Scale relativity, fractal space-time and morphogenesis of structures

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Abstract

The theory of scale relativity extends Einstein's principle of relativity to scale transformations of resolutions. It is based on the giving up of the axiom of differentiability of the space-time continuum. Three consequences arise from this withdrawal.

(i) The geometry of space-time becomes fractal, i.e., explicitly resolution-dependent : this allows one to describe a non-differentiable physics in terms of differential equations acting in the scale space. The requirement that these equations satisfy the principle of scale relativity leads to introduce scale laws having a Galilean form (constant fractal dimension), then a log-Lorentzian form. In this framework, the Planck length-time scale becomes a minimal impassable scale, invariant under dilations, and the cosmic length-scale (related to the cosmological constant) a maximal one. Recent measurements of the cosmological constant have confirmed the theoretically predicted value.

Then we attempt to construct a generalized scale relativity which includes non-linear scale transformations and scale-motion coupling. In this last framework, one can reinterpret gauge invariance as scale invariance on the internal resolutions. This approach has allowed us to make theoretical predictions concerning coupling constants and elementary particle masses (electron, Higgs boson, vacuum energy of the Higgs field), which we update in the present contribution. These predictions are successfully checked using recently improved experimental values.

(ii) The geodesics of a non-differentiable space-time are fractal and in infinite number: this leads one to use a fluid-like description and implies adding new terms in the differential equations of mean motion.

(iii) Time reversibility is broken at the infinitesimal level: this can be described in terms of a two-valuedness of the velocity vector, for which we use a complex representation.

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These three effects can be combined to construct a covariant time derivative operator, which transforms the fundamental equations of classical dynamics into a generalized Schrödinger equation. This provides us with a theory of morphogenesis and self-organization, since the solutions of this equation yield probability densities, which are interpreted as a tendency for the system to make structures. Several new theoretical predictions can be made by applying this approach to the equations of motion of test-particles in various gravitational potentials of astrophysical relevance. These predictions are supported by a comparison with observational data on a wide range of scales, from planetary systems to cosmological structures.

1 Introduction

The theory of scale relativity [13] is an attempt to study the consequences of giving up the hypothesis of space-time differentiability. One can show [13] [14] that a continuous but nondifferentiable space-time is necessarily *fractal*. Here the word fractal [11] is taken in a general meaning, as defining a set, object or space that shows structures at all scales, or on a wide range of scales. More precisely, one can demonstrate that a continuous but nondifferentiable function is explicitly resolution-dependent, and that its length $\mathcal{L}$ tends to infinity when the resolution interval tends to zero, i.e. $\mathcal{L} = \mathcal{L}(\varepsilon)_{\varepsilon \to 0} \to \infty$. This theorem and other properties of non-differentiable curves have been recently analysed in detail by Ben Adda and Cresson [4]. It naturally leads to the proposal that the concept of *fractal space-time* [19] [24] [13] [6] is the geometric tool adapted to the research of such a new description based on non-differentiability. In such a generalized framework including all continuous functions, the usual differentiable functions remain included, but as very particular and rare cases.

Since a nondifferentiable, fractal space-time is explicitly resolution-dependent, the same is a priori true of all physical quantities that one can define in its framework. We thus need to complete the standard laws of physics (which are essentially laws of motion in classical physics) by laws of scale, intended to describe the new resolution dependence. We have suggested [12] that the principle of relativity can be extended to constrain also these new scale laws.

Namely, we generalize Einstein's formulation of the principle of relativity, by requiring *that the laws of nature be valid in any reference system, whatever its state*. Up to now, this principle has been applied to changes of state of the coordinate system that concerned the origin, the axes orientation, and the motion (measured in terms of velocity and acceleration)

In scale relativity, we assume that the space-time resolutions are not only a characteristic of the measurement apparatus, but acquire a universal status. They are considered as essential variables, inherent to the physical description. We define them as characterizing the "state of scale" of the reference system, in the same way as the velocity characterizes its state of motion. The principle of scale relativity consists of applying the principle of relativity to such a scale-state. Then we set a principle of *scale-covariance*, requiring that the equations of physics keep their form under resolution transformations.

In the present paper, we shall review various levels of development of the theory, then consider some of its consequences in the domains of elementary particles, cosmology and gravitational structure formation.

2 Galilean scale relativity

2.1 Standard fractal laws

Scaling laws have already been studied at length in several domains of science. A power-law scale dependence is frequently encountered in a lot of natural systems, it is described geometrically in terms of fractals, and algebrically in terms of the renormalization group. As we shall see now, such simple scale-invariant laws can be identified with a "Galilean" version of scale-relativistic laws.

Indeed, let us consider a non-differentiable coordinate $\mathcal{L}$. Our basic theorem that links non-differentiability to fractality implies that $\mathcal{L}$ is an explicit function $\mathcal{L}(\varepsilon)$ of the resolution interval ε. As a first step, one can assume that $\mathcal{L}(\varepsilon)$ satisfies the simplest possible scale differential equation one may write, namely, the first order equation:

$$\frac{d \ln \mathcal{L}}{d \ln(\lambda/\varepsilon)} = \delta, \tag{1}$$

where δ is a constant. The solution is a fractal, power-law dependence:

$$\mathcal{L} = \mathcal{L}_0 (\lambda/\varepsilon)^{\delta}, \tag{2}$$

where δ is the scale dimension, i.e., $\delta = D - D_T$, the fractal dimension minus the topological dimension. The Galilean structure of the group of scale transformation that corresponds to this law can be verified in a straightforward manner from the fact that it transforms in a scale transformation $\varepsilon \to \varepsilon'$ as

$$\ln \frac{\mathcal{L}(\varepsilon')}{\mathcal{L}_0} = \ln \frac{\mathcal{L}(\varepsilon)}{\mathcal{L}_0} + \delta(\varepsilon) \ln \frac{\varepsilon}{\varepsilon'} \quad ; \quad \delta(\varepsilon') = \delta(\varepsilon). \tag{3}$$

This transformation has exactly the structure of the Galileo group, as confirmed by the law of composition of dilations $\varepsilon \to \varepsilon' \to \varepsilon''$, which writes $\ln \rho'' = \ln \rho + \ln \rho'$, with $\rho = \varepsilon'/\varepsilon$, $\rho' = \varepsilon''/\varepsilon'$ and $\rho'' = \varepsilon''/\varepsilon$.

2.2 Breaking of the scale symmetry

More generally, one can write a first order equation where the scale variation of $\mathcal{L}$ depends on $\mathcal{L}$ only, $d\mathcal{L}/d \ln \varepsilon = \beta(\mathcal{L})$. The function $\beta(\mathcal{L})$ is a priori unknown but, always taking the simplest case, we may consider a perturbative approach and take its Taylor expansion. We obtain the equation:

$$\frac{d\mathcal{L}}{d \ln \varepsilon} = a + b\mathcal{L} + \ldots \tag{4}$$

This equation is solved in terms of a standard power law of power $\delta = -b$, broken at some relative scale λ (which is a constant of integration):

$$\mathcal{L} = \mathcal{L}_0 \, [1 + (\frac{\lambda}{\varepsilon})^{\delta}]. \tag{5}$$

Depending on the sign of δ, this solution represents either a small-scale fractal behavior (in which the scale variable is a resolution), broken at larger scales, or a large-scale fractal behavior (in which the scale variable ε would now represent a changing window for a fixed resolution λ), broken at smaller scales.

2.3 Euler-Lagrange scale equations

In the previous approach, we have considered as primary variables the position $\mathcal{L}$ and the resolution ε. However, we are naturally led, in the scale-relativistic approach, to reverse the definition and the meaning of variables. The scale dimension δ can be generalized in terms of an essential, fundamental *variable*, that would remain constant only in very particular situations (namely, in the case of scale invariance, that corresponds to "scale-freedom"). It plays for scale laws the same role as played by time in motion laws. We have proposed to call "djinn" this varying scale dimension. The new approach amounts to work in a "space-time-djinn" rather than only in space-time, thus including motion and scale behaviour in the same 5-dimensional description. The resolution can now be defined as a derived quantity in terms of the fractal coordinate and of the djinn:

$$\bar{V} = \ln(\lambda/\varepsilon) = \frac{d \ln \mathcal{L}}{d\delta}. \tag{6}$$

Our identification of standard fractal behavior as Galilean scale laws can now be fully proven. We assume that, as in the case of motion laws, scale laws can be constructed from a Lagrangian approach. A scale Lagrange function $\bar{L}(\ln \mathcal{L}, \bar{V}, \delta)$ is introduced, from which a scale-action is constructed:

$$\bar{S} = \int_{\delta_1}^{\delta_2} \bar{L}(\ln \mathcal{L}, \bar{V}, \delta) d\delta. \tag{7}$$

The action principle, applied on this action, yields a scale-Euler-Lagrange equation that writes:

$$\frac{d}{d\delta}\frac{\partial \bar{L}}{\partial \bar{V}} = \frac{\partial \bar{L}}{\partial \ln \mathcal{L}}. \tag{8}$$

The simplest possible form for the Lagrange function is the equivalent for scales of what inertia is for motion, i.e., $\bar{L} \propto \bar{V}^2$ and $\partial \bar{L}/\partial \ln \mathcal{L} = 0$ (no scale "force"). The Lagrange equation writes in this case:

$$\frac{d\bar{V}}{d\delta} = 0 \Rightarrow \bar{V} = cst. \tag{9}$$

The constancy of $\bar{V} = \ln(\lambda/\varepsilon)$ means here that it is independent of the scale-time δ. Then Eq. (6) can be integrated in terms of the usual power law behavior,

$\mathcal{L} = \mathcal{L}_0(\lambda/\varepsilon)^\delta$. This reversed viewpoint has several advantages which allow a full implementation of the principle of scale relativity:
(i) The scale dimension takes its actual status of "scale-time", and the logarithm of resolution $\bar{V}$ its status of "scale-velocity", $\bar{V} = d\ln\mathcal{L}/d\delta$.
(ii) This leaves open the possibility of generalizing our formalism to the case of four independent space-time resolutions. Indeed, from $\mathcal{L}^\mu, \mu = 0,1,2,3$ and δ one can now build a 4-component resolution vector, $\bar{V}^\mu = \ln(\lambda^\mu/\varepsilon^\mu) = d\ln\mathcal{L}^\mu/d\delta$.
(iii) As we shall see in what follows, scale laws more general than the simplest self-similar ones can be derived from more generalized scale-Lagrangians.

3 Special and generalized scale-relativity

3.1 Special scale relativity

It is well known that the Galileo group of motion is only a degeneration of the more general Lorentz group. The same is true for scale laws. Indeed, if one looks for the general linear scale laws that come under the principle of scale relativity, one finds that they have the structure of the Lorentz group [12]. Therefore, in special scale relativity, we have suggested to substitute to the Galilean law of composition of dilations $\ln(\varepsilon'/\lambda) = \ln\rho + \ln(\varepsilon/\lambda)$ the more general log-Lorentzian law:

$$\ln\frac{\varepsilon'}{\lambda} = \frac{\ln\rho + \ln(\varepsilon/\lambda)}{1 + \ln\rho\ln(\varepsilon/\lambda)/\ln^2(\lambda_P/\lambda)}, \tag{10}$$

while the scale dimension becomes a variable according to the law:

$$\delta(\varepsilon) = \frac{1}{\sqrt{1 - \ln^2(\varepsilon/\lambda)/\ln^2(\lambda_P/\lambda)}}, \tag{11}$$

where λ is the fractal / nonfractal transition scale. In the microphysical domain, the invariant length-scale is naturally identified with the Planck scale, $\lambda_P = (\hbar G/c^3)^{1/2}$, that now becomes impassable and plays the physical role that was previously devoted to the zero point. The same is true in the cosmological domain, with an inversion of the scale laws: there appears a maximal, impassable scale of resolution that plays the physical role of the infinite, that we have identified with the length-scale $\mathbb{L} = \Lambda^{-1/2}$ related to the cosmological constant Λ. The consequences of this new interpretation of the cosmological constant have been considered in [13] [14].

3.2 From scale dynamics to general scale relativity

The whole of our previous discussion indicates to us that the scale invariant behavior corresponds to freedom in the framework of a scale physics. However, in the same way as there exists forces in nature that imply departure from inertial, rectilinear uniform motion, we expect most natural fractal systems to also present distorsions in their scale behavior respectively to pure scale invariance. Such distorsions may be, as a first step, attributed to the effect of a scale "dynamics", i.e. to "scale-forces". In this case the Lagrange scale-equation takes the form of Newton's equation of dynamics:

$$\bar{F} = \mu \frac{d^2 \ln \mathcal{L}}{d\delta^2}, \tag{12}$$

where μ is a "scale-mass", which measures the way the system resists to the scale-force.

3.2.1 Constant scale-force

Let us first consider the case of a constant scale-force. We set $\bar{G} = \bar{F}/\mu =$ constant. Equation (12) is easily integrated as:

$$\delta = \delta_0 + \frac{1}{\bar{G}} \ln\left(\frac{\lambda}{\varepsilon}\right) \quad ; \quad \ln\frac{\mathcal{L}}{\mathcal{L}_0} = \frac{1}{2\bar{G}} \ln^2\left(\frac{\lambda}{\varepsilon}\right). \tag{13}$$

The scale dimension δ becomes a linear function of resolution, and the $(\ln \mathcal{L}, \ln \varepsilon)$ relation is now parabolic rather than linear as in the standard power-law case. There are several physical situations that could come under such a "scale-dynamical" description, where a clear curvature appears in the $(\ln \mathcal{L}, \ln \varepsilon)$plane (e.g., turbulence, sand piles,...). In these cases it might be interesting to identify and study the scale-force responsible for the scale distorsion.

3.2.2 Harmonic oscillator

Another interesting case of scale-potential is that of a harmonic oscillator $\phi = -(1/2)(\ln \mathcal{L}/\alpha)^2$. It is solved as

$$\ln\frac{\mathcal{L}}{\mathcal{L}_0} = \alpha\sqrt{\ln^2\left(\frac{\lambda}{\varepsilon}\right) - \frac{1}{\alpha^2}}. \tag{14}$$

For $\varepsilon \ll \lambda$ it gives the standard Galilean case $\mathcal{L} = \mathcal{L}_0(\lambda/\varepsilon)^\alpha$, but its large-scale behavior is particularly interesting, since it does not permit the existence of resolutions larger than a scale $\lambda_{max} = \lambda\, e^{-1/\alpha}$. Such a behavior could provide a model of confinement in QCD.

More generally, we shall be led to look for the general non-linear scale laws that satisfy the principle of scale relativity. Such a generalized framework implies working in a five-dimensional fractal space-time. The development of such a "general scale-relativity" lies outside the scope of the present paper and will be considered in forthcoming works.

3.3 Scale-motion coupling and mass-charge relations

The theory of scale relativity also allows to get new insights about the physical meaning of gauge invariance [14]. In the scale laws recalled hereabove, only scale transformations at a given point were considered. But we may also wonder about what happens to the structures in scale-space of a scale-dependent object such as an electron or another charged particle, when it is displaced. Consider anyone of these structures, lying at some (relative) resolution ε (such that $\varepsilon < \lambda$, where λ is the Compton length of the particle) for a given position of the particle. In a displacement, the relativity of scales implies that the resolution at which this given structure appears in the new position will a priori be different from the initial one. In other words, $\varepsilon = \varepsilon(x, t)$ is now a function of the space-time coordinates, and we expect the occurrence of *dilations of resolutions induced by translations*, so that we are led to introduce a covariant derivative:

$$e \frac{D\varepsilon}{\varepsilon} = e \frac{d\varepsilon}{\varepsilon} - A_\mu dx^\mu, \tag{15}$$

where a four-vector A_μ must be introduced since dx^μ is itself a four-vector and $d \ln \varepsilon$ a scalar (in the case of a global dilation).

However, if one wants such a "field" A_μ to be physical, it should be defined whatever the initial scale from which we started. Starting from another scale $\varepsilon' = \rho\varepsilon$, we get the same expression as in Eq.15, but with A_μ replaced by A'_μ. Therefore, we obtain the relation:

$$A'_\mu = A_\mu + e\, \partial_\mu \ln \rho, \tag{16}$$

which depends on the relative "state of scale", $\bar{V} = \ln \rho = \ln(\varepsilon/\varepsilon')$, that is now a function of the coordinates.

One may therefore identify A_μ with the electromagnetic potential, and Eq.(16) with the property of gauge invariance. Now we know that applying a gauge transformation to the electromagnetic field implies to change also the wave function of the electron, that becomes:

$$\psi' = \psi\, e^{i4\pi\alpha \ln \rho} \tag{17}$$

where α is the square of charge in units of $\hbar c$, i.e., a coupling constant. While in Galilean scale relativity, the scale ratio ρ is unlimited, in the more general framework of special scale relativity it is limited by the Planck-scale/Compton-scale ratio. This limitation implies the quantization of charge, following the general mass-charge relation [14]:

$$\alpha \ln \left(\frac{m_P}{m} \right) = k/2, \tag{18}$$

where k is integer. Such a relation between the electron mass and the electroweak coupling $8\alpha/3$ (where $\alpha^{-1} = 137.036$) is implemented with a relative precision of 2×10^{-3}, becoming 10^{-4} when accounting for threshhold effects [14].

This approach can be generalized, since we can define four different and independent dilations along the four space-time resolutions instead of only one global dilation. The above U(1) field is then expected to be embedded into a larger field, in agreement with the electroweak and grand unification theories, and the charge e to be one element of a more complicated, "vectorial" charge. Some hints about such a generalization will be given in what follows.

4 Theoretical predictions of masses and couplings

In the new framework, theoretical predictions of some of the free parameters of the standard model become possible. We have presented and checked such predictions in previous works [12] [13] [14]. But in the recent years, there has been an improvement of several experimental measurements [27], so that it may now be interesting to check them again with these new values. They are, respectively for the top quark mass, Higgs boson mass, W and Z boson masses, strong coupling constant at Z scale, fine structure constant at Z scale, and $\sin^2\theta$ of weak mixing angle at Z scale in the modified minimal substraction scheme (where it is defined through the SU(2) charge g and the U(1) charge g'):

$m_t = 174.3 \pm 5.1\,\mathrm{GeV}$; $m_H = 108 - 220\,\mathrm{GeV}$
$m_W = 80.42 \pm 0.04\,\mathrm{GeV}$; $m_Z = 91.1872 \pm 0.0021$
$\alpha_S(m_Z)^{-1} = 0.118 \pm 0.002$; $\alpha(m_Z)^{-1} = 128.92 \pm 0.03$
$\hat{s}_Z^2 = \frac{g'^2}{g^2+g'^2} = 0.23117 \pm 0.00016$

4.1 Fine structure constant

In [14], we derived a prediction of the fine structure constant (i.e. the electromagnetic coupling). It was based on the suggestion that the bare (infinite energy) value of the electroweak coupling (which becomes finite in special scale-relativity) is $4\pi^2$. The fact that 3 among the 4 gauge bosons acquire mass through the Higgs mechanisms leads to a multiplying factor 8/3, so that one expects that $\alpha_\infty^{-1} = 32\pi^2/3$. The difference between the infinite energy and Z or low energy values was computed using the solutions to the renormalization group equation for the running coupling. The prediction at the Z value for 1 Higgs doublet was:

$$\alpha(m_Z)^{-1} = \frac{32\pi^2}{3} + \frac{11}{6\pi} \ln\left(\frac{m_P}{m_Z}\right) + 2\text{nd order term}, \tag{19}$$

where m_P is the Planck mass ($= 1.2210(9) \times 10^{19} GeV/c^2$). The second order term is given by Eq. 112 of [14]. Now we can combine this expression with another prediction of the theory, according to which the electroweak scale and the inverse coupling $\alpha_\infty^{-1} = 4\pi^2$ are linked by a mass-charge relation:

$$\ln\left(\frac{m_P}{m_Z}\right) \approx 4\pi^2. \tag{20}$$

Replacing this expression in the first and second order terms we obtain:

$$\alpha(m_Z)^{-1} = \frac{32\pi^2}{3} + \frac{22\pi}{3} + \frac{6}{\pi^2} = 128.922\,. \tag{21}$$

Though Eq. 20 is only an approximation for the Z scale (see below), it occurs in Eq. 19 as a first order correction and in terms of the logarithm of the mass ratio, so that the final result (Eq. 21) finally gives a good approximation of our theoretical prediction. It indeed compares very well with the experimental value, 128.92 ± 0.03.

4.2 Strong coupling

From the conjecture that the strong coupling value reaches the critical value $1/4\pi^2$ at unification scale (i.e. $m_P/2\pi$ in the special scale-relativistic modified standard model), we obtained a predicted value $\alpha_S(m_Z)^{-1} = 0.1155 \pm 0.0002$ from the solution to the renormalization group equation of the running coupling [12] [14]. This expectation remains in agreement (within about one σ) with the recently improved experimental value 0.118 ± 0.002.

4.3 SU(2) coupling

In ref. [14], we also attempted to apply the mass-charge relation to the SU(2) coupling α_2. We found that the relation

$$3\,\alpha_{2Z}\,C_Z = 4 \tag{22}$$

was precisely achieved at the Z scale. However the factor 3 was not accounted for in that work. The solution to this problem relies on the generalization of scale (i.e. gauge) transformations to dilations which are no longer global, but instead may be different on the resolutions corresponding to the various coordinates. The group SU(2) corresponds to rotations in a 3-dimensional scale space. Therefore the phase term in a fermion field will write:

$$\alpha_2\,\ln(\frac{\varepsilon_x}{\lambda}) + \alpha_2\,\ln(\frac{\varepsilon_y}{\lambda}) + \alpha_2\,\ln(\frac{\varepsilon_z}{\lambda}) < 3\,\alpha_2\,\ln(\frac{\lambda_P}{\lambda}), \tag{23}$$

since the same coupling applies to the three variables, and since all three resolutions are limited at small scales by the Planck scale. From Eq. (22) we expect a value $\alpha_{2Z}^{-1} = 29.8169 \pm 0.0002$. The present precise experimental value is:

$$\alpha_{2Z}^{-1} = \alpha_Z^{-1} \times \hat{s}_Z^2 = 29.802 \pm 0.027, \tag{24}$$

which lies within 1σ of the theoretical prediction.

4.4 Vacuum expectation value of the Higgs field

As recalled hereabove, there are fundamental arguments for introducing a bare inverse coupling at infinite energy (i.e., in special scale relativity, at Planck length-scale) given by the critical value $4\pi^2$. Moreover, our re-interpretation of gauge invariance as scale-invariance on space-time resolution led us to construct general relations between couplings and scale ratios. Therefore one expects the emergence of a new fundamental scale given by:

$$\ln\left(\frac{\lambda}{\lambda_P}\right) = 4\pi^2, \tag{25}$$

where λ_P is the Planck length-scale. This relation may provide a solution to the hierarchy problem, according to which there is a misunderstood factor $\approx 10^{17}$ between the electroweak scale and the Planck scale (expected to be the full unification scale). Indeed the scale λ defined above is $e^{4\pi^2} = 1.397 \times 10^{17}$ larger than the Planck scale. As a first approximation, we can apply this relation to mass ratios. This leads a mass scale of 87.39 GeV, intermediate between the Z and W masses. However, mass-scales and length-scales are no longer directly inverse in the scale-relativity framework. There is a "log-Lorentz" factor between them (when they are referred to low energies). Namely, by taking as reference the electron Compton scale, the new mass-scale is more precisely given by:

$$\ln\left(\frac{m}{m_e}\right) = \frac{\ln(\lambda_e/\lambda)}{\sqrt{1 - \ln^2(\lambda_e/\lambda)/C_e^2}}. \tag{26}$$

With the currently accepted value of the gravitational constant (for which the error is now thought to be 12 times larger than previously given, see [27]), we obtain for the fundamental constant $C_e = \ln(\lambda_e/\lambda_P) = 51.52797(70)$. Then the new theoretically predicted mass scale is

$$m_v = 123.23 \pm 0.09\,\text{GeV}, \tag{27}$$

which is closely linked to the vacuum expectation value v of the Higgs field, since the present experimental value of $v/\sqrt{2} = m_W/g$ (where g is the SU(2) weak charge) is 123.11 ± 0.03 GeV. Now some work remains to be done to really understand why the new mass-scale should have precisely this interpretation.

Let us finally note that the previously pointed out coincidence of the top quark mass (174.3 ± 5.1 GeV) and of the vacuum expectation value of the Higgs field (174.10 ± 0.05 GeV) remains remarkable.

4.5 Mass of the Higgs boson

The framework of generalized scale-relativity provides one with possibilities to make theoretical predictions of the value of the Higgs boson mass. The (summarized) argument is as follows.

In today's electroweak scheme, the Higgs boson is considered to be separated from the electroweak field. Moreover, a more complete unification is mainly seeked in terms of attempts of "grand" unifications with the strong field. However, one may wonder whether, maybe in terms of an effective, intermediate energy, theory, one could not achieved a more tightly unified purely electroweak theory. Recall indeed that in the present standard model, the weak and electromagnetic fields are mixed, but there remains four free parameters, which can e.g. be taken to be the Higgs boson mass, the vacuum expectation value of the Higgs field and the Z and W masses.

The structure of the present electroweak boson content is as follows. There is an SU(2) gauge field, then involving three fields of null mass (i.e. $2 \times 3 = 6$ degrees of freedom), a U(1) null mass field (2 d.f.) and a Higgs boson complex doublet (4 d.f.), which makes 12 d.f. in all. Through the Glashow-Salam-Weinberg mechanism, 3 of the 4 components of the Higgs doublet become longitudinal components of the weak field which therefore acquires mass ($3 \times 3 = 9$ d.f.), while the photon remains massless (2 d.f.), so that there remains a Higgs scalar which is nowadays experimentally searched (1 d.f.).

Now, we have suggested a new interpretation of gauge invariance as being scale invariance on the internal resolutions, considered as intrinsic to the description of the particle-fields (at scale smaller than their Compton length in restframe). As a first step we considered only global dilations, which led us to a U(1) invariance and to the relations between mass scale and coupling constant recalled above. But more generally one may consider four independant scale transformations on the four space-time resolutions, i.e., $(\ln \varepsilon_x, \ln \varepsilon_y, \ln \varepsilon_z, \ln \varepsilon_t)$. This means that the scale space (i.e., here the gauge space) is at least four-dimensional (but note that this is not the final word on the subject, since this does not yet include the fifth "djinn" dimension δ). Moreover, the mixing relation between the B (U(1)) and W_3 (SU(2)) fields may also be interpreted as a rotation in the full gauge space. We therefore expect the appearance of a 6 component antisymmetric tensor field (linked to the rotations in this space), corresponding in the simplest case to a SO(4) group. Such a zero mass field would yield 12 degrees of freedom by itself alone.

What about the Higgs boson in such a unified framework ? We shall tentatively explore the possibility that it appears as a separated scalar only as a low energy approximation, while in the new framework it would be one of the components of the unified field (in analogy with energy appearing as scalar at low velocity, while it is ultimately a component of the energy-momentum four-vector in the relativistic framework).

Such an attempt is supported by the form of the electroweak Lagrangian (we adopt Aitchison's [3] notations). Its Higgs scalar boson part writes:

$$L_H = \frac{1}{2}\, \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2}\, m_H^2 \sigma^2 - \frac{1}{8} \lambda^2 \sigma^4. \tag{28}$$

The vacuum expectation value v of the Higgs field is computed from the square (mass term) and quartic term, so that the Higgs mass is related to v and λ as:

$$m_H = \sqrt{2}\, v\, \lambda. \tag{29}$$

A prediction of the constant λ would therefore lead to a prediction of the Higgs mass. Now, a non-Abelian field writes in terms of its potential :

$$F^{\alpha\mu\nu} = \partial^\mu W^{\alpha\nu} - \partial^\nu W^{\alpha\mu} - g\, c^\alpha_{\beta\gamma} W^{\beta\mu} W^{\gamma\nu}, \tag{30}$$

where g is the (now unique) charge and $c^\alpha_{\beta\gamma}$ the structure coefficients of the Lie algebra associated to the gauge group. Its Lagrangian writes:

$$L_W = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}. \tag{31}$$

It therefore includes W^4 terms coming from the W^2 terms in the field. Now our ansatz consists of identifying some of these W^4 terms, of coefficient $-\frac{1}{4}\, g^2\, (c^\alpha_{\beta\gamma})^2$, with the Higgs boson σ^4 term of coefficient $-\frac{1}{2}\, \lambda^2$. This allows a determination of the constant λ:

$$\lambda^2 = \frac{g^2(\Sigma c^2)}{2}, \tag{32}$$

where the sum is on the terms that contribute to the final effective Higgs boson. Provided the global charge is identical to the SU(2) charge, and since the W mass is given by $m_W = gv/\sqrt{2}$, one finally obtains a Higgs boson mass :

$$m_H = \sqrt{2(\Sigma c^2)}\, m_W. \tag{33}$$

For a large class of groups (like e.g. SO(4)), the c's take the values 0, ± 1, so that we expect $m_H = \sqrt{2k}\, m_W$ with k integer. In particular, the case $k = 1$, yields a theoretical prediction:

$$m_H = \sqrt{2}\, m_W = 113.73 \pm 0.06\,\mathrm{GeV}, \tag{34}$$

which is in agreement with current constraints. Although this calculation is still incomplete and although the self-consistency of this model remains to be established, we hope that at least some of its ingredients could reveal to be useful in more complete attempts [Lehner and Nottale, in preparation].

4.6 Cosmological constant and gravitational coupling

In [14], we were able to make a theoretical prediction of the value of the cosmological constant. Recall that, in the special scale-relativistic framework, new dilation laws having a log-Lorentz form have been introduced [12], that lead to re-interpret the length-scale of the cosmological constant $\mathbb{L} = \Lambda^{-1/2}$ and the Planck length-scale λ_P as impassable, respectively maximal and minimal length-scales, invariant under dilations of resolutions.

Their ratio defines a fundamental pure number, $\mathbb{K} = \mathbb{L}/\lambda_P$. From an analysis of the vacuum energy density problem, the logarithm of this ratio has been found to have the numerical value $C_U = \ln \mathbb{K} = 139.83 \pm 0.01$, i.e. $\mathbb{K} = 5.3 \times 10^{60}$ [13] [14]. This value corresponds to a reduced cosmological constant $\Omega_\Lambda = 0.36 h^{-2}$, where $h = H_0/100$ km/s.Mpc. Now the Hubble constant has been recently determined with an improved precision to be $H_0 = 70 \pm 10$ km/s.Mpc. Therefore we predicted a reduced cosmological constant $\Omega_\Lambda = 0.70 \pm 0.25$. Recent measurements using the Hubble diagram of SNe I [8] [29] [30] and the angular power spectrum of the cosmic microwave radiation [5] point precisely toward the same value, 0.7 ± 0.2.

5 Gravitational structuration

5.1 Generalized Schrodinger equation

One can demonstrate [13] [14] [16] that Newton's fundamental equation of dynamics can be integrated in the form of a Schrödinger-like equation under the three following hypotheses:

(i) The test-particles can follow an infinity of potential trajectories: this leads one to use a fluid-like description, $v = v(x(t), t)$.

(ii) The geometry of each trajectory is fractal (of dimension 2). Each elementary displacement is then described in terms of the sum, $dX = dx + d\xi$, of a mean, classical displacement $dx = v\,dt$ and of a fractal fluctuation $d\xi$ whose behavior satisfies the principle of scale relativity (in its simplest "Galilean" version). It is such that $< d\xi >= 0$ and $< d\xi^2 >= 2\mathcal{D}dt$. The existence of this fluctuation implies introducing new second order terms in the differential equations of motion.

(iii) The motion is assumed to be locally irreversible, i.e., the $(dt \leftrightarrow -dt)$ reflection invariance is broken, leading to a two-valuedness of the velocity vector that we represent in terms of a complex velocity, $\mathcal{V} = (v_+ + v_-)/2 - i(v_+ - v_-)/2$.

These three effects can be combined to construct a complex time-derivative operator which writes

$$\frac{\acute{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{V} \cdot \nabla - i\,\mathcal{D}\,\triangle \tag{35}$$

where the mean velocity $\mathcal{V} = \acute{d}\,x/dt$ is now complex and $\mathcal{D}$ is a parameter characterizing the fractal behavior of trajectories (namely, it defines the fractal-nonfractal transition in scale space).

Since the mean velocity is complex, the same is true of the Lagrange function, then of the generalized action $\mathcal{S}$. Setting $\psi = e^{i\mathcal{S}/2m\mathcal{D}}$, Newton's equation of dynamics becomes $m\,\acute{d}\,\mathcal{V}/dt = -\nabla\phi$, and can be integrated in terms of a generalized Schrödinger equation [13]:

$$\mathcal{D}^2 \triangle \psi + i\mathcal{D}\frac{\partial}{\partial t}\psi = \frac{\phi}{2m}\,\psi. \tag{36}$$

This equation becomes, for a Kepler potential and in the time-independent case:

$$2\mathcal{D}^2 \triangle \psi + \left(\frac{E}{m} + \frac{GM}{r}\right)\psi = 0. \tag{37}$$

Since the imaginary part of this equation is the equation of continuity, $\rho = \psi\psi^\dagger$ can be interpreted as giving the probability density of the particle position.

Even though it takes this Schrödinger-like form, this equation is still in essence an equation of gravitation, so that it must keep the fundamental properties it owns in Newton's and Einstein's theories. Namely, it must agree with the equivalence principle [15] [9] [1], i.e., it must be independent of the mass of the test-particle and GM must provide the natural length-unit of the system under consideration. As a consequence, the parameter $\mathcal{D}$ takes the form:

$$\mathcal{D} = \frac{GM}{2w}, \tag{38}$$

where w is a fundamental constant that has the dimension of a velocity.

Actually, the ratio

$$\alpha_g = \frac{w}{c}. \tag{39}$$

stands out as a macroscopic gravitational coupling constant [1] [2] [22]. This can be seen from the fact that w is the average velocity in the fundamental orbit, in the same manner as $v_0 = \alpha\, c$ (where α is the fine structure constant, i.e., the electromagnetic coupling) gives the mean velocity of an electron in a Bohr orbital. Moreover, contrarily to what happens in the classical theory, the equation of motion (Eq. 36) can be shown to be gauge invariant. If the potential ϕ is replaced by $\phi + GMm\, \partial\chi(t)/c\partial t$, where the factor GMm ensures a correct dimensionality, then Eq. (36) remains invariant provided ψ is replaced by $\psi\, e^{-i\alpha_g \chi}$, with α_g related to $\mathcal{D}$ by:

$$\alpha_g \times 2m\mathcal{D} = \frac{GMm}{c}, \tag{40}$$

which is the previously established relation for $\alpha_g = w/c$.

As an example, let us briefly show how such an approach can be applied to formation of planetary systems. We assume that the probability density solution of Eq.37 describes the distribution of planetesimals in a protoplanetary nebula. Then they form a planet by accretion as in the standard models of planetary formation. But the new point here is that only some particular orbitals are allowed, so that the semi-major axes of the orbits of the resulting planets are quantized according to the law:

$$a_n = \frac{GMn^2}{w^2}, \tag{41}$$

where n is an integer. In an equivalent way, using Kepler's third law that relates the semimajor axis a to the orbital period P, $(a/GM)^3 = (P/2\pi GM)^2$, the average velocity of the planet, $v = 2\pi a/P = (GM/a)^{1/2}$, is expected to have a distribution peaked at $v_n = w/n$. Therefore, we predict that the values of $\tilde{n}$ defined, in Solar System units (AU, year, $M_\odot$ and Earth velocity) as:

$$\tilde{n} = w\, (a/M)^{1/2} = w\, (P/M)^{1/3} = \frac{w}{v} \tag{42}$$

be clustered around integer numbers.

5.2 Comparison with observational data

We have shown [13], that this approach accounts for several structures observed in the Solar System, including planet distances, eccentricities, and mass distribution [21], obliquities and inclinations of planets and satellites [17]), giant planet

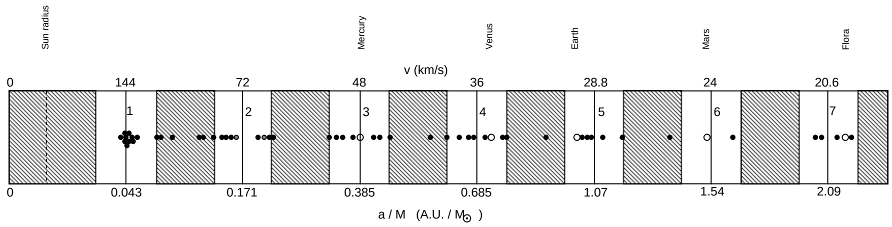

Figure 1: Observed distribution of $\tilde{n} = 4.83\,(P/M)^{1/3} = 144/v$, where v is the average planet velocity in km/s, and where the orbital period P and the star mass M are taken in Solar System units (AU and $M_\odot$), for the recently discovered exoplanet candidates (black dots) and for the planets of our inner Solar System (white dots). The grey zone stands for the theoretically predicted low probability density of planets and the white zones for high probability. The error bars are typically of the order of $0.03\,\tilde{n}$. The probability to obtain such a non uniform distribution by chance is about 10^{-4}.

satellite distances [10], parabolic comet perihelions [Nottale & Schumacher, in preparation]. Moreover, it also allows one to predict and understand structures observed on a large range of scales, from binary stars [20], to binary galaxies [14], [Tricottet & Nottale, in preparation], and the distribution of galaxies at the scale of the local supercluster [20]. A similar kind of approach has been applied by Perdang [28] to a statistical description of HR diagrams.

It has been also demonstrated that the first newly discovered extra-solar planetary systems come under the same structures, in terms of the same universal constant as in our own Solar System [15] [22] [23] (see Fig. 1). Their distribution shows peaks of probability density that are consistent with the law $a/GM = n^2/{w_0}^2$, where the constant w_0 takes the value 144 km/s as in our own inner Solar System and in extragalactic data. Moreover, most of these exoplanets (51 Peg-type objects) fall in the fundamental probability density peaks ($n = 1$, $a/M = 0.043$ AU/$M_\odot$) and in the second orbital ($n = 2$, $a/M = 0.17$ AU/$M_\odot$) predicted by the theory. The system of three planets discovered around the pulsar PSR B1257+12 also agree with the theoretical prediction with a very high precision of some 10^{-4} [15] [18].

6 Conclusion

After having summarized the main lines of development of the scale-relativity theory, we have, in the present contribution, updated some of its theoretical predictions, then we have shown that they continue to agree with recently improved experimental results.

Moreover, we have recalled that scale relativity, when combined with the laws of gravitation, provides us with a general theory of the structuring of gravitational systems [14] [16]. In this new approach, we do not any longer follow individual trajectories, but we jump to a statistical description in terms of probability amplitudes. Indeed, under only three simple hypotheses (large number of potential trajectories, fractal geometry of each trajectory and differential irreversibility), Newton's equation of dynamics can be transformed and integrated in terms of a generalized Schrödinger equation. This result suggests, in accordance with recent similar conclusions [31] [25] [26] [7] that the Schrödinger equation could be universal, i.e. that it may have a larger domain of application than previously thought, but with an interpretation different from that of standard quantum mechanics.

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