}$$

In this expression, we have combined the twin velocities in terms of a unique complex velocity, while, in a general way, the Lagrange function is expected to be a function of the variables $x$ and their time derivatives $\dot{x}$. We have found that the number of velocity components $\dot{x}$ is doubled, so that we are led to write

$$L = L(x, \dot{x}_+, \dot{x}_-, t). \tag{2.21}$$

Instead, we have made the choice to write the Lagrange function as $L = L(x, \mathcal{V}, t)$. We now justify this choice by the covariance principle. Re-expressed in terms of $\dot{x}_+$ and $\dot{x}_-$ , this Lagrange function writes

$$L = L\left(x, \frac{1-i}{2}\,\dot{x}_+ + \frac{1+i}{2}\,\dot{x}_-, t\right). \tag{2.22}$$

Therefore we obtain

$$\frac{\partial L}{\partial \dot{x}_+} = \frac{1-i}{2}\,\frac{\partial L}{\partial \mathcal{V}} \quad ; \quad \frac{\partial L}{\partial \dot{x}_-} = \frac{1+i}{2}\,\frac{\partial L}{\partial \mathcal{V}}, \tag{2.23}$$

while the new covariant time derivative operator writes

$$\frac{\bar{d}}{dt} = \frac{1-i}{2}\,\frac{d_+}{dt} + \frac{1+i}{2}\,\frac{d_-}{dt}\,. \tag{2.24}$$

Let us write the stationary action principle in terms of the Lagrange function of Eq. (2.21)

$$\delta S = \delta \int_{t_1}^{t_2} L(x, \dot{x}_+, \dot{x}_-, t)\; dt = 0. \tag{2.25}$$

It becomes

$$\int_{t_1}^{t_2} \left( \frac{\partial L}{\partial x}\,\delta x + \frac{\partial L}{\partial \dot{x}_+}\,\delta \dot{x}_+ + \frac{\partial L}{\partial \dot{x}_-}\,\delta \dot{x}_- \right)\, dt = 0. \tag{2.26}$$

Since $\delta \dot{x}_+ = d_+(\delta x)/dt$ and $\delta \dot{x}_- = d_-(\delta x)/dt$, it takes the form

$$\int_{t_1}^{t_2} \left( \frac{\partial L}{\partial x}\,\delta x + \frac{\partial L}{\partial \mathcal{V}}\, \left[ \frac{1-i}{2}\,\frac{d_+}{dt} + \frac{1+i}{2}\,\frac{d_-}{dt} \right] \delta x \right)\, dt = 0, \tag{2.27}$$

i.e.,

$$\int_{t_1}^{t_2} \left( \frac{\partial L}{\partial x}\,\delta x + \frac{\partial L}{\partial \mathcal{V}}\,\frac{\bar{d}}{dt}\,\delta x \right)\, dt = 0. \tag{2.28}$$

The subsequent demonstration of the Lagrange equations from the stationary action principle relies on an integration by part. This integration by part cannot be performed in the usual way without a specific analysis, because it involves the new covariant derivative.

The first point to be considered is that such an operation involves the Leibniz rule for the covariant derivative operator $\bar{d}/dt$. Since $\bar{d}/dt = \partial/dt + \mathcal{V} . \nabla - i\mathcal{D}\Delta$ is a linear combination of first and second order derivatives, the same is true of its Leibniz rule. This implies the appearance of an additional term in the expression for the derivative of a product (see [16]):

$$\frac{\bar{d}}{dt}\left(\frac{\partial L}{\partial \mathcal{V}}\,.\,\delta x\right) = \frac{\bar{d}}{dt}\left(\frac{\partial L}{\partial \mathcal{V}}\right)\,.\,\delta x + \frac{\partial L}{\partial \mathcal{V}}\,.\,\frac{\bar{d}}{dt}\delta x - 2i\,\mathcal{D}\nabla\left(\frac{\partial L}{\partial \mathcal{V}}\right)\,.\,\nabla \delta x. \tag{2.29}$$

Since $\delta x(t)$ is not a function of $x$, the additional term vanishes. Therefore the above integral becomes

$$\int_{t_1}^{t_2} \left[ \left( \frac{\partial L}{\partial x} - \frac{\bar{d}}{dt} \frac{\partial L}{\partial \mathcal{V}} \right) \delta x + \frac{\bar{d}}{dt} \left( \frac{\partial L}{\partial \mathcal{V}} \, . \, \delta x \right) \right] dt = 0. \tag{2.30}$$

The second point is concerned with the integration of the covariant derivative itself. We define a new integral as being the inverse operation of the covariant derivation, i.e.,

$$⨍ \bar{d} f = f \tag{2.31}$$

in terms of which one obtains

$$⨍_{t_1}^{t_2} \bar{d} \left( \frac{\partial L}{\partial \mathcal{V}} \, . \, \delta x \right) = \left[ \frac{\partial L}{\partial \mathcal{V}} \, . \, \delta x \right]_{t_1}^{t_2} = 0, \tag{2.32}$$

since $\delta x(t_1) = \delta x(t_2) = 0$ by definition of the variation principle. Therefore the action integral becomes

$$\int_{t_1}^{t_2} \left( \frac{\partial L}{\partial x} - \frac{\bar{d}}{dt} \frac{\partial L}{\partial \mathcal{V}} \right) \delta x \, dt = 0. \tag{2.33}$$

And finally we obtain generalized Euler-Lagrange equations that read

$$\frac{\bar{d}}{dt} \frac{\partial L}{\partial \mathcal{V}} = \frac{\partial L}{\partial x}. \tag{2.34}$$

Therefore, thanks to the transformation $d/dt \to \bar{d}/dt$, they take exactly their standard classical form. This result reinforces the identification of our tool with a "quantum-covariant" representation, since, as we have shown in previous works and as we recall in what follows, this Euler-Lagrange equation can be integrated in the form of a Schrödinger equation.

Since we now consider only the 'classical parts' of the variables (while the effects on them of the fractal parts are included in the covariant derivative) the basic symmetries of classical physics hold. From the homogeneity of standard space one defines a generalized complex momentum given by

$$\mathcal{P} = \frac{\partial \mathcal{L}}{\partial \mathcal{V}}. \tag{2.35}$$

If we now consider the action as a functional of the upper limit of integration in Eq. (2.20), the variation of the action from a trajectory to another nearby trajectory yields a generalization of another well-known relation of standard mechanics,

$$\mathcal{P} = \nabla \mathcal{S}. \tag{2.36}$$

As concerns the generalized energy, its expression involves an additional term [16, 17, 19]: namely it write for a Newtonian Lagrange function and in the absence of exterior potential, $\mathcal{E} = (1/2)m(\mathcal{V}^2 - 2i\,\mathcal{D}\,\mathrm{div}\mathcal{V})$.

*Geodesics equation*

Let us now specialize our study, and consider Newtonian mechanics, i.e., the general case when the structuring external scalar field is described by a potential energy $\Phi$. The Lagrange function of a closed system, $L = \frac{1}{2}mv^2 - \Phi$, is generalized, in the large-scale domain, as $\mathcal{L}(x, \mathcal{V}, t) = \frac{1}{2}m\mathcal{V}^2 - \Phi$. The Euler-Lagrange equations keep the form of Newton's fundamental equation of dynamics

$$m\frac{\bar{d}}{dt}\mathcal{V} = -\nabla\Phi, \tag{2.37}$$

which is now written in terms of complex variables and complex operators.

In the case when there is no external field, the strong covariance is explicit, since Eq. (2.37) takes the form of the equation of inertial motion, i.e., of a geodesics equation,

$$\bar{d}\mathcal{V}/dt = 0. \tag{2.38}$$

In the case when $\Phi$ is a gravitational potential, one can combine the scale-relativity and motion-relativity covariant derivative and equation (2.37) also becomes a geodesics equation [18].

In both cases, with or without external field, the complex momentum $\mathcal{P}$ reads

$$\mathcal{P} = m\mathcal{V}, \tag{2.39}$$

so that, from Eq. (2.36), the complex velocity $\mathcal{V}$ appears as a gradient, namely the gradient of the complex action

$$\mathcal{V} = \nabla\mathcal{S}/m. \tag{2.40}$$

We now introduce a complex wave function $\psi$ which is nothing but another expression for the complex action $\mathcal{S}$

$$\psi = e^{i\mathcal{S}/\mathcal{S}_0}. \tag{2.41}$$

The factor $\mathcal{S}_0$ has the dimension of an action (i.e., an angular momentum) and must be introduced for dimensional reasons. When this formalism is applied to microphysics, $S_0$ is nothing but the fundamental constant $\hbar$. The function $\psi$ is related to the complex velocity appearing in Eq. (2.40) as follows

$$\mathcal{V} = -i\,\frac{\mathcal{S}_0}{m}\,\nabla(\ln\psi). \tag{2.42}$$

We have now at our disposal all the mathematical tools needed to write the fundamental equation of dynamics of Eq. (2.37) in terms of the new quantity $\psi$. It takes the form

$$i\mathcal{S}_0\frac{\bar{d}}{dt}(\nabla\ln\psi) = \nabla\Phi. \tag{2.43}$$

Now one should be aware that $\bar{d}$ and $\nabla$ do not commute. However, as we shall see in the following, $\bar{d}(\nabla \ln \psi)/dt$ is nevertheless always a gradient.

Replacing $\bar{d}/dt$ by its expression, given by Eq. (2.19), yields

$$\nabla \Phi = i\mathcal{S}_0 \left( \frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\mathcal{D}\Delta \right) (\nabla \ln \psi), \tag{2.44}$$

and replacing once again $\mathcal{V}$ by its expression in Eq. (2.42), we obtain

$$\nabla \Phi = i\mathcal{S}_0 \left[ \frac{\partial}{\partial t} \nabla \ln \psi - i \left\{ \frac{\mathcal{S}_0}{m} (\nabla \ln \psi.\nabla)(\nabla \ln \psi) + \mathcal{D}\Delta(\nabla \ln \psi) \right\} \right]. \tag{2.45}$$

*Improved remarkable identity*

Let us now show that one can derive without any additional hypothesis both the Compton relation and the Schrödinger equation.

Let us first prove the following remarkable identity:

$$\frac{1}{\alpha}\, \nabla \left( \frac{\Delta R^\alpha}{R^\alpha} \right) = 2\alpha (\nabla \ln R.\nabla)(\nabla \ln R) + \Delta(\nabla \ln R), \tag{2.46}$$

which is a generalisation of the identity used in previous works [2] to derive a generalized Schrödinger equation. Let us indeed start from the relation

$$(\nabla \ln R)^2 + \Delta \ln R = \frac{\Delta R}{R}\,, \tag{2.47}$$

which proceeds from the following derivation

$$\begin{aligned} \partial_\mu \partial^\mu \ln R + \partial_\mu \ln R\, \partial^\mu \ln R &= \partial_\mu \frac{\partial^\mu R}{R} + \frac{\partial_\mu R}{R} \frac{\partial^\mu R}{R} \\ &= \frac{R\, \partial_\mu \partial^\mu R - \partial_\mu R\, \partial^\mu R}{R^2} + \frac{\partial_\mu R\, \partial^\mu R}{R^2} \\ &= \frac{\partial_\mu \partial^\mu R}{R}\,. \end{aligned} \tag{2.48}$$

By taking its gradient, we finally obtain

$$\nabla \left( \frac{\Delta R}{R} \right) = 2(\nabla \ln R.\nabla)(\nabla \ln R) + \Delta(\nabla \ln R). \tag{2.49}$$

Its generalization to equation (2.46) is easy to obtain thanks to the fact that $R$ appears only through its logarithm in the right-hand side of the above equation. By replacing in it $R$ by $R^\alpha$, we get the general remarkable identity (Eq. 2.46) in a straighforward way.

*Derivation of Schrödinger equation and Compton relation*

Let us now write equation (2.45) under the form

$$\nabla\phi = i\mathcal{S}_0 \left[\frac{\partial}{\partial t}\nabla \ln\psi - i\mathcal{D}\left\{\frac{\mathcal{S}_0}{m\mathcal{D}}(\nabla \ln\psi.\nabla)(\nabla \ln\psi) + \Delta(\nabla \ln\psi)\right\}\right]. \tag{2.50}$$

Then using the generalized identity (2.46) in which we set $\alpha = S_0/2m\mathcal{D}$, we obtain

$$\frac{\mathcal{S}_0}{m\mathcal{D}}\,(\nabla \ln\psi.\nabla)(\nabla \ln\psi) + \Delta(\nabla \ln\psi) = \frac{2m\mathcal{D}}{S_0}\,\nabla\left(\frac{\Delta\psi^{\frac{S_0}{2m\mathcal{D}}}}{\psi^{\frac{S_0}{2m\mathcal{D}}}}\right). \tag{2.51}$$

Therefore the right-hand side of equation (2.50) becomes a gradient whatever the value of $S_0$, namely

$$\nabla\phi = i\mathcal{S}_0\,\nabla\left[\frac{\partial}{\partial t}\ln\psi - i\,\frac{2m\mathcal{D}^2}{S_0}\,\frac{\Delta\psi^{\frac{S_0}{2m\mathcal{D}}}}{\psi^{\frac{S_0}{2m\mathcal{D}}}}\right]. \tag{2.52}$$

Let us now set

$$\tilde{\psi} = \psi^{\frac{S_0}{2m\mathcal{D}}} = e^{i\frac{S}{2m\mathcal{D}}}. \tag{2.53}$$

We have $\ln\psi = (2m\mathcal{D}/S_0)\ln\tilde{\psi}$ and equation (2.52) becomes

$$\nabla\phi = i\,2m\mathcal{D}\,\nabla\left[\frac{\partial\tilde{\psi}/\partial t - i\,\mathcal{D}\Delta\tilde{\psi}}{\tilde{\psi}}\right], \tag{2.54}$$

which can finally be integrated as a generalized Schrödinger equation

$$\mathcal{D}^2\,\Delta\tilde{\psi} + i\mathcal{D}\frac{\partial}{\partial t}\tilde{\psi} - \frac{\phi}{2m}\tilde{\psi} = 0. \tag{2.55}$$

As concerns this new function $\tilde{\psi}$, the constant $S_0$ is now given by:

$$S_0 = 2m\mathcal{D}. \tag{2.56}$$

Therefore, in this general proof, the form of this constant, that appears in the initial definition of the wave function $\tilde{\psi}$, which is itself solution of this linear Schrödinger-type equation, is no longer assumed but is instead now derived.

This relation is nothing but a generalized Compton relation, to which a geometric interpretation is now provided. Indeed, the standard quantum case corresponds to the particular case when $S_0$ is a fundamental constant, $S_0 = \hbar$, while the parameter $\mathcal{D} = \langle d\xi^2\rangle/2dt$, which defines the amplitude of the fractal fluctuations, may be re-expressed in terms of a length-scale $\lambda = 2\mathcal{D}/c$. In this case equation (2.56) becomes $\lambda = \hbar/mc$, i.e. the Compton relation.

In other words, both the generalized Schrödinger equation and the generalized Compton relation are obtained together in this derivation.

*Derivation of Born's and Von Neumann's postulates*

The Born postulate, according to which $P = |\psi|^2$ gives the probability of presence of the particle, can now be inferred from the scale-relativity construction.

Indeed, we have identified the "particle" with the various geometric properties of a sub-set of the fractal geodesics of a non-differentiable space-time. In such an interpretation, a measurement (and more generally any knowledge about the system) amounts to a selection of the sub-sample of the geodesics family in which are kept only the geodesics having the geometric properties corresponding to the measurement result. Therefore, just after the measurement, the system is in the state given by the measurement result, in accordance with the von Neumann postulate of quantum mechanics.

As a consequence, the probability for the "particle" to be found at a given position must be proportional to the density of the geodesics fluid. We already know its velocity field, which is expected to be given by $V(x,t)$, identified, at the classical limit, with a classical velocity field. The geodesics density $\rho$ has not yet been introduced at this level of the construction (contrarily to most stochastic approaches where it is introduced from the very beginning and is used to define averages). In order to calculate it, we remark that it is expected to be a solution of a fluid-like Euler + continuity system of equations, namely,

$$\left(\frac{\partial}{\partial t} + V \cdot \nabla\right) V = -\nabla\left(\phi + Q\right), \tag{2.57}$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho V) = 0, \tag{2.58}$$

where $\phi$ is an external scalar potential and $Q$ is the potential that is expected to appear as a manifestation of the fractal geometry of space (in analogy with the appearance of the Newtonian potential as a manifestation of the curved geometry in general relativity). This is a system of four equations for four unknowns, $(\rho, V_x, V_y, V_z)$, that would be therefore completely determined by such a system.

Now, by separating the real and imaginary parts of the Schrödinger equation and by using a mixed representation, $(P, V)$ [instead of $(V, U)$ in the geodesics form and of $\psi = \sqrt{P} \times e^{i\theta}$ in the Schrödinger form], one obtains precisely such a standard system of fluid dynamics equations, namely,

$$\left(\frac{\partial}{\partial t} + V \cdot \nabla\right) V = -\nabla\left(\phi - 2\mathcal{D}^2 \frac{\Delta\sqrt{P}}{\sqrt{P}}\right), \tag{2.59}$$

$$\frac{\partial P}{\partial t} + \operatorname{div}(PV) = 0. \tag{2.60}$$

The comparison with the expected equations (2.57,2.58) allows one to univoquely identify $P = |\psi|^2$ with the probability density of the geodesics and therefore with the probability of presence of the 'particle'. This generalizes the derivation of Ref. [10] which was restricted to the case of separable variables. Moreover, one identifies

$$Q = -2\mathcal{D}^2 \frac{\Delta\sqrt{P}}{\sqrt{P}} \tag{2.61}$$

with the new potential which emerges from the fractal geometry [19]. Numerical simulations [20, 21], in which the expected probability density can be obtained directly from the distribution of geodesics without writing the Schrödinger equation, have confirmed this result. Moreover, we have recently suggested [21] to achieve a new kind of quantum-like system in a laboratory experiment by applying such a generalized quantum potential to a classical fluid though a retro-active loop involving real time measurements of its density. Numerical simulations of such an experiment have given encouraging results [Nottale and Lehner, in preparation].

*Generalization to non-differentiable wave function*

In the above derivation of the Schrödinger equation, only the classical part of the velocity was taken into account when defining the wave function. However, the full velocity field of the fractal space-time geodesics actually contains a formally infinite term, which manifests the nondifferentiability of space-time. We shall now show that this non-differentiability of space-time is expected to manifest itself in terms of a possible nondifferentiability of wave functions. Such a result agrees with Berry's [22] and Hall's [23] similar finding in the framework of standard quantum mechanics.

Each of the two fractal velocity fields can be decomposed in terms of a classical (differentiable) part and a fractal (nondifferentiable) part:

$$V_+(t, dt) = v_+(t) + w_+(t, dt), \quad V_-(t, dt) = v_-(t) + w_-(t, dt). \tag{2.62}$$

The next step of the demonstration consists of combining the two velocities into a velocity doublet. We can write the velocity doublet under the form

$$\tilde{\mathcal{V}} = \mathcal{V} + \mathcal{W} = \left( \frac{v_+ + v_-}{2} - i\,\frac{v_+ - v_-}{2} \right) + \left( \frac{w_+ + w_-}{2} - i\,\frac{w_+ - w_-}{2} \right), \tag{2.63}$$

that now includes the classical and the fractal parts.

Let us now consider the full complex action, now defined in terms of the classical part *and of the fractal (divergent) part* of the velocity,

$$d\tilde{\mathcal{S}} = \frac{1}{2} m (\mathcal{V} + \mathcal{W})^2 \, dt, \tag{2.64}$$

which is equal (see Section 3) to the previous mean action $d\mathcal{S} = (1/2)m\mathcal{V}^2 dt$ plus terms of zero stochastic average.

We can define a complete wavefunction $\tilde{\psi}$ from this full action $\tilde{\mathcal{S}}$:

$$\tilde{\psi} = e^{i\tilde{\mathcal{S}}/2m\mathcal{D}}, \tag{2.65}$$

and the relation of the complete complex velocity to the complete wavefunction therefore reads

$$\tilde{\mathcal{V}} = \mathcal{V} + \mathcal{W} = \nabla\tilde{\mathcal{S}}/m = -2i\mathcal{D}\nabla \ln \tilde{\psi}. \tag{2.66}$$

Under the standard point of view, the complex fluctuation $\mathcal{W}$ is infinite and therefore $\nabla \ln \tilde{\psi}$ is undefined, so that equation (2.66) would be meaningless. In the scale-relativity approach, on the contrary, this equation keeps a mathematical and physical meaning, in terms of fractal functions, which are explicitly dependent on the scale interval $dt$ and divergent when $dt \to 0$.

In both cases, this behavior points to the fact that the wavefunction $\tilde{\psi}$ defined hereabove is non-differentiable.

The last step consists of proving that this nondifferentiable wavefunction remains solution of a Schrödinger equation.

Let us write the fractal parts of the velocities under the form:

$$w_+ = \eta_+ \sqrt{\frac{2\mathcal{D}}{dt}}, \quad w_- = \eta_- \sqrt{\frac{2\mathcal{D}}{dt}}, \tag{2.67}$$

where $\eta_+$ and $\eta_-$ are stochastic variables such that $<\eta_+>=<\eta_->= 0$ and $<\eta_+^2>=<\eta_-^2>= 1$. Though $w_+(t,0)$ and $w_-(t,0)$ are infinite (which is the manifestation of the nondifferentiability), $w_+(t,dt)$ and $w_-(t,dt)$ are nevertheless completely defined as explicit functions of $dt$.

The two $(+)$ and $(-)$ derivatives read

$$\frac{d_+ f}{dt} = \frac{\partial f}{\partial t} + (v_+ + w_+)\nabla f + \mathcal{D}\,\eta_+^2\,\Delta f + ..., \tag{2.68}$$

$$\frac{d_- f}{dt} = \frac{\partial f}{\partial t} + (v_- + w_-)\nabla f - \mathcal{D}\,\eta_-^2\,\Delta f + ..., \tag{2.69}$$

where the next terms are infinitesimals. Let us now define the following complex stochastic variables:

$$\tilde{\eta} = \frac{\eta_+ + \eta_-}{2} - i\,\frac{\eta_+ - \eta_-}{2}, \tag{2.70}$$

$$1 + \tilde{\zeta} = \frac{\eta_+^2 + \eta_-^2}{2} + i\,\frac{\eta_+^2 - \eta_-^2}{2}, \tag{2.71}$$

which are such that $<\tilde{\eta}>= 0$ and $<\tilde{\zeta}>= 0$. We can now combine the two derivatives in terms of a further generalized complex covariant derivative

$$\frac{\widehat{d}}{dt} = \frac{\partial}{\partial t} + (\mathcal{V} + \mathcal{W}).\nabla - i\mathcal{D}(1 + \tilde{\zeta})\Delta \tag{2.72}$$

plus infinitesimal terms that vanish when $dt \to 0$. This generalization can be written under the form

$$\frac{\widehat{d}}{dt} = \left[\frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\mathcal{D}\Delta\right] + \sqrt{\frac{2\mathcal{D}}{dt}}\,\tilde{\eta}.\nabla - i\mathcal{D}\,\tilde{\zeta}\,\Delta. \tag{2.73}$$

We therefore recover the mean covariant derivative introduced in previous works [2, 3, 10], namely $\bar{d}/dt = \partial/\partial t + \mathcal{V}.\nabla - i\mathcal{D}\Delta$, plus two additional stochastic terms of zero mean, the first being $\mathcal{W}.\nabla$, which is infinite at the limit $dt \to 0$, and the second $-i\mathcal{D}\,\tilde{\zeta}\,\Delta$, in which $\tilde{\zeta}$ remains finite. The first of these terms was already introduced in Ref.[24], while the second was neglected since their ratio is an infinitesimal: indeed, their sum can be written under the form $\sqrt{2\mathcal{D}/dt}\,(\tilde{\eta}.\nabla - i\sqrt{\mathcal{D}/2}\,dt^{1/2}\,\tilde{\zeta}\,\Delta)$.

We are now able, using this covariant derivative, to write a complete equation of motion for a free "particle" in terms of a geodesics equation that keeps the form of the free Galilean inertial motion equation [24]:

$$\frac{\widehat{d}}{dt}\tilde{\mathcal{V}} = 0. \tag{2.74}$$

In the presence of a potential $\phi$, it can be easily generalized in terms of a covariant equation which keeps the form of Newton's fundamental equation of dynamics:

$$\frac{\widehat{d}}{dt}\tilde{\mathcal{V}} = -\frac{\nabla\phi}{m}. \tag{2.75}$$

After expansion of this equation, new terms of zero stochastic mean are now added [24] with respect to the previous incomplete form of the equation $\bar{d}\mathcal{V}/dt = -\nabla\phi/m$ [2]:

$$\frac{\bar{d}\mathcal{V}}{dt} + \left(\frac{\partial\mathcal{W}}{\partial t} + \nabla(\mathcal{V}.\mathcal{W}) + \mathcal{W}.\nabla\mathcal{W} - i\mathcal{D}\Delta\mathcal{W}\right) - i\mathcal{D}\,\tilde{\zeta}\,\Delta(\mathcal{V} + \mathcal{W}) = -\frac{\nabla\phi}{m}. \tag{2.76}$$

Starting from the full geodesics equation (2.74), and more generally from (2.75), we now expand the covariant derivative and we find

$$\left(\frac{\partial}{\partial t} + \tilde{\mathcal{V}}.\nabla - i\mathcal{D}(1 + \tilde{\zeta})\Delta\right)\tilde{\mathcal{V}} = -\frac{\nabla\phi}{m}, \tag{2.77}$$

of which equation (2.76) is an expansion. Now, we have seen that the stochastic term $\tilde{\zeta}\Delta\tilde{\mathcal{V}}$ is infinitesimal with respect to the other stochastic term $\tilde{\mathcal{W}}.\nabla\tilde{\mathcal{V}}$, so that we can neglect it as we did in Ref. [24]. Now introducing the full wavefunction $\tilde{\psi}$ in this equation thanks to equation (2.66), we obtain

$$\left(\frac{\partial}{\partial t} + (-2i\mathcal{D}\nabla \ln \tilde{\psi}).\nabla - i\mathcal{D}\Delta\right)(-2i\mathcal{D}\nabla \ln \tilde{\psi}) = -\frac{\nabla\phi}{m}. \tag{2.78}$$

In the standard framework, the very writing of this equation would be forbidden since $\tilde{\psi}$ is nondifferentiable and therefore its derivatives are formally infinite. But, as recalled above, the fundamental tool used in the scale-relativity approach, which was definitely constructed to solve this kind of problems (at the level of fractal space-time coordinates), can now be used in a similar manner at the level of the wavefunction. Namely, in terms of a wavefunction $\tilde{\psi}(t, dt)$, the various terms of equation (2.78) remain finite for all values of $dt \neq 0$. We are therefore in the same conditions as in previous calculations involving a differentiable wave function [2, 10], so that it can finally be integrated in terms of a generalized Schrödinger equation that keeps the same form as in the differentiable wave function case, namely,

$$\mathcal{D}^2\Delta\tilde{\psi} + i\mathcal{D}\,\frac{\partial\tilde{\psi}}{\partial t} - \frac{\phi}{2m}\tilde{\psi} = 0. \tag{2.79}$$

This generalized Schrödinger equation now allows nondifferentiable solutions, which come, in our framework, as a direct manifestation of the nondifferentiability of space. The research in laboratory experiments of such a behavior constitute an interesting new challenge for quantum physics .

## 2.5 Dirac Equation

*Reflection symmetry breaking of coordinates differentials*

We have seen that in the framework of the theory of scale relativity, the non-differentiable geometry of space involves a symmetry breaking of the reflection invariance $dt \leftrightarrow -dt$, and therefore a two-valuedness of the classical velocity vector.

Going to motion-relativistic quantum mechanics amounts to introduce not only a fractal space, but a fractal space-time. The invariant parameter becomes in this case the proper time $s$ instead of the time $t$. As a consequence the complex nature of the four-dimensional wave function in the Klein-Gordon equation comes from the discrete symmetry breaking $ds \leftrightarrow -ds$.

However, this is not the last word of the new structures implied by the non-differentiability. The total derivative of a physical quantity also involves partial derivatives with respect to the space variables, $\partial/\partial x^\mu$. Once again, from the very definition of derivatives, the discrete symmetry under the reflection $dx^\mu \leftrightarrow -dx^\mu$ should also be broken at a more profound level of description. Therefore, in a general description, one expects a new two-valuedness of the generalized velocity.

At this level one should also account for parity, as in the standard quantum theory (this is the origin of the passage from Pauli spinors to Dirac bi-spinors). Finally, we have suggested that the three discrete symmetry breakings

$$ds \leftrightarrow -ds \qquad dx^\mu \leftrightarrow -dx^\mu \qquad x^\mu \leftrightarrow -x^\mu$$

can be accounted for by the introduction of a bi-quaternionic velocity. It has been subsequently shown [10] that one can derive in this way the Dirac equation, namely, as an integral of a geodesics equation. In other words, this means that this new two-valuedness is at the origin of the bi-spinor nature of the electron wave function. This demonstration is summarized in what follows.

*Spinors as bi-quaternionic wave-function*

Since $\mathcal{V}^\mu$ is now bi-quaternionic, the Lagrange function is also bi-quaternionic and, therefore, the same is true of the action. Moreover, it has been shown [10] that, for $s$-stationary processes, the bi-quaternionic generalisation of the quantum-covariant derivative keeps the same form as in the complex number case, namely,

$$\frac{\bar{d}\!\!\!/}{ds} = \mathcal{V}^\nu \partial_\nu + i\frac{\lambda}{2}\partial^\nu \partial_\nu, \tag{2.80}$$

where $\lambda = 2\mathcal{D}/c$.

A generalized equivalence principle, as well as a strong covariance principle, allows us to write the equation of motion under a free-motion form, i.e., under the form of a differential geodesics equation

$$\frac{\bar{d}\!\!\!/ \mathcal{V}_\mu}{ds} = 0, \tag{2.81}$$

where $\mathcal{V}_\mu$ is the bi-quaternionic four-velocity, e.g., the covariant counterpart of $\mathcal{V}^\mu$.

The elementary variation of the action, considered as a functional of the coordinates, keeps the usual form

$$\delta\mathcal{S} = -mc\,\mathcal{V}_\mu\,\delta x^\mu. \tag{2.82}$$

We thus obtain the bi-quaternionic four-momentum, as

$$\mathcal{P}_\mu = mc\mathcal{V}_\mu = -\partial_\mu \mathcal{S}. \tag{2.83}$$

We are now able to introduce the wave function. We define it as a re-expression of the bi-quaternionic action by

$$\psi^{-1}\partial_\mu\psi = \frac{i}{cS_0}\partial_\mu\mathcal{S}, \tag{2.84}$$

using, in the left-hand side, the quaternionic product. The bi-quaternionic four-velocity is derived from Eq. (2.83), as

$$\mathcal{V}_\mu = i\frac{S_0}{m}\psi^{-1}\partial_\mu\psi. \tag{2.85}$$

This is the bi-quaternionic generalization of the definition used in the Schrödinger case, $\psi = e^{iS/S_0}$.

Finally, the isomorphism which can be established between the quaternionic and spinorial algebrae through the multiplication rules applying to the Pauli spin matrices allows us to identify the wave function $\psi$ to a Dirac bispinor. Indeed, spinors and quaternions are both a representation of the SL(2,C) group. This identification is reinforced by the result [10] that follows, according to which the geodesics equation written in terms of bi-quaternions is naturally integrated under the form of the Dirac equation.

*Free-particle bi-quaternionic Klein-Gordon equation*

The equation of motion, Eq. (2.81), writes

$$\left(\mathcal{V}^\nu\partial_\nu + i\frac{\lambda}{2}\partial^\nu\partial_\nu\right)\mathcal{V}_\mu = 0. \tag{2.86}$$

We replace $\mathcal{V}_\mu$, (respectively $\mathcal{V}^\nu$), by their expressions given in Eq. (2.85) and obtain

$$i\frac{S_0}{m}\left(i\frac{S_0}{m}\psi^{-1}\partial^\nu\psi\partial_\nu + i\frac{\lambda}{2}\partial^\nu\partial_\nu\right)(\psi^{-1}\partial_\mu\psi) = 0. \tag{2.87}$$

The choice $\mathcal{S}_0 = m\lambda$ allows us to simplify this equation and we get

$$\psi^{-1}\partial^\nu\psi\ \partial_\nu(\psi^{-1}\partial_\mu\psi) + \frac{1}{2}\partial^\nu\partial_\nu(\psi^{-1}\partial_\mu\psi) = 0. \tag{2.88}$$

The definition of the inverse of a quaternion

$$\psi\psi^{-1} = \psi^{-1}\psi = 1, \tag{2.89}$$

implies that $\psi$ and $\psi^{-1}$ commute. But this is not necessarily the case for $\psi$ and $\partial_\mu\psi^{-1}$ nor for $\psi^{-1}$ and $\partial_\mu\psi$ and their contravariant counterparts. However, when we derive Eq. (2.89) with respect to the coordinates, we obtain

$$\begin{aligned}\psi\ \partial_\mu\psi^{-1} &= -(\partial_\mu\psi)\psi^{-1},\\ \psi^{-1}\partial_\mu\psi &= -(\partial_\mu\psi^{-1})\psi,\end{aligned} \tag{2.90}$$

and identical formulae for the contravariant analogues.

Developing Eq. (2.88), using Eqs. (2.90) and the property $\partial^\nu \partial_\nu \partial_\mu = \partial_\mu \partial^\nu \partial_\nu$, we obtain, after some calculations,

$$\partial_\mu [(\partial^\nu \partial_\nu \psi)\psi^{-1}] = 0. \tag{2.91}$$

We integrate this four-gradient as

$$\partial^\nu \partial_\nu \psi + C\psi = 0. \tag{2.92}$$

We therefore recognize the Klein-Gordon equation for a free particle with a mass $m$, after the identification $C = m^2 c^2/\hbar^2 = 1/\lambda^2$. But, in this equation, $\psi$ is now a biquaternion, i.e. a Dirac bispinor.

*Derivation of the Dirac equation*

We now use a long-known property of the quaternionic formalism, which allows to obtain the Dirac equation for a free particle as a mere square root of the Klein-Gordon operator (see Refs. in [10]).

We first develop the Klein-Gordon equation as

$$\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{m^2 c^2}{\hbar^2}\psi. \tag{2.93}$$

Thanks to the property of the quaternionic and complex imaginary units $e_1^2 = e_2^2 = e_3^2 = i^2 = -1$, we can write Eq. (2.93) under the form

$$\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = e_3^2 \frac{\partial^2 \psi}{\partial x^2} e_2^2 + i e_1^2 \frac{\partial^2 \psi}{\partial y^2} i + e_3^2 \frac{\partial^2 \psi}{\partial z^2} e_1^2 + i^2 \frac{m^2 c^2}{\hbar^2} e_3^2 \psi e_3^2. \tag{2.94}$$

We now take advantage of the anticommutative property of the quaternionic units ($e_i e_j = -e_j e_i$ for $i \neq j$) to add to the right-hand side of Eq. (2.94) six vanishing couples of terms which we rearrange as

$$\begin{aligned}
\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \psi}{\partial t}\right) = {} & e_3 \frac{\partial}{\partial x}\left(e_3 \frac{\partial \psi}{\partial x} e_2 + e_1 \frac{\partial \psi}{\partial y} i + e_3 \frac{\partial \psi}{\partial z} e_1 - i\frac{mc}{\hbar} e_3 \psi e_3\right) e_2 \\
& + e_1 \frac{\partial}{\partial y}\left(e_3 \frac{\partial \psi}{\partial x} e_2 + e_1 \frac{\partial \psi}{\partial y} i + e_3 \frac{\partial \psi}{\partial z} e_1 - i\frac{mc}{\hbar} e_3 \psi e_3\right) i \\
& + e_3 \frac{\partial}{\partial z}\left(e_3 \frac{\partial \psi}{\partial x} e_2 + e_1 \frac{\partial \psi}{\partial y} i + e_3 \frac{\partial \psi}{\partial z} e_1 - i\frac{mc}{\hbar} e_3 \psi e_3\right) e_1 \\
& - i\frac{mc}{\hbar} e_3 \left(e_3 \frac{\partial \psi}{\partial x} e_2 + e_1 \frac{\partial \psi}{\partial y} i + e_3 \frac{\partial \psi}{\partial z} e_1 - i\frac{mc}{\hbar} e_3 \psi e_3\right) e_3.
\end{aligned} \tag{2.95}$$

We see that Eq. (2.95) is obtained by applying two times to the biquaternionic wavefunction $\psi$ the operator

$$\frac{1}{c}\frac{\partial}{\partial t} = e_3 \frac{\partial}{\partial x} e_2 + e_1 \frac{\partial}{\partial y} i + e_3 \frac{\partial}{\partial z} e_1 - i\frac{mc}{\hbar} e_3 (\quad) e_3. \tag{2.96}$$

The three first Conway matrices $e_3(\quad)e_2$, $e_1(\quad)i$ and $e_3(\quad)e_1$ [25], appearing in the right-hand side of Eq. (2.96), can be written in the compact form $-\alpha^k$, with

$$\alpha^k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix},$$

the $\sigma_k$'s being the three Pauli matrices, while the fourth Conway matrix

$$e_3(\quad)e_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

can be recognized as the Dirac $\beta$ matrix.

Finally, we can therefore write Eq. (2.96) as the non-covariant Dirac equation for a free fermion

$$\frac{1}{c}\frac{\partial\psi}{\partial t} = -\alpha^k \frac{\partial\psi}{\partial x^k} - i\frac{mc}{\hbar}\beta\psi. \tag{2.97}$$

The covariant form, in the Dirac representation, can be recovered by applying $ie_3(\quad)e_3$ to Eq. (2.97).

*Pauli equation*

Finally it is easy to derive the Pauli equation and Pauli spinors, since it is known that it can be obtained as a non-relativistic approximation of the Dirac equation [26]. Two of the components of the Dirac bi-spinor become negligible when $v \ll c$, so that they become Pauli spinors and the Dirac equation is transformed in a Schrödinger equation for these spinors with a magnetic dipole additional term which is doubled with respect to the non-relativistic expectation. Such an equation is the Pauli equation.

# 3 Origin of complex numbers and quaternions in quantum mechanics

## 3.1 Algebra doubling

Let us return to the step of our demonstration where complex numbers are introduced. All we know is that each component of the velocity now takes two values instead of one. This means that each component of the velocity becomes a vector in a two-dimensional space, or, in other words, that the velocity becomes a two-index tensor. So let us introduce generalized velocities

$$V^k_\sigma = (V^k, U^k) \;\; ; \;\; k = 1, 2, 3 \;\; ; \;\; \sigma = 1, 2. \tag{3.1}$$

This can be generalized to all other physical quantities affected by the two-valuedness: namely, scalars $A$ of the position space become vectors $A_\sigma$ of the new 2D-space, etc... . While the generalization of the sum of these quantities is straighforward, $C^k_\sigma = A^k_\sigma + B^k_\sigma$, the generalisation of the product remains at this level an open question, which should be derived from first principles.

The problem can be put in a general way: it amounts to find a generalization of the standard product that keeps its fundamental physical properties, or at least that keeps the most important ones (e.g., internal composition law), while some of them cannot escape to be lost (e.g., commutativity when jumping to quaternions).

From the mathematical point of view, we are here exactly confronted to the well-known problem of the doubling of algebra (see, e.g., Ref. [27]). Indeed, the effect of the symmetry breaking $dt \leftrightarrow -dt$ (or $ds \leftrightarrow -ds$) is to replace the algebra $\mathcal{A}$ in which the classical physical quantities are defined, by a direct sum of two exemplaries of $\mathcal{A}$, i.e., the space of the pairs $(a, b)$ where $a$ and $b$ belong to $\mathcal{A}$. The new vectorial space $\mathcal{A}^2$ must be supplied with a product in order to become itself an algebra (of doubled dimension). The same problem is asked again when one takes also into account the symmetry breakings $dx \leftrightarrow -dx$ and $x^\mu \leftrightarrow -x^\mu$: this leads to new algebra doublings. The mathematical solution to this problem is well-known: the standard algebra doubling amounts to supply $\mathcal{A}^2$ with the complex product. Then the doubling $\mathbb{R}^2$ of $\mathbb{R}$ is the algebra $\mathbb{C}$ of complex numbers, the doubling $\mathbb{C}^2$ of $\mathbb{C}$ is the algebra $\mathbb{H}$ of quaternions, the doubling $\mathbb{H}^2$ of quaternions is the algebra of Graves-Cayley octonions.

More generally, this means that such a process of complexification of the physical description tool, which is expected to be continued when including additional discrete quantum numbers (such as isospin, hypercharge, color, etc...) is naturally accounted for by the mean of Clifford algebrae. This allows one to bring a justification from first principles of the fact, which was recognized long time ago [28], that Clifford algebra provides a very useful tool for description of geometry and physics, as examplified by the development of the concept of Clifford space [29].

The problem with algebra doubling is that the iterative doubling leads to a progressive deterioration of the algebraic and other properties. Namely, one loses order relation in the complex plane (so that one can no longer use a least action principle, though this principle remains preserved in terms of its primary form of stationary action principle). When jumping to quaternions and biquaternions, one loses commutativity: this degradation was to be expected, since the non-commutativity of the geometry is indeed one of the main properties of a quantum geometry [30], a property that is therefore also derived from non-differentiability. Octonion algebra becomes non-associative: the question whether one will be led to develop a non-associative physics in the future remains an open problem. Now, an important positive result for physical applications is that the doubling of a metric algebra is a metric algebra [27].

These theorems about algebra doubling fully justify the use of complex numbers, then of quaternions, in order to describe the successive two-valuedness due to discrete symmetry breakings at the infinitesimal level, which are themselves more and more profound consequences of space-time non-differentiability.

However, we give in what follows complementary arguments of a physical nature, which show that the use of the complex product in the first algebra doubling have a simplifying and covariant effect (we use here the word "covariant" in the original and general meaning given to it by Einstein [31], namely, the requirement of the form invariance of fundamental equations).

### 3.2 Origin of complex numbers

In order to simplify the argument, let us consider the generalization of scalar quantities, for which the product law is the standard product in $\mathbb{R}$. Scalar quantities $A,\ B$ are replaced by multiplets $A^\alpha = \{A^1, A^2, A^3, \dots\}$, $B^\beta = \{B^1, B^2, B^3, \dots\}$.

The first constraint is that the new product must remain an internal composition law. We also make the simplifying assumption that it remains linear in terms of each of the components of the two quantities to be multiplied. This problem may be addressed in a general way for any length of the multiplets by writing a bilinear internal product under the form (using Einstein's summation convention on the indices),

$$C^\gamma = A^\alpha\, \omega^\gamma_{\alpha\beta}\, B^\beta, \tag{3.2}$$

where the three-dimensional "$\omega$-matrix" $\omega^\gamma_{\alpha\beta}$ is a tensor (analog for some of its aspects to the structure constants of a Lie group) that defines completely the new product.

In the case of doublets as considered here, the product is determined by eight real numbers. For example, the complex product is given by $\omega^1_{11} = 1$, $\omega^1_{22} = -1$, $\omega^2_{21} = 1$, $\omega^2_{12} = 1$ (and the four other numbers $= 0$). Study numbers (' hyperbolic complexes' such as $z = r \times e^{j\theta} = x + jy = r(\cosh\theta + j\sinh\theta)$ with $j^2 = 1$) are given by $\omega^1_{11} = 1$, $\omega^1_{22} = 1$, $\omega^2_{21} = 1$, $\omega^2_{12} = 1$.

In three dimensions (triplets) there are 27 numbers: for example, the non-zero coefficients that define the vectorial product are $\omega^1_{23} = 1$, $\omega^1_{32} = -1$, $\omega^2_{13} = 1$, $\omega^2_{31} = -1$, $\omega^3_{12} = 1$, $\omega^3_{21} = -1$. More generally, one may define the general properties of bilinear products in terms of relations between the $\omega^\gamma_{\alpha\beta}$ [LN, unpublished], but such a study goes beyond the scope of the present contribution.

The second physical constraint is that we recover the classical variables and the classical product at the classical limit. The mathematical equivalent of this constraint is the requirement that $\mathcal{A}$ still be a sub-algebra of $\mathcal{A}^2$. Therefore we identify $a_0 \in \mathcal{A}$ with $(a_0, 0)$ and we set $(0, 1) = \alpha$. This allows us to write the new two-dimensional vectors in the simplified form $a = a_0 + a_1\alpha$, so that the product now writes

$$c = (a_0 + a_1\alpha)\,(b_0 + b_1\alpha) = a_0b_0 + a_1b_1\alpha^2 + (a_0b_1 + a_1b_0)\alpha. \tag{3.3}$$

The problem is now reduced to find $\alpha^2$, i.e., only two $\omega$ coefficients instead of eight are to be determined, namely,

$$\alpha^2 = \omega_0 + \omega_1\alpha. \tag{3.4}$$

Let us now come back to the beginning of our construction. We have introduced two elementary displacements, each of them made of two terms, a classical part and a fluctuation,

$$\begin{aligned} dX_+(t) &= v_+\ dt + d\xi_+(t), \\ dX_-(t) &= v_-\ dt + d\xi_-(t). \end{aligned} \tag{3.5}$$

Therefore, one can define velocity fluctuations $w_+ = d\xi_+/dt$ and $w_- = d\xi_-/dt$, then a complete velocity in the doubled algebra [24]

$$\mathcal{V} + \mathcal{W} = \left(\frac{v_+ + v_-}{2} - \alpha\,\frac{v_+ - v_-}{2}\right) + \left(\frac{w_+ + w_-}{2} - \alpha\,\frac{w_+ - w_-}{2}\right). \tag{3.6}$$

Note that in terms of standard methods, this writing would be forbidden since the velocity $\mathcal{W}$ is infinite from the viewpoint of usual differential calculus. But we recall that we give meaning to this concept by considering it as an explicit function of the differential element $dt$, which becomes itself a variable (it is proportional to $dt^{-1/2}$).

Now, from the covariance principle, the classical part of the Lagrange function in the Newtonian case should strictly be written:

$$\mathcal{L} = \frac{1}{2}m\ \langle(\mathcal{V} + \mathcal{W})^2\rangle = \frac{1}{2}m\ \left(\langle\mathcal{V}^2\rangle + \langle\mathcal{W}^2\rangle\right) \tag{3.7}$$

We have $\langle\mathcal{W}\rangle = 0$, by definition, and $\langle\mathcal{V}.\mathcal{W}\rangle = 0$, because they are mutually independent. But what about $\langle\mathcal{W}^2\rangle$ ? The presence of this term would greatly complicate all the subsequent developments toward the Schrödinger equation, since it would imply a fundamental divergence of non-relativistic quantum mechanics. Note however that such a divergence finally happens in relativistic quantum field theories and leads to the renormalisation and renormalisation group approachs. Moreover, we have seen in the previous section that non-differentiable solutions of the Schrödinger equation, whose derivatives are divergent, were allowed and could be easily described in terms of explicitly scale-dependent fractal functions. Let us expand the $\langle\mathcal{W}^2\rangle$ term:

$$\begin{aligned} \langle\mathcal{W}^2\rangle &= \frac{1}{4}\langle[(w_+ + w_-) - \alpha\,(w_+ - w_-)]^2\rangle \\ &= \frac{1}{4}\langle(w_+^2 + w_-^2)(1+\alpha^2) - 2\alpha(w_+^2 - w_-^2) + 2w_+w_-(1-\alpha^2)\rangle. \end{aligned} \tag{3.8}$$

Since $\langle w_+^2 \rangle = \langle w_-^2 \rangle$ and $\langle w_+ w_- \rangle = 0$ (they are mutually independent), we finally find that $\langle \mathcal{W}^2 \rangle$ can only vanish provided

$$\alpha^2 = -1. \tag{3.9}$$

Therefore we have shown that the choice of the complex product in the algebra doubling plays an essential physical role, since it allows to suppress what would be additional infinite terms in the final equations of motion.

### 3.3 Origin of bi-quaternions

A last point that must be justified is the use of complex quaternions (bi-quaternions) for describing the new two-valuedness that leads to bi-spinors and the Dirac equation. One could think that the argument given in Sec. 3.1 (algebra doubling) leads to use Graves-Cayley octonions (and therefore to give up associativity) in the case of three successive doublings as considered here. However, these three doublings are not on the same footing from a physical point of view:

(i) The first two-valuedness comes from a discrete symmetry breaking at the level of the differential invariant, namely, $dt$ in the case of a fractal space (yielding the Schrödinger equation) and $ds$ in the case of a fractal space-time (yielding the Klein-Gordon equation). This means that it has an effect on the total derivatives $d/dt$ and $d/ds$. This two-valuedness is achieved by the introduction of complex variables.

(ii) The second two-valuedness (differential parity and time reversal violation, "dX"), comes from a new discrete symmetry breaking (expected from the giving up of the differentiability hypothesis) on the space-time differential element $dx^\mu \leftrightarrow -dx^\mu$. It is subsequent to the first two-valuedness, since it has an effect on the partial derivative $\partial/\partial x^\mu$ that intervenes in the complex covariant derivative operator, namely,

$$\frac{\bar{d}}{ds} = (\mathcal{V}^\mu + i\frac{\lambda}{2}\partial^\mu)\,\partial_\mu. \tag{3.10}$$

(iii) The third two-valuedness is a standard effect of parity ($P$ symmetry) and time reversal ($T$ symmetry) in the motion-relativistic situation, which is not specific of our approach and is already used in the standard construction of Dirac spinors.

Therefore, from the second and third doublings, complex numbers, then quaternions, can be introduced, which will affect variables which are already complex due to the first, more fundamental, doubling. This leads to the bi-quaternionic tool we use in the present work.

Let us conclude this section by noting that these symmetry breakings are effective only at the level of the underlying description of elementary displacements (namely, in the non-differentiable fractal space-time). The effect of introducing a two-valuedness of variables in terms of double symmetrical processes precisely amounts to recover symmetry in terms of the bi-process, and therefore in terms of the quantum tools which are built from it. But reversely, this remark opens a possible way of investigation for future research about the origin of other features characteristic of the microphysical world.

## 4 Conclusion

In this contribution, we have given an improved derivation of the physical and mathematical tools of quantum mechanics (complex then bi-spinorial wave functions) and of the equations they satisfy (Schrödinger then Dirac equation) in the scale relativity framework, which involves a nondifferentiable and fractal space-time geometry.

In particular, such an approach based on first principles allows one to suggest an answer to the question of the origin of complex and biquaternionic numbers, and more generally of Clifford algebra in quantum mechanics. We have attributed it to a direct consequence of nondifferentiability, which implies a fundamental two-valuedness of the fractal velocity field in the non-relativistic case, followed by new additional doublings in the relativistic case.

Several improvements of the previous construction of the theory are given in this contribution: they include a description of the metric of a fractal space-time, a twin derivation of both generalized Compton relation and Schödinger equation, a new generalization of the Schrödinger equation which allows for nondifferentiable solutions, a full derivation of the Born postulate, and new proposal to put the theory to the test in laboratory experiments.

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