The Theory of Scale Relativity: Non-Differentiable Geometry and Fractal Space-Time[1]

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Abstract. The aim of the theory of scale relativity is to derive the physical behavior of a non-differentiable and fractal space-time and of its geodesics (with which particles are identified), under the constraint of the principle of the relativity of scales. We mainly study in this contribution the effects induced by internal fractal structures on the motion in standard space. We find that the main consequence is the transformation of classical mechanics in a quantum mechanics. The various mathematical quantum tools (complex wave functions, spinors, bi-spinors) are built as manifestations of the non-differentiable geometry. Then the Schrödinger, Klein-Gordon and Dirac equations are successively derived as integrals of the geodesics equation, for more and more profound levels of description. Finally we tentatively suggest a new development of the theory, in which quantum laws would hold also in the scale-space: in such an approach, one naturally defines a new conservative quantity, named 'complexergy', which measures the complexity of a system as regards its internal hierarchy of organization. We also give some examples of applications of these proposals in various sciences, and of their experimental and observational tests.

Keywords: relativity, scales, quantum mechanics, fractal geometry, non-differentiable space-time.

1. INTRODUCTION

The theory of scale-relativity is an attempt to extend today's theories of relativity, by applying the principle of relativity not only to motion transformations, but also to scale transformations of the reference system. Recall that, in the formulation of Einstein [1], the principle of relativity consists of requiring that 'the laws of nature be valid in every systems of coordinates, whatever their state'. Since Galileo, this principle had been applied to the states of position (origin and orientation of axes) and of motion of the system of coordinates (velocity, acceleration). These states are characterized by their relativity, namely, they are never definable in a absolute way. This means that the state of any system (including reference systems) can be defined only in comparison with another system.

We have suggested that the observation scale (i.e., the resolution at which a system is observed or experimented) should also be considered as characterizing the state of reference systems. It is an experimental fact known for long that the scale of a system can be defined only in a relative way: only scale ratios do have a physical meaning, never absolute scales. This led us to propose that the principle of relativity should be generalized to apply also to the transformations of the scale of reference systems. In this new approach, one re-interprets the resolutions, not only as a property of the measuring device and / or of the measured system, but more generally as a property that is intrinsic to the geometry of space-time itself: in other words, space-time is considered to be fractal. The principle of relativity of scale then consists of requiring that 'the fundamental laws of nature apply whatever the state of scale of the coordinate system'.

Such a first principle allows one to generalize the current description of the geometry of space-time, which is usually reduced to at least two-time differentiable manifolds). So a way of generalization of today's physics consists of trying to abandon the hypothesis of differentiability of space-time coordinates. This means to consider general continuous manifolds. These manifolds include as a sub-set the usual differentiable ones,

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and therefore all the Riemannian geometries that subtend Einstein's generalized relativity of motion. Then in such an approach, the standard classical physics will be naturally recovered.

But new physics also emerges in this framework. Indeed one can prove that a continuous and non-differentiable space is fractal, under Mandelbrot's general definition of this concept [2, 3]: namely, the length of a non-differentiable curve acquires an explicit dependence on resolutions and diverge when the resolution interval tends to zero [4, 6].

The introduction of non-differentiable trajectories in physics dates back to pioneering works by Feynman in the framework of quantum mechanics [8]. Namely, Feynman has demonstrated that the typical quantum mechanical paths that contribute in a dominant way to the path integral are non-differentiable curves of fractal dimension 2 [9, 10]. Now one is naturally led to consider the reverse question: does quantum mechanics itself find its origin in the fractality and non-differentiability of space-time ? Such a suggestion, first made twenty years ago [11, 10], has been subsequently developed by several authors [12, 13, 14, 15, 16, 4, 5, 17, 18].

The introduction of non-differentiable trajectories was also underlying the various attempts of construction of a stochastic mechanics [19, 20]. But this theory is now known to have problems of self-consistency [21, 27], and, moreover, to be in contradiction with quantum mechanics [22]). Scale relativity, even if it shares some common features with stochastic mechanics, is fundamentally different and is not subjected to the same difficulties [27].

In the present contribution, we shall recall how the description of the effects on motion of the internal non-differentiable structures of 'particles' lead to write a geodesics equation equivalent to the equations of quantum mechanics (Sections 3-5). As we shall see, the Schrödinger equation is derived in the (motion) non-relativistic case, that corresponds to a space-time of which only the spatial part is fractal. Looking for the motion-relativistic case amounts to work in a full fractal space-time, in which the Klein-Gordon equation is derived. Finally the Pauli and Dirac equation are derived when accounting for still more profound effects of the non-differentiability.

Now, the three minimal conditions under which this result is obtained (i) infinity of trajectories (which are identified to the geodesics of the non-differentiable space-time), (ii) fractality of the trajectories, and (iii) breaking of differential time reflexion invariance, may be achieved in more general systems than only the microscopic realm. As a consequence, new fundamental laws having some quantum properties may apply to different realms. We shall give some examples of applications of these new quantum mechanics (which are not based on the Planck constant $\hbar$, but on new constants which are specific of the system under consideration), in the domains of gravitation and of sciences of life.

We finally consider hints of a new tentative extension of the theory (Section 6), in which quantum mechanical laws are written in the scale space. A new quantized conservative quantity, that we have called 'complexergy', is defined, whose increase corresponds to an increase of the level of complexity of a system.

2. SCALE LAWS

The theory of scale relativity is constructed by first completing the standard laws of classical physics (laws of motion in space, i.e. of displacement in space-time) by new scale laws (in which the space-time resolutions are considered as variables intrinsic to the description, which are defined in a 'scale-space').

In a second step, the effects induced on motion of the internal fractal and non-differentiable structures of the geodesics are considered.

The third step (which will not be considered in this contribution) accounts for scale-motion coupling, i.e. the effects of dilations induced by displacements, that we interpret as gauge fields, in the Abelian case (electromagnetism) [26, 23, 73] and non-Abelian case [38].

Since the present contribution is mostly devoted to the construction of the induced quantum mechanics in standard space-time, we shall only give here a brief summary of the various scale laws that can be constructed as coming under the principle of scale relativity. But one should keep in mind that this is a huge domain of investigation in itself, with several line of research that have their own developments, applications and results in various sciences: we refer the interested reader to the quoted references.

Several levels of the description of scale laws can be considered. These levels are quite parallel to that of the historical development of the theory of motion laws:

(i) *Galilean scale-relativity*: standard laws of dilation, that have the mathematical structure of a Galileo group (fractal power law with constant fractal dimension). When the fractal dimension of trajectories is $D_F = 2$, the induced motion laws are that of standard quantum mechanics [4, 37, 23]. We shall consider only this case in the present paper; we refer the reader to Ref. [23] and references therein for a study of the more general situation $D_F \neq 2$.

(ii) *Special scale-relativity*: generalization of the laws of dilation to a Lorentzian form [15]. The fractal dimension itself becomes a variable, and plays the role of a fifth dimension, that we have called 'djinn'. It is combined, not with the standard space-time coordinates, that keep their four-dimensional nature of signature $(+,-,-,-)$, but with the four fractal fluctuations. Two impassable length-time scales, invariant under dilations, appear in the theory; they replace the zero (and the infinite), and play for scale laws the same role as played by the speed of light for motion. We have identified the minimal horizon scale with the Planck length-scale [15, 4], and the maximal one with the scale of the cosmological constant [4]. Such a proposal has several implications for high energy physics and for cosmology, which have allowed us to make new theoretical predictions and to put the theory to the test with success [23, 29, 30].

(iii) *Non-linear scale laws and scale-dynamics*: while the first two cases correspond to "scale freedom", one can also consider distorsion from strict self-similary, as described by second-order differential equations of scale transformations. This generalisation includes log-periodic corrections to scale invariance. It has been applied to a large number of critical phenomena [31], including species evolution [32, 33, 34], embryogenesis [35] and economic and historical evolution [33, 36]. Still more general distorsions from self-similarity can also be described in terms of a 'scale-dynamics', i.e. of the effect of a "scale-force" (that is a mere Newton-like way to describe geometric effects in the scale space) [27, 28, 73].

(iv) *General scale-relativity*: in analogy with the field of gravitation being ultimately attributed to the geometry of space-time, a more profound description of the scale-field can be done in terms of geometry of the scale 'space-djinn' and its couplings with the standard classical space-time. The account of scale-motion couplings, that leads to a new interpretation of gauge fields (third step hereabove), is a part of such a general theory of scale-relativity [26, 23, 38].

(v) *Quantum scale-relativity*: the above cases assume differentiability of the scale transformations in the scale-space. Giving up this hypothesis leads one to construct a new quantum mechanics in scale-space. A hint of such an approach and of its potential applications is given in Sec. 6 of this contribution.

3. FRACTAL SPACE AND INDUCED QUANTUM MECHANICS

3.1. Introduction

The question addressed in what follows is: what are the consequences on motion of the internal fractal structures of a non-differentiable space-time? This is a huge question that cannot be solved in one time. We therefore proceed by first studying the induced effects of the simplest scale laws, namely, self-similar laws of fractal dimension 2 for trajectories, under more and more general conditions: only fractal space, then fractal space and time, breaking of local discrete symmetry on time, then also on space. As recalled in the following Sections, we successively recover in this way more and more profound levels of quantum mechanical laws, including the quantum tools and their equations: namely, non-relativistic quantum mechanics (complex wave functions and Schrödinger equation), relativistic quantum mechanics without spin (Klein-Gordon equation), then for spinors (bi-quaternionic wave function and Dirac equation).

3.2. Scale-dependent velocity

Strictly, the non-differentiability of the coordinates means that, under its standard definition, the velocity

$$V = \frac{dX}{dt} = \lim_{dt \to 0} \frac{X(t+dt) - X(t)}{dt} \tag{1}$$

is undefined, i.e., when $dt \to 0$, either the ratio dX/dt tends to infinity, or it fluctuates without reaching any limit.

However, as recalled in the introduction, we have proved the following fundamental theorem [4, 23]: if $X(t)$ is continuous and non-differentiable, then X becomes an explicit function of the scale (which is here defined by the time element dt now considered and treated as an independent and explicit variable), i.e. $X = X(t, dt)$, and $\int |dX| \to \infty$ when $dt \to 0$. As a consequence, the velocity, V is itself re-defined as an explicitly scale-dependent function $V(t, dt)$. In the simplest case, we expect that it is solution of a first order scale differential equation like

$$\frac{dV}{d\ln(dt)} = \beta(V) = a + bV + ... \tag{2}$$

whose solution, after redefinition of the constants, can be written under the form

$$V = v + w = v\left[1 + \zeta\left(\frac{\tau}{dt}\right)^{1-1/D_F}\right]. \tag{3}$$

This means that we expect the velocity to be the sum of two independent terms, a scale-independent, differentiable one and a fractal, explicitly scale-dependent one. These two terms are of different orders of differentiation, since their ratio v/w is, from the standard viewpoint, infinitesimal. Their combination involves two regimes with a spontaneous transition between them: beyond the transition scale the effects of nondifferentiablility are smoothed out and one recovers the standard differentiable description.

3.3. Infinite number of geodesics

The above description strictly applies for an individual fractal trajectory. Now, one of the geometric consequences of the non-differentiability and of the subsequent fractal character of space itself is that there is an infinity of fractal geodesics relating any couple of points of this fractal space [4]. As a consequence, we are led to replace the velocity $V(t, dt)$ on a particular geodesic by the velocity field $V[x(t), t, dt]$ of the whole infinite ensemble of geodesics, and therefore to jump to a fluid-like approach.

We have therefore suggested [12] that the description of a quantum mechanical particle, including its property of wave-particle duality, could be reduced to the geometric properties of the set of fractal geodesics that corresponds to a given state of this "particle". In such an interpretation, we do not have to endow the "particle" with internal properties such as mass, spin or charge, since the "particle" is not identified with a point mass which would follow the geodesics, but with the geodesics themselves. Namely, its "internal" properties can now be defined as global geometric properties of the fractal geodesics. As a consequence, any measurement is interpreted as a sorting out (or selection) of the geodesics: for example, if the "particle" has been observed at a given position with a given resolution, this means that the geodesics which pass through this domain have been selected [4, 12].

3.4. 'Classical part' and 'fractal part' of differentials

The transition scale appearing in Eq. (3) yields two distinct behaviors for the system depending on the resolution at which it is considered. Equation (3) multiplied by dt gives the elementary displacement, dX, of the system as a sum of two infinitesimal terms of different orders

$$dX = dx + d\xi. \tag{4}$$

The variable

$$dx = \mathcal{C}\ell\langle dX\rangle \tag{5}$$

is defined as the "classical" part of the full deplacement dX. By 'classical', we do not mean that this is necessarily a variable of classical physics (for example, as we shall see hereafter, thedx will become two-valued due to non-differentiability, which is clearly not a classical property). We mean that it remains differentiable, and therefore come under classical differentiable equations.

Here $d\xi$ represents the fractal fluctuations or "fractal part" of the displacement dX : due to the definitive loss of information implied by the non-differentiability, we represent it in terms of a stochastic variable. We therefore write:

$$dx = v\, dt, \quad d\xi = \eta\sqrt{2\mathcal{D}}(dt^2)^{1/2D}. \tag{6}$$

The fractal fluctuation becomes, for $D=2$,

$$d\xi = \eta\sqrt{2\mathcal{D}}dt^{1/2}, \tag{7}$$

where $2\mathcal{D}=\tau v^2$, and where η is a stochastic variable such that $<\eta>=0$ and $<\eta^2>=1$. As we shall see further on, $2\mathcal{D}$ is a scalar quantity which will be identified with the Compton scale of the particle (up to fundamental constants), since we shall find that $\mathcal{D}=\hbar/2m$ in the microphysical domain. We are therefore led to define an operator $\mathcal{C}\ell\langle\quad\rangle$, which we apply to the fractal variables or functions each time we are drawn to the classical domain, for which $dx \gg d\xi$.

3.5. Discrete symmetry breaking

One of the most fundamental consequences of the non-differentiable nature of space (more generally, of space-time) is the breaking of a discrete symmetry, namely, of the reflection invariance on the differential element of (proper) time. As we shall see in what follows, it implies a two-valuedness of velocity which can be subsequently shown to be the origin of the complex nature of the quantum tool.

The derivative with respect to the time t of a differentiable function f can be written twofold

$$\frac{df}{dt} = \lim_{dt\to 0}\frac{f(t+dt)-f(t)}{dt} = \lim_{dt\to 0}\frac{f(t)-f(t-dt)}{dt}\,. \tag{8}$$

The two definitions are equivalent in the differentiable case. In the non-differentiable situation, both definitions fail, since the limits are no longer defined. In the new framework of scale relativity, the physics is related to the behavior of the function during the “zoom” operation on the time scale-variable dt. The nondifferentiable function $f(t)$ is replaced by an explicitly scale-dependent fractal function $f(t,dt)$, which is therefore a function of two variables, t (in space-time) and dt (in scale-space). We therefore define two functions f'_+ and f'_- of the two variables t and dt

$$f'_+(t,dt) = \frac{f(t+dt,dt)-f(t,dt)}{dt}\,,\quad f'_-(t,dt) = \frac{f(t,dt)-f(t-dt,dt)}{dt}\,. \tag{9}$$

One passes from one definition to the other by the transformation $dt \leftrightarrow -dt$ (differential time reflection invariance), which actually was an implicit discrete symmetry of differentiable physics. When applied to fractal space coordinates $x(t,dt)$, these definitions yield, in the non-differentiable domain, two velocity fields instead of one, that are fractal functions of the resolution, $V_+[x(t),t,dt]$ and $V_-[x(t),t,dt]$. In order to go back to the classical domain and to derive the classical velocities, we smooth out each fractal geodesic in the bundles selected by the zooming process with balls of radius larger than τ. This amounts to carry out a transition from the non-differentiable to the differentiable domain and leads to define two classical velocity fields which are now resolution-independent: $V_+[x(t),t,dt>\tau] = \mathcal{C}\ell\langle V_+[x(t),t,dt]\rangle = v_+[x(t),t]$ and $V_-[x(t),t,dt>\tau] = \mathcal{C}\ell\langle V_-[x(t),t,dt]\rangle = v_-[x(t),t]$. The important new fact appearing here is that, after the transition, there is no longer any reason for these two velocity fields to be the same. While, in standard mechanics, the concept of velocity was one-valued, we must introduce, for the case of a non-differentiable space, two velocity fields instead of one, even when going back to the classical domain. In recent papers, Ord [39] also insists on the importance of introducing ‘entwined paths’ for understanding quantum mechanics: however, this two-valuedness is not postulated in the scale-relativity approach, but established as a mere consequence of the non-differentiability.

3.6. ‘Covariant’ total derivative operator

We are now lead to describe the elementary displacements for both processes, $dX_\pm$, as the sum of a $\mathcal{C}\ell$ part, $dx_\pm = v_\pm\, dt$, and a fluctuation about this $\mathcal{C}\ell$ part, $d\xi_\pm$, which is, by definition, of zero classical part, $\mathcal{C}\ell\langle d\xi_\pm\rangle = 0$

$$dX_+(t) = v_+\,dt + d\xi_+(t),\quad dX_-(t) = v_-\,dt + d\xi_-(t). \tag{10}$$

Considering first the large-scale displacements, two derivatives, d_+/dt and d_-/dt, are defined, using the $\mathcal{C}\ell$ part extraction procedure. Applied to the position vector, x, they yield the twin large-scale velocities

$$\frac{d_+}{dt}x(t) = v_+ \ , \qquad \frac{d_-}{dt}x(t) = v_- \ . \tag{11}$$

As regards the fluctuations, the generalization to three dimensions of Eq. (6) writes (for $D_F = 2$)

$$\mathcal{C}\ell\langle d\xi_{\pm i}\, d\xi_{\pm j}\rangle = \pm 2\, \mathcal{D}\, \delta_{ij}\, dt \qquad i,j = x,y,z, \tag{12}$$

as the $d\xi(t)$'s are of null $\mathcal{C}\ell$ part and mutually independent. The Krönecker symbol, δ_{ij}, in Eq. (12), implies indeed that the $\mathcal{C}\ell$ part of every crossed product $\mathcal{C}\ell\langle d\xi_{\pm i}\, d\xi_{\pm j}\rangle$, with $i \neq j$, is null.

3.6.1. Origin of complex numbers in quantum mechanics

We now know that each component of the velocity takes two values instead of one. This means that it becomes itself a vector in a two-dimensional space. The generalization of the sum of these quantities is straighforward, but one also needs to define a generalized product. The problem can be put in a general way: it amounts to find a generalization of the standard product that keeps its fundamental physical properties.

From the mathematical point of view, we are here exactly confronted to the well-known problem of the doubling of algebra (see, e.g., Ref. [65]). Indeed, the effect of the symmetry breaking $dt \leftrightarrow -dt$ (or $ds \leftrightarrow -ds$) is to replace the algebra $\mathcal{A}$ in which the classical physical quantities are defined, by a direct sum of two exemplaries of $\mathcal{A}$, i.e., the space of the pairs (a,b) where a and b belong to $\mathcal{A}$. The new vectorial space $\mathcal{A}^2$ must be supplied with a product in order to become itself an algebra (of doubled dimension). The same problem is asked again when one takes also into account the symmetry breakings $dx^\mu \leftrightarrow -dx^\mu$ and $x^\mu \leftrightarrow -x^\mu$ (see [56]): this leads to new algebra doublings. The mathematical solution to this problem is well-known: the standard algebra doubling amounts to supply $\mathcal{A}^2$ with the complex product. Then the doubling $\mathbb{R}^2$ of $\mathbb{R}$ is the algebra $\mathbb{C}$ of complex numbers, the doubling $\mathbb{C}^2$ of $\mathbb{C}$ is the algebra $\mathbb{H}$ of quaternions, the doubling $\mathbb{H}^2$ of quaternions is the algebra of Graves-Cayley octonions. The problem with algebra doubling is that the iterative doubling leads to a progressive deterioration of the algebraic properties. Namely, one loses the order relation of reals in the complex plane, while the quaternion algebra is non-commutative, and the octonion algebra is also non-associative. But an important positive result for physical applications is that the doubling of a metric algebra is a metric algebra [65].

These mathematical theorems fully justify the use of complex numbers, then of quaternions, in order to describe the successive doublings due to discrete symmetry breakings at the infinitesimal level, which are themselves more and more profound consequences of space-time non-differentiability.

Moreover, we have given elsewhere complementary arguments of a physical nature to this conclusion [24]. We have indeed shown that the use of the complex product has a simplifying and covariant effect on the equations (we use here the word “covariant” in the original meaning given to it by Einstein [1], namely, the requirement of the form invariance of fundamental equations). Indeed, the choice of the complex product allows one to suppress infinite terms in the final equations of motion (see [24] for more detail).

3.6.2. Complex velocity

We now combine the two derivatives to obtain a complex derivative operator, that allows us to recover local differential time reversibility in terms of the new complex process [4]:

$$\frac{\hat{d}}{dt} = \frac{1}{2}\left(\frac{d_+}{dt} + \frac{d_-}{dt}\right) - \frac{i}{2}\left(\frac{d_+}{dt} - \frac{d_-}{dt}\right) \ . \tag{13}$$

Applying this operator to the position vector yields a complex velocity

$$\mathcal{V} = \frac{\hat{d}}{dt}x(t) = V - iU = \frac{v_+ + v_-}{2} - i\,\frac{v_+ - v_-}{2} \ . \tag{14}$$

The real part, V, of the complex velocity, $\mathcal{V}$, represents the standard classical velocity in the fluid-like description (in terms of velocity fields). At the usual classical limit, the velocity fields $v_+ = v_- = v$, so that $V = v$ and $U = 0$. When one now considers the classical deterministic limit (that involves definite and specified initial conditions), it is given by $v_+ = -v_-$, and therefore $V = 0$ and $U = v$ (see [59]).

3.6.3. Complex time-derivative operator

Contrary to what happens in the differentiable case, the total derivative with respect to time of a fractal function $f(x(t),t)$ of integer fractal dimension contains finite terms up to higher order [58]. In the special case of fractal dimension $D_F = 2$ (which we only consider in the present contribution), the total derivative writes

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\frac{dX_i}{dt} + \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}\frac{dX_i dX_j}{dt} \,. \tag{15}$$

Let us now consider the '$\mathcal{C}\ell$ part' of this expression. By definition, $\mathcal{C}\ell\langle dX\rangle = dx$, so that the second term is reduced to $v.\nabla f$. Now with regards to the term $dX_i dX_j/dt$, it is usually infinitesimal, but here its $\mathcal{C}\ell$ part reduces to $\mathcal{C}\ell\langle d\xi_i \, d\xi_j\rangle/dt$. Therefore, thanks to Eq. (12), the last term of the $\mathcal{C}\ell$ part of Eq. (15) amounts to a Laplacian, and we obtain

$$\frac{d_\pm f}{dt} = \left(\frac{\partial}{\partial t} + v_\pm.\nabla \pm \mathcal{D}\Delta\right) f \,. \tag{16}$$

Substituting Eqs. (16) into Eq. (13), we finally obtain the expression for the complex time derivative operator [4]

$$\frac{\acute{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\mathcal{D}\Delta \,. \tag{17}$$

The passage from standard classical mechanics to the new non-differentiable theory can now be implemented by replacing the standard time derivative d/dt by the new complex operator $\acute{d}/dt$ [4] (while remaining cautious with the fact that it involves a combination of first order and second order derivatives, see Sec. 5.1.2). In other words, this means that $\acute{d}/dt$ plays the role of a "covariant derivative operator".

It should be remarked, before going on with this construction, that we use here the word 'covariant' in analogy with the covariant derivative $D_j A^k = \partial_j A^k + \Gamma^k_{jl} A^l$ replacing $\partial_j A^k$ in Einstein's general relativity. But one shoud be cautious with this analogy, since the two situations are different. Indeed, the problem posed in the construction of general relativity was that of a new geometry, in a framework where the differential calculus was not affected. Therefore the Einstein covariant derivative amounts to substracting the new geometric effect $-\Gamma^k_{jl} A^l$ in order to recover the mere inertial motion, for which the Galilean law of motion $Du^k/ds = 0$ naturally holds [68]. In the scale relativity theory there is an additional question to be treated: new effects come not only from the geometry, but also from the non-differentiability and from its consequences on the differential calculus.

Therefore the true status of the new derivative is actually an extension of the concept of total derivative. Already in standard physics, the passage from the free Galileo-Newton's equation to its Euler form was a case of conservation of the form of equations in a more complicated situation, namely, $dv/dt = 0 \rightarrow dv/dt = (\partial/\partial t + v.\nabla)v = 0$. In the non-differentiable situation considered here, the three consequences (infinity of geodesics, fractality and two-valuedness) lead to three new terms in the total derivative operator instead of only one, namely $V.\nabla$, $-iU.\nabla$ and $-i\mathcal{D}\Delta$. Note also that a scale-covariant derivative acting in the same way as that of general relativity (i.e., as an effect of the fractal geometry itself) is introduced in the framework of the scale-relativistic identification of gauge fields as a manifestation of fractality [26, 23, 38])

3.7. Covariant mechanics induced by scale laws

Let us now summarize the main steps by which one may generalize the standard classical mechanics using this covariance. We are now searching for the laws of motion in the standard space. In what follows, we therefore consider only the 'classical parts' of the variables, which are differentiable and independent of resolutions. The effects of the internal non-differentiable structures are now contained in the covariant derivative. We assume that the '$\mathcal{C}\ell$ part' of the mechanical system under consideration can be characterized by a Lagrange function that keeps the usual form but now in terms of the complex velocity, $\mathcal{L}(x, \mathcal{V}, t)$, from which an action $\mathcal{S}$ is defined

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}(x, \mathcal{V}, t) dt. \tag{18}$$

In this expression, we have combined the two velocities in terms of a unique complex velocity. We have recalled, in the previous section, that this choice is a simplifying and covariant choice. Moreover, we have proved in Ref. [24] that it also allows one to conserve the standard form of the Euler-Lagrange equations as a consequence of a generalized stationary action principle. They read

$$\frac{\acute{d}}{dt} \frac{\partial L}{\partial \mathcal{V}} = \frac{\partial L}{\partial x}. \tag{19}$$

Since we now consider only the 'classical parts' of the variables (while the effects on them of the fractal parts are included in the covariant derivative) the basic symmetries of classical physics hold. From the homogeneity of standard space one defines a generalized complex momentum given by

$$\mathcal{P} = \frac{\partial \mathcal{L}}{\partial \mathcal{V}}. \tag{20}$$

If we now consider the action as a functional of the upper limit of integration in Eq. (18), the variation of the action from a trajectory to another yields a generalization of another well-known relation of standard mechanics:

$$\mathcal{P} = \nabla \mathcal{S}. \tag{21}$$

As concerns the generalized energy, its expression involves an additional term [40, 60]: namely it write for a Newtonian Lagrange function and in the absence of exterior potential, $\mathcal{E} = (1/2)m(\mathcal{V}^2 - 2i\mathcal{D}\,\mathrm{div}\mathcal{V})$.

3.8. Newton-Schrödinger Equation

3.8.1. Geodesics equation

Let us now specialize our study, and consider Newtonian mechanics, i.e., the general case when the structuring external scalar field is described by a potential energy Φ. The Lagrange function of a closed system, $L = \frac{1}{2}mv^2 - \Phi$, is generalized, in the large-scale domain, as $\mathcal{L}(x, \mathcal{V}, t) = \frac{1}{2}m\mathcal{V}^2 - \Phi$. The Euler-Lagrange equations keep the form of Newton's fundamental equation of dynamics

$$m \frac{\acute{d}}{dt} \mathcal{V} = -\nabla \Phi, \tag{22}$$

which is now written in terms of complex variables and complex operators.

In the case when there is no external field (and more generally when the field is gravitational, see following section), the covariance is explicit, since Eq. (22) takes the form of the equation of inertial motion, i.e., of a geodesics equation,

$$\acute{d}\,\mathcal{V}/dt = 0, \tag{23}$$

This is analog to Einstein's general relativity, where the equivalence principle and the strong covariance principle both lead to write the equation of motion of a free particle in the form of a geodesics equation, $Du_\mu = 0$, that keeps the form of free inertial motion in terms of the general-relativistic covariant derivative D and of the four-velocity vector u_μ.

The covariance induced by scale effects leads to an analogous transformation of the equation of motions, which, as we show below, become after integration the Schrödinger equation, (then the Klein-Gordon and Dirac equations in the motion-relativistic case), which we can therefore consider as the integral of a geodesics equation (they actually have the status of Hamilton-Jacobi equations).

In both cases, with or without external field, the complex momentum $\mathcal{P}$ reads

$$\mathcal{P} = m\mathcal{V}, \tag{24}$$

so that, from Eq. (21), the complex velocity $\mathcal{V}$ appears as a gradient, namely the gradient of the complex action

$$\mathcal{V} = \nabla \mathcal{S}/m. \tag{25}$$

3.8.2. Complex wave function

We now introduce a complex wave function ψ which is nothing but another expression for the complex action $\mathcal{S}$

$$\psi = e^{i\mathcal{S}/\mathcal{S}_0}. \tag{26}$$

The factor $\mathcal{S}_0$ has the dimension of an action (i.e., an angular momentum) and must be introduced for dimensional reasons. We show in what follows, that, when this formalism is applied to the standard quantum mechanics (in microphysics), S_0 is nothing but the fundamental constant $\hbar$. But it may also take more general forms (see in what follows the application to gravitational structures and living systems). The function ψ is related to the complex velocity appearing in Eq. (25) as follows

$$\mathcal{V} = -i\,\frac{\mathcal{S}_0}{m}\,\nabla(\ln\psi). \tag{27}$$

3.8.3. Schrödinger equation

We have now at our disposal all the mathematical tools needed to write the fundamental equation of dynamics of Eq. (22) in terms of the new quantity ψ. It takes the form [4]

$$i\mathcal{S}_0 \frac{\acute{d}}{dt}(\nabla \ln\psi) = \nabla\Phi. \tag{28}$$

Now one should be aware that $\acute{d}$ and ∇ do not commute. However, as we shall see in the following, there is a particular choice of the arbitrary constant $\mathcal{S}_0$ for which $\acute{d}(\nabla\ln\psi)/dt$ is nevertheless a gradient.

Replacing $\acute{d}/dt$ by its expression, given by Eq. (17), yields

$$\nabla\Phi = i\mathcal{S}_0\left(\frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\mathcal{D}\Delta\right)(\nabla\ln\psi), \tag{29}$$

and replacing once again $\mathcal{V}$ by its expression in Eq. (27), we obtain

$$\nabla\Phi = i\mathcal{S}_0\left[\frac{\partial}{\partial t}\nabla\ln\psi - i\left\{\frac{\mathcal{S}_0}{m}(\nabla\ln\psi.\nabla)(\nabla\ln\psi) + \mathcal{D}\Delta(\nabla\ln\psi)\right\}\right]. \tag{30}$$

Consider now the remarkable identity [4]

$$(\nabla\ln f)^2 + \Delta\ln f = \frac{\Delta f}{f}\ , \tag{31}$$

which proceeds from the following tensorial derivation

$$\partial_\mu\partial^\mu\ln f + \partial_\mu\ln f\partial^\mu\ln f = \partial_\mu\frac{\partial^\mu f}{f} + \frac{\partial_\mu f}{f}\frac{\partial^\mu f}{f} = \frac{f\partial_\mu\partial^\mu f - \partial_\mu f\partial^\mu f}{f^2} + \frac{\partial_\mu f\partial^\mu f}{f^2} = \frac{\partial_\mu\partial^\mu f}{f}\ . \tag{32}$$

When we apply this identity to ψ and take its gradient, we obtain

$$\nabla\left(\frac{\Delta\psi}{\psi}\right) = 2(\nabla\ln\psi.\nabla)(\nabla\ln\psi) + \Delta(\nabla\ln\psi). \tag{33}$$

We recognize, in the right-hand side of this equation, the two terms of Eq. (30), which were respectively in factor of $\mathcal{S}_0/m$ and $\mathcal{D}$. Therefore, the particular choice

$$\mathcal{S}_0 = 2m\mathcal{D} \tag{34}$$

allows us to simplify the right-hand side of Eq. (30). This is more general than standard quantum mechanics, in which S_0 is restricted to the only value $S_0 = \hbar$. Eq. 34 is actually a generalization of the Compton relation (see next section): this means that the function ψ becomes a wave function only provided it is accompanied by a Compton-de Broglie relation. Without this condition, the equation of motion would remain of third order, with no general prime integral. Indeed, the simplification brought by this choice is twofold: (i) several complicated terms are compacted into a simple one; (ii) the final remaining term is a gradient, which means that the fundamental equation of dynamics can now be integrated in a universal way. The function ψ in Eq. (26) is therefore defined as

$$\psi = e^{i\mathcal{S}/2m\mathcal{D}}, \tag{35}$$

and it is solution of the fundamental equation of dynamics, Eq. (22), which we write

$$\frac{\hat{d}}{dt}\mathcal{V} = -2\mathcal{D}\nabla\left\{i\frac{\partial}{\partial t}\ln\psi + \mathcal{D}\frac{\Delta\psi}{\psi}\right\} = -\nabla\Phi/m. \tag{36}$$

Integrating this equation finally yields

$$\mathcal{D}^2\Delta\psi + i\mathcal{D}\frac{\partial}{\partial t}\psi - \frac{\Phi}{2m}\psi = 0, \tag{37}$$

up to an arbitrary phase factor which may be set to zero by a suitable choice of the ψ phase.

Arrived at that point, several steps have been already made toward the final identification of the function ψ with a wave function: it is complex, solution of a Schrödinger equation, so that its linearity is also ensured: namely, if ψ_1 and ψ_2 are solutions, $a_1\psi_1 + a_2\psi_2$ is also a solution. Let us complete the proof by giving new insights about other basic axioms of quantum mechanics.

3.8.4. Compton length

In the case of standard quantum mechanics, as applied to microphysics, the necessary choice $\mathcal{S}_0 = 2m\mathcal{D}$ means that there is a natural link between the Compton relation and the Schrödinger equation. In this case, indeed, $\mathcal{S}_0$ is nothing but the fundamental action constant $\hbar$, while $\mathcal{D}$ is the constant on which the fractal/non-fractal transition relies. Therefore, the relation $\mathcal{S}_0 = 2m\mathcal{D}$ becomes a relation between mass and the transition from fractal behavior to scale-independence, which writes

$$\lambda_c = \frac{\hbar}{mc}\,. \tag{38}$$

We recognize here the definition of the Compton length. Therefore we have reached a new definition of the inertial mass of a particle: in our framework it has acquired a geometric meaning. This length-scale is to be understood as a structure of scale-space, not of standard space. The de Broglie length (which the true fractal-nonfractal transition) can now be easily recovered: the fractal fluctuation reads $< d\xi^2 >= \hbar dt/m$ in function of dt, and then $< d\xi^2 >= \lambda_x dx$ in function of the space differential elements. This implies $\lambda_x = \hbar/mv$, which is the non-relativistic de Broglie length.

3.8.5. Born postulate

The statistical meaning of the wave function (Born postulate) can now be deduced in the one-dimensional and stationary case from the very construction of the theory. Even in the case of only one particle, the virtual geodesic family is infinite (this remains true even in the zero particle case, i.e., for the vacuum field). The particle properties are assimilated to those of a random subset of the geodesics in the family, and its probability to be found at a given position must be proportional to the density of the geodesics fluid. This density can easily be calculated in our formalism.

Indeed, we have found up to now two equivalent representations of the same equations: (i) a geodesics equation (in the free case) $\hat{d}\mathcal{V}/dt = 0$, depending on two variables V and U, i.e. the real and imaginary parts of the complex velocity; (ii) a Schrödinger equation, that depends on two variables $\sqrt{P}$ and θ, i.e. the modulus and the phase of ψ.

Now, a third mixed equivalent representation is possible (see Sec. 4.1.3), in terms of the couple of variables (P, V). This opens the possibility to get a derivation of Born's postulate in this context. This question has already been considered by Hermann [59], who obtained numerical solutions of the equation of motion (22) in terms of a large number of explicit trajectories (in the case of a free particle in a box). He constructed a probability density from these trajectories and recovered in this way solutions of the Schrödinger equation without writing it.

In function of these variables, the imaginary part of Eq. (37) writes

$$\frac{\partial P}{\partial t} + \mathrm{div}(PV) = 0, \tag{39}$$

where V is identified, at the classical limit, with the classical velocity. This equation is recognized as an equation of continuity. This implies that $P = \psi\psi^\dagger$ is related to the probability density ρ (which is also subjected to an equation of continuity) by a relation $P = K\rho$, such that $dK/dt = \partial K/\partial t + V.\nabla K = 0$. In the one-dimensional stationary case this implies that K =cst, thus ensuring the validity of Born's postulate in this case and in all those that can be brought back to it (separation of variables, etc...). The general case will be considered in a forthcoming work.

3.8.6. Von Neumann postulate

The von Neumann postulate (i.e. the axiom of wave function collapse) is also easily recovered in such a geometric interpretation. Indeed, we may identify a measurement with a selection of the sub-sample of the geodesics family that keeps only the geodesics having the geometric properties corresponding to the measurement result. Therefore, just after the measurement, the system is in the state given by the measurement result.

3.8.7. Schrödinger form of other fundamental equations of physics

The general method described above can be applied to any physical situation where the three basic conditions (namely, infinity of trajectories, each trajectory is a fractal curve of fractal dimension 2, breaking of differential time reflexion invariance) are achieved in an exact or in an approximative way. Several fundamental equations of classical physics can be transformed to take a generalized Schrödinger form under these conditions: namely, equation of motion in the presence of an electromagnetic field, the Euler and Navier-Stokes equations in the case of potential motion and for incompressible and isentropic fluids; the equation of the rotational motion of solids, the motion equation of dissipative systems; field equations (scalar field for one space variable). We cannot enter here into the detail of these generalizations, so we refer the interested reader to Ref. [27].

4. APPLICATIONS TO VARIOUS SCIENCES

4.1. Application to gravitation

4.1.1. Curved and fractal space

In several physical macroscopic situations (in particular those involving strong chaos and irreversibility), the three conditions upon which the derivation of a generalized Schrödinger equation relies (infinity or large number of potential trajectories, Brownian-like character of each trajectory and discrete symmetry breaking of local time reflexion invariance) can be considered as satisfied. This remark has led us to suggest that a generalized Schrödinger approach could be used in such situations.

In particuler, applications of the theory to the problem of the formation and evolution of gravitational structures have been presented in several previous works [4, 23, 41, 27, 42, 43, 44, 45]. A recent review about the comparison between the theoretically predicted structures and the observational data, from the scale of planetary systems to extragalactic scales, has been given in [46]. We shall only briefly sum up here the principles and the methods used in such an attempt, then give an example of application to the Solar system and to the extra-solar planetary systems.

In its present acceptance, gravitation is understood as the various manifestations of the geometry of space-time at large scales. Up to now, in the framework of Einstein's theory, this geometry was considered to be Riemannian, i.e. curved. However, in the new framework of scale relativity, the geometry of space-time is assumed to be characterized not only by curvature, but also by fractality beyond a new relative time-scale and/or space-scale of transition, which is an horizon of predictibility for the classical deterministic description.

4.1.2. Gravitational Schrödinger equation

We shall briefly consider, in what follows, only the Newtonian limit. In this case the equation of geodesics keeps the form of Newton's fundamental equation of dynamics in a gravitational field, namely,

$$\frac{\bar{D}\mathcal{V}}{dt} = \frac{d\mathcal{V}}{dt} + \nabla\left(\frac{\phi}{m}\right) = 0, \tag{40}$$

where ϕ is the Newtonian potential energy. As demonstrated hereabove, once written in terms of ψ, this equation can be integrated to yield a gravitational Newton-Schrödinger equation :

$$\mathcal{D}^2 \triangle \psi + i\mathcal{D}\frac{\partial}{\partial t}\psi = \frac{\phi}{2m}\psi. \tag{41}$$

The situation and therefore part of the interpretation are different in this macroscopic case from the application of the theory to the microphysical domain. The two main differences are:

(i) While in the microscopic realm elementary "particles" can be defined as the geodesics themselves, in the macroscopic domain there does exist actual particles that follow the geodesics.

(ii) While differentiability is definitively lost toward the small scales in the microphysical domain, the macroscopic quantum theory is valid only beyond some time-scale transition which is an horizon of predictibility. Therefore in this last case there is an underlying classical theory toward small time scales, which means that the quantum macroscopic approach is not expected to violate Bell inequalities [27], so that it remains "classical" in several of its aspects.

Even though it takes this Schrödinger-like form, equation (41) is still in essence an equation of gravitation, so that it must keep the fundamental properties it owns in Newton's and Einstein's theories. Namely, it must agree with the equivalence principle [41, 48], i.e., it is independent of the mass of the test-particle. In the Kepler central potential case ($\phi = -GMm/r$), GM provides the natural length-unit of the system under consideration. As a consequence, the parameter $\mathcal{D}$ takes the form:

$$\mathcal{D} = \frac{GM}{2w}, \tag{42}$$

where w is a fundamental constant that has the dimension of a velocity. The ratio $\alpha_g = w/c$ actually plays the role of a macroscopic gravitational coupling constant [48, 45]).

4.1.3. Formation and evolution of gravitational structures

Let us now compare our approach with the standard theory of gravitational structure formation and evolution. Instead of the Euler-Newton equation and of the continuity equation which are used in the standard approach, we write the only above Newton-Schrödinger equation. In both cases, the Newton potential is given by the Poisson equation. Two situations can arise: (i) when the 'orbitals', which are solutions of the motion equation, can be considered as filled with the particles (e.g., planetesimals in the case of planetary systems formation, interstellar gas and dust in the case of star formation, etc...), the mass density ρ is proportional to the probability density $P = \psi\psi^{\dagger}$: this situation is relevant in particular for addressing problems of structure formation; (ii) another possible situation concerns test bodies which are not in sufficiently large number to change the matter density, but whose motion is nevertheless submitted to the Newton-Schrödinger equation: this case may be relevant for the anomalous dynamical effects which have up to now been attributed to unseen, "dark" matter.

By separating the real and imaginary parts of the Schrödinger equation we obtain respectively a generalized Euler-Newton equation and a continuity equation that adds to the Poisson equation:

$$m\left(\frac{\partial}{\partial t} + V \cdot \nabla\right)V = -\nabla(\phi + Q), \tag{43}$$

$$\frac{\partial P}{\partial t} + \mathrm{div}(PV) = 0, \tag{44}$$

$$\Delta\phi = 4\pi G\rho m. \tag{45}$$

In the case $P \propto \rho$ this system of equations is equivalent to the classical one used in the standard approach of gravitational structure formation, except for the appearance of an extra potential energy term Q that writes:

$$Q = -2m\mathcal{D}^2 \frac{\Delta\sqrt{P}}{\sqrt{P}}. \tag{46}$$

The existence of this potential energy, is, in our approach a very manifestation of the fractality of space [40], in similarity with Newton's potential being a manifestation of curvature in the general relativity framework.

4.1.4. Example of application: solar and extrasolar planetary systems

The theory has been able to predict in a quantitative way a large number of new effects in the domain of gravitational structures [46]. Moreover, these predictions have been successfully checked in various systems on a large range of scales and in terms of a common gravitational coupling constant (or one of its multiples or submultiples) whose value averaged on these systems was found to be $w_0 = c\alpha_g = 144.7 \pm 0.7$ km/s [41]. New structures have been theoretically predicted, then checked by the observational data in a statistically significant way, for our solar system, including distances of planets [4, 42] and satellites [44], sungrazer comet perihelions [46], obliquities and inclinations of planets and satellites [49], exoplanets semi-major axes [41, 45] (see Fig. 1) and eccentricities [46] (see Fig. 2), including planets around pulsars, for which a high precision is reached [41, 50], double stars [43], planetary nebula [46], binary galaxies [23], our local group of galaxies [46], clusters of galaxies and large scale structures of the universe [43, 46].

Let us consider in more detail the application of the theory to the formation of planetary systems. The standard model of formation of planetary systems can be reconsidered in terms of a fractal description of the motion of planetesimals in the protoplanetary nebula. On length-scales much larger than that their mean free path, we have assumed [4] that their highly chaotic motion satisfy the three conditions upon which the derivation of a Schrödinger equation is based (large number of trajectories, fractality and time symmetry breaking).

We assume that such a method can be applied to the distribution of planetesimals in a protoplanetary nebula which has formed in the potential $\phi = -GM/r$ of a star of mass M. During the planetesimal era, there is no defined orbital parameter such as semi-major axis a or eccentricity e. But the solutions of the Schrödinger equation describe stationary states for which conservative quantities, such as the energy E, the projection on a given axis of the angular momentum, L_z, and of the Runge-Lenz vector, A_z, can have determined values. Once the planet formed from a distribution of planetesimals described by such a state and once the system stabilized, the planet recovers classical orbital parameters. With regards to, e.g., semi-major axes, the planetesimals are expected to fill the 'orbital' characterized by a conservative energy E, then they finally accrete to yield a planet whose semi-major axis will be, with highest probability, given by $a = -GMm/2E$, according to conservation laws.

It is noticeable that this proposal, made more than ten years ago [4], has anticipated the concept of migration and also the recent results of numerical simulations according to which the migration is not monotonous but Brownian-like. Migration is the result of the coupling between a protoplanet and the protoplanetary disk, while such a coupling is exactly what is described by the additional terms that lead to the Schrödinger representation.

This description applies to the distribution of planetesimals in the proto-planetary nebula at several embedded levels of hierarchy. Each hierarchical level (k) is characterized by a length-scale defining the parameter $\mathcal{D}_k$ (and therefore the velocity w_k) that appears in the generalized Schrödinger equation describing this sub-system. The matching of the wave functions implies that the ratios of the velocity parameters w_k/w_{k-1} be integer.

In each subsystem characterized by w_k, one expects the occurence of peaks of probability density for semimajor axes $a_n = GM(n/w_k)^2$, where n is integer and M is the star mass [41]. Through Kepler's third law, this is equivalent to peaks of probability density of periods at $P_n = 2\pi GM(n/w_k)^3$, and of effective velocities at $v_n = 2\pi a_n/P_n = w_k/n$.

With regards to the theoretical prediction of the eccentricity distribution, one remarks that the solutions of the Schrödinger equation in parabolic coordinates describe states of fixed E, L_z and A_z, where A is the Runge-Lenz vector, which is a conservative quantity that expresses a dynamical symmetry specific of the Kepler problem (see e.g. [61]). This vector identifies with the major axis of the orbit and is directed toward the perihelion while its modulus is the eccentricity itself. Therefore the theoretical expectation for the eccentricity distribution is obtained by taking as z axis the major axis of the orbit. One finds $A_z = e = k/n$, where the number k is an integer and varies from 0 to $n-1$.

This hierarchical model has allowed us to recover the mass distribution of planets and small planets in the inner and outer solar systems [42]. It is generally supported by the structure of our own solar system, which is made of several subsystems embedded one in anothers:

***The Sun**. Through Kepler's third law, the velocity $w = 3 \times 144.7 = 434.1$ km/s is very closely the Keplerian velocity at the Sun radius ($R_\odot = 0.00465$ AU corresponds to $w = 437.1$ km/s). This result opens the possibility that the whole structure of the solar system be ultimately brought back to the Sun radius itself. Such a result is not unexpected in the scale-relativity approach. Indeed, the fundamental equation of stellar structure is the Euler equation, which can also be transformed in a Schrödinger equation [27], yielding preferential values for star radii. Matching conditions between the probability amplitude that describes the interior matter distribution (the Sun) and the exterior solution (the Solar System) during the formation epoch imply a matching of the positions of the probability peaks.

Moreover, one can also apply our approach to the organization of the solar plasma itself. In such a framework, one expect the distribution of the various relevant physical quantities that characterize the solar activity at the Sun surface (sun spot number, magnetic field, etc...) to be described by a wave function whose stationary solutions read

$$\psi = \psi_0 \, e^{iEt/2m\mathcal{D}}. \tag{47}$$

The energy E results from the rotational velocity and, to be complete, should also include the turbulent velocity, so that $E = (v_{rot}^2 + v_{turb}^2)/2$. This means that we expect the solar surface activity to be subjected to a fundamental period (which is nothing but the macroscopic equivalent of a de Broglie period for the Sun) given by:

$$\tau = \frac{2\pi m\mathcal{D}}{E} = \frac{4\pi\mathcal{D}}{v_{rot}^2 + v_{turb}^2}, \tag{48}$$

(including a factor of 2 when going from the probability amplitude ψ to the probability density $|\psi|^2$). Now we can use our knowledge of the parameter $\mathcal{D}$ at the Sun radius, $\mathcal{D} = GM_\odot/2w_\odot$ to finally obtain:

$$\tau = \frac{2\pi GM_\odot}{w_\odot(v_{rot}^2 + v_{turb}^2)}. \tag{49}$$

If the matter of the Sun was still rotating with its Keplerian velocity, one would have $v_{rot} = w_\odot$, so that one would recover (neglecting the turbulent velocity) the Keplerian period at the distance of the Sun radius, $P = 2\pi GM_\odot/w_\odot^3$. However, as many other Sun-like stars, the Sun has been subjected to an important angular momentum loss since its formation (see e.g. [52]). As a consequence, its rotational velocity is about 200 times smaller. The average sideral rotation period is 25.38 days, yielding a velocity of 2.01 km/s at equator [53]. The turbulent velocity has been found to be $v_{turb} = 1.4 \pm 0.2$ km/s [54]. Therefore we find numerically

$$\tau = (10.2 \pm 1.0)\,\text{yr}. \tag{50}$$

The observed value of the period of the Solar activity cycle, $\tau_{obs} = 11.0$ yr, supports this theoretical prediction. We shall in future works test this proposal by a more detailed study of the solar and star activity.

*__The intramercurial system__, organized on the constant $w_\odot = 3 \times 144 = 432$ km/s. The existence of an intramercurial subsystem is supported by various stable and transient structures observed in dust, asteroid and comet distributions (see [46]). We have in particular suggested the existence of a new ring of asteroids (of size less than ≈ 10 km), the 'Vulcanoid belt', at a preferential distance of about 0.17 AU from the Sun (orbital $n = 2$ based on $w_{in} = 144$ km/s, i.e. $n = 6$ in terms of $w_\odot = 432$ km/s). Recent numerical simulations have shown that there should exist a stable zone between 0.1 and 0.2 AU, and, moreover, that some of the known Earth-crossing asteroids could well come from this zone (see[46] and references therein).

*__The inner solar system__ (earth-like planets), organized with a constant $w_i = 144$ km/s (see Fig. 1). Hence, the velocity of Mercury ($n = 3$ is $47.9 \approx 144/3$ km/s, of Venus ($n = 4$) $35.0 \approx 144/4$ km/s, of the Earth ($n = 5$) $29.8 \approx 144/5$ km/s, of Mars ($n = 6$) $24.1 \approx 144/6$ km/s and of Ceres ($n = 8$) $144/8$ km/s.

*__The outer solar system__ (Jovian planets), organized with a constant $w_o = 144/5 = 29$ km/s, as deduced from the fact that the mass peak of the inner solar system lies at the Earth distance ($n = 5$). The whole inner solar system ranks $n = 1$ in the outer system, while Jupiter, Saturn, Uranus, Neptune and Pluto rank respectively at $n = 2$, 3, 4, 5 and 6. The recently discovered Kuiper and scattered Kuiper belt objects show peaks of probability at $n = 6$ to 9 [46], as predicted before their discovery [51].

We have suggested more than ten years ago [4, 51], (i.e. before the first discovery of exoplanets), that the theoretical predictions from this approach should apply to all planetary systems, not only our own solar system. Meanwhile more than 120 exoplanets have been discovered, and the theoretical predictions about the semi-major axes and the eccentricities have been put to the test. The observations support them in a highly statistically significant way (see [41, 45, 46] and Figs. 1 and 2).

A full account of this new domain would be too long to be included in the present contribution. We have given here only few typical examples of these effects (see the figures) and we refer the interested reader to the review paper Ref. [46] and references therein for more detail.

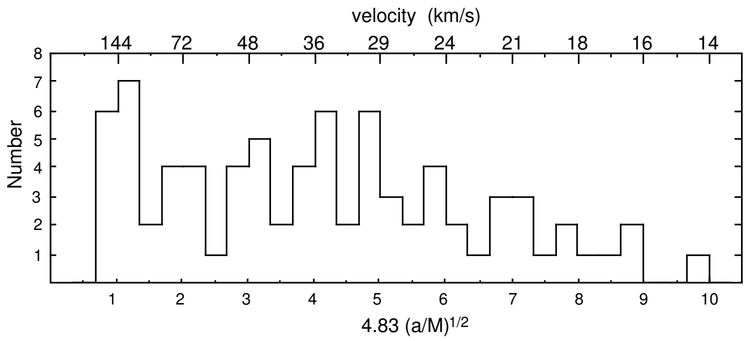

FIGURE 1. Observed distribution of semi-major axes and velocities of inner solar system planets and of recently discovered exoplanets, compared with the theoretical prediction from the scale-relativity / Schrödinger approach (peaks of probability for $v_n = 144/n$ km/s, see text). The data supports the theoretical prediction in a statistically significant way (the probability to obtain such an agreement by chance is $P = 4 \times 10^{-5}$).

4.1.5. A new approach to the "dark matter" problem

We have recently suggested [29, 46, 30] that the additional scalar field Q (Eq. 46), which is a manifestation of the fractality of space, may be responsible for the various dynamical and lensing effects which are usually attributed to unseen "dark matter". Recall that up to now two hypotheses have been formulated in order to account for these effects (which are far larger than those due to visible matter): (i) The existence of a very large amount of unseen matter in the Universe: but, despite intense and continuous efforts, it has escaped detection. (ii) A modification of Newton's law of force: but such an ad hoc hypothesis seems impossible to reconcile with its geometric origin in general relativity, which lets no latitude for modification. In the scale-relativity proposal, there is no need for additional matter, and Newton's potential is unchanged since it remains linked to curvature, but there is an additional potential linked to fractality.

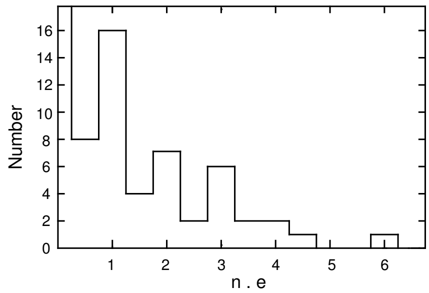

FIGURE 2. Observed distribution of the eccentricities of exoplanets. The theory predicts that the product of the eccentricity e by the quantity $\tilde{n} = 4.83(a/M)^{1/2}$, where a is the semi-major axis and M the parent star mass, should cluster around integers. The data support this theoretical prediction at a probability level $P = 10^{-4}$ [46].

This result can be applied, as an example, to the motion of bodies in the outer regions of spiral galaxies. In these regions there is practically no longer any visible matter, so that the Newtonian potential (in the absence of additional dark matter) is Keplerian. While the standard Newtonian theory predicts for the velocity of the halo bodies $v \propto \phi^{1/2}$, i.e. $v \propto r^{-1/2}$, we predict in our theory $v \propto |(\phi+Q)/m|^{1/2}$, i.e., $v =$ constant. More specifically, assuming that the gravitational Schrödinger equation is solved for the halo objects in terms of the fundamental level wave function, one finds $Q_{pred} = -(GMm/2r_B)(1-2r_B/r)$, where $r_B = GM/w_0^2$. This is exactly the result which is systematically observed in spiral galaxies (i.e., flat rotation curves) and which has motivated (among other effects) the assumption of the existence of dark matter.

4.2. Application to sciences of life

Self-similar fractal laws have already been used as models for the description of a huge number of biological systems (lungs, blood network, brain, cells, vegetals, etc..., see e.g. [7, 71], previous volumes, and references therein).

The scale-relativistic tools may also be relevant for a description of behaviors and properties which are typical of living systems. Some examples have been given in [32, 33, 34, 35, 70], with regards to halieutics, morphogenesis, log-periodic branching laws and cell "membrane" models. As we shall see in what follows, scale relativity may also provide a physical and geometric framework for the description of additional properties such as formation, duplication, morphogenesis and imbrication of hierarchical levels of organization. This approach does not mean to dismiss the importance of chemical and biological laws in the determination of living systems, but on the contrary to attempt to establish a geometric foundation that could underlie them.

4.2.1. Morphogenesis

The Schrödinger equation can be viewed as a fundamental equation of morphogenesis. Scale-relativity extends the potential domain of application of Schrödinger-like equations to systems in which the three conditions (infinite or very large number of trajectories, fractal dimension 2 of individual trajectories, local irreversibility) are fulfilled. These three above conditions seem to be particularly well adapted to the description of living systems. Macroscopic Schrödinger equations can therefore be constructed, which are not based on Planck's constant $\hbar$, but on constants that are specific of each system and may emerge from their self-organization.

Let us give a simple example of such an application to flower-like morphogenesis. In living systems, morphologies are acquired through growth processes. One can attempt to describe such a growth in terms of an infinite family of virtual, fractal and locally irreversible, trajectories. Their equation can therefore be written under the form (22), then it can be integrated in terms of a Schrödinger equation (37).

We now look for solutions describing a growth from a center. This problem is formally identical to that of planetary nebulae [46, 47] (which are stars that eject their outer shells), and, from the quantum point of view, to the problem of particle scattering. The solutions looked for correspond to the case of the outgoing spherical probability wave.

Depending on the potential, on the boundary and on the symmetry conditions, a large family of solutions can be obtained. Let us consider here only the simplest ones, i.e., central potential and spherical symmetry. The probability density distribution of the various possible values of the angles are given in this case by the spherical harmonics:

$$P(\theta, \varphi) = |Y_{lm}(\theta, \varphi)|^2. \tag{51}$$

These functions show peaks of probability for some angles, depending on the quantized values of the square of angular momentum L^2 quantum number l) and of its projection L_z on axis z (quantum number m).

Finally a more probable morphology is obtained by making the structure grow along angles of maximal probability. The solutions obtained in this way, show floral 'tulip'-like shape (see Ref. [46, 47, 70]). Now the spherical symmetry is broken in the case of living systems. One jumps to discrete cylindrical symmetry: this leads in the simplest case to introduce a periodic quantization of angle θ (measured by an additional quantum number k), that gives rise to a separation of discretized petals. Moreover there is a discrete symmetry breaking along axis z linked to orientation (separation of 'up' and 'down' due to gravity, growth from a stem). This results in floral shapes such as given in Fig. 3.

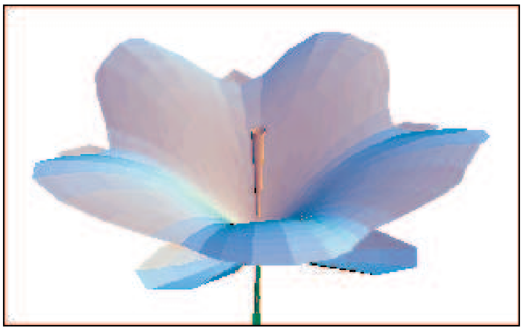

FIGURE 3. Morphogenesis of flower-like structure, solution of a Schrödinger equation that describes a growth process ($l = 5$, $m = 0$). The 'petals' and 'sepals' and 'stamen' are traced along angles of maximal probability density. A constant force of 'tension' has been added, involving an additional curvature of petals, and a quantization of the angle θ that gives an integer number of petals (here, $k = 5$).

4.2.2. Formation, duplication and bifurcation

A fondamentally new feature of the scale-relativity approach with regards to problems of formation is that the Schrödinger form taken by the geodesics equation can be interpreted as a general tendency to make structures, i.e., to self-organization. In the framework of a classical deterministic approach, the question of the formation of a system is always posed in terms of initial conditions. In the new framework, structures are formed whatever the initial conditions, in correspondance with the field, the boundary conditions and the symmetries, and in function of the values of the various conservative quantities that characterize the system.

A typical example is given by the formation of gravitational structures from a background medium of strictly constant density. This problem has no classical solution: no structure can form and grow in the absence of large initial fluctuations. On the contrary, in the present quantum-like approach, the stationary Schrödinger equation for an harmonic oscillator potential (which is the form taken by the gravitational potential in this case) does have confined stationary solutions. The 'fundamental level' solution ($n = 0$ is made of one object with Gaussian distribution, the second level ($n = 1$) is a pair of objects (see Fig. 4), then one obtains chains, trapezes, etc... for higher levels. It is remarkable that, whatever the scales in the large scale Universe (stars, clusters of stars, galaxies, clusters of galaxies) the zones of formation show in a systematic way this kind of double, aligned or trapeze-like structures [46].

Now these solutions may also be meaningful in other domains than gravitation, because the harmonic oscillator potential is encountered in a wide range of conditions. It is the general force that appears when a system is displaced from its equilibrium conditions, and, moreover, it describes an elementary clock. For these reasons, it is well adapted to an attempt of description of living systems, at first in a rough preliminary way.

Firstly, such an approach could allow one to ask the question of the origin of life in a renewed way. This problem is the analog of the 'vacuum' (lowest energy) solutions, i.e. of the passage from a non-structured medium to the simplest, fundamental level structures. Provided the description of the prebiotic medium comes under the three basic conditions (complete information loss on angles, position and time), we suggest that it could be subjected to a Schrödinger equation (with a coefficient $\mathcal{D}$ self-generated by the system itself). Such a possibility is supported by the symplectic formal structure of thermodynamics [55], in which the state equations are analogous to Hamilton-Jacobi equations. But clearly much pluridisciplinary work is needed in order to implement such a working program.

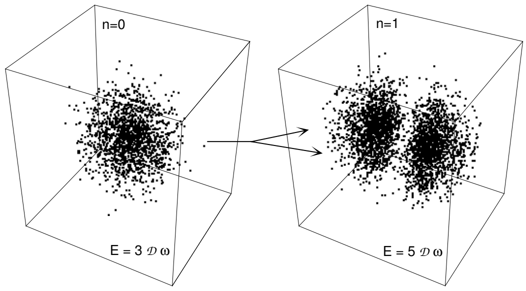

FIGURE 4. Model of duplication. The stationary solutions of the Schrödinger equation in a 3D harmonic oscillator potential can take only discretized morphologies in correspondence with the quantized value of the energy. Provided the energy increases from the one-object case ($E_0 = 3\mathcal{D}\omega$), no stable solution can exist before it reaches the second quantized level at $E_1 = 5\mathcal{D}\omega$. The solutions of the time-dependent equation show that the system jumps from the one object to the two-object morphology.

Secondly, the analogy can be pushed farther, since the passage from the fundamental level to the first excited level provides us with a (rough) model of duplication (see [70], Figs. 4 and 5). Once again, the quantization implies that, in case of energy increase, the system will not increase its size, but will instead be lead to jump from a one-object structure to a two-object structure, with no stable intermediate step between the two stationary solutions $n = 0$ and $n = 1$. Moreover, if one comes back to the level of description of individual trajectories, one finds that from each point of the initial one body-structure there exist trajectories that go to the two final structures. We expect, in this framework, that duplication needs a discretized and precisely fixed jump in energy.

Such a model can also be applied to the description of a branching process (Fig. 5), e.g. in the case of a tree growth when the previous structures remain and add themselves along a z axis instead of disappearing as in cell duplication.

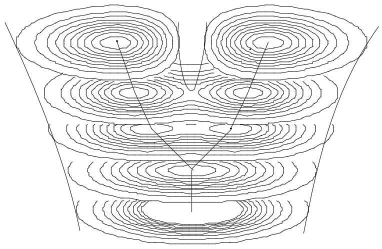

FIGURE 5. Model of branching / bifurcation. Successive solutions of the time-dependent 2D Schrödinger equation in an harmonic oscillator potential are plotted as isodensities. The energy varies from the fundamental level ($n = 0$) to the first excited level ($n = 1$), and as a consequence the system jumps from a one-object to a two-object morphology.

5. FRACTAL SPACE-TIME AND RELATIVISTIC QUANTUM MECHANICS

5.1. Klein-Gordon equation

5.1.1. Theory

Let us now come back to standard quantum mechanics, but in the motion-relativistic case (i.e., classical Minkowski space-time). We shall recall here how one can get the free and electromagnetic Klein-Gordon equations, as already presented in [26, 23, 60].

Most elements of our approach as described hereabove remain correct, with the time differential element dt replaced by the proper time differential element, ds. Now not only space, but the full space-time continuum, is considered to be nondifferentiable, and therefore fractal. We chose a metric of signature $(+,-,-,-)$. The elementary displacement along geodesics now writes (in the standard case $D_F = 2$):

$$dX^{\mu}_{\pm} = dx^{\mu}_{\pm} + d\xi^{\mu}_{\pm}. \tag{52}$$

Due to the breaking of the reflection symmetry ($ds \leftrightarrow -ds$) issued from non-differentiability, we still define two 'classical' derivatives, d_+/ds and d_-/ds, which, once applied to x^μ, yield two classical 4-velocities,

$$\frac{d_+}{ds} x^{\mu}(s) = v^{\mu}_{+} \quad ; \quad \frac{d_-}{ds} x^{\mu}(s) = v^{\mu}_{-}. \tag{53}$$

These two derivatives can be combined in terms of a complex derivative operator

$$\frac{\acute{d}}{ds} = \frac{(d_+ + d_-) - i\,(d_+ - d_-)}{2ds}, \tag{54}$$

which, when applied to the position vector, yields a complex 4-velocity

$$\mathcal{V}^{\mu} = \frac{\acute{d}}{ds} x^{\mu} = V^{\mu} - i\,U^{\mu} = \frac{v^{\mu}_{+} + v^{\mu}_{-}}{2} - i\,\frac{v^{\mu}_{+} - v^{\mu}_{-}}{2}. \tag{55}$$

We are lead to a stochastic description, due to the infinity of geodesics of the fractal space-time. This forces us to consider the question of the definition of a Lorentz-covariant stochasticity in space-time. This problem has been addressed by several authors in the framework of a relativistic generalization of Nelson's stochastic quantum mechanics. Two mutually independent fluctuation fields, $d\xi^{\mu}_{\pm}(s)$, are defined, such that ($< d\xi^{\mu}_{\pm} >= 0$), and

$$< d\xi^{\mu}_{\pm}\, d\xi^{\nu}_{\pm} >= \mp \lambda\, \eta^{\mu\nu} ds. \tag{56}$$

The constant λ is another writing for the coefficient $2\mathcal{D} = \lambda c$, with $\mathcal{D} = \hbar/2m$ in the standard quantum case: namely, it is the Compton length of the particle. This process makes sense only in $\mathbb{R}^4$, i.e. the "metric" $\eta^{\mu\nu}$ should be positive definite: indeed, the fractal fluctuations are of the same nature as uncertainties and 'errors', so that the space and the time fluctuations add quadratically. The sign corresponds to a choice of space-like fluctuations.

Dohrn and Guerra [62] introduce the above "Brownian metric" and a kinetic metric $g_{\mu\nu}$, and obtain a compatibility condition between them which reads $g_{\mu\nu}\eta^{\mu\alpha}\eta^{\nu\beta} = g^{\alpha\beta}$. An equivalent method was developed by Zastawniak [63], who introduces, in addition to the covariant drifts v^{μ}_{+} and v^{μ}_{-}, new drifts b^{μ}_{+} and b^{μ}_{-} (note that our notations are different from his). Serva [64] gives up Markov processes and considers a covariant process which belongs to a larger class, known as "Bernstein processes".

All these proposals are equivalent, and amount to transforming a Laplacian operator in $\mathbb{R}^4$ into a Dalembertian. Namely, the two $(+)$ and $(-)$ differentials of a function $f[x(s), s]$ can be written:

$$d_{\pm} f/ds = (\partial/\partial s + v^{\mu}_{\pm}\,\partial_{\mu} \mp \frac{1}{2}\lambda\,\partial^{\mu}\partial_{\mu}) f. \tag{57}$$

In what follows, we shall only consider s-stationary functions, i.e., that are not explicitly dependent on the proper time s. In this case the covariant time derivative operator reduces to [26]:

$$\frac{\acute{d}}{ds} = \left(\mathcal{V}^{\mu} + \frac{1}{2} i\,\lambda\,\partial^{\mu}\right)\partial_{\mu}. \tag{58}$$

Let us assume that the system under consideration can be characterized by an action $\mathcal{S}$, which is complex because the four-velocity is now complex. The same reasoning as in classical mechanics leads us to write $d\mathcal{S} = -mc\mathcal{V}_\mu dx^\mu$ (see [60] for another equivalent choice). The least-action principle applied on this action yields the equations of motion of a free particle, that takes the form of a geodesics equation, $\acute{d}\,\mathcal{V}_\alpha/ds = 0$. Such a form is also directly obtained from the 'strong covariance' principle and the generalized equivalence principle. We can also write the variation of the action as a functional of coordinates. We obtain the usual result (but here generalized to complex quantities):

$$\delta\mathcal{S} = -mc\mathcal{V}_\mu \delta x^\mu \;\Rightarrow\; \mathcal{P}_\mu = mc\mathcal{V}_\mu = -\partial_\mu \mathcal{S}, \tag{59}$$

where $\mathcal{P}_\mu$ is now a complex 4-momentum. As in the nonrelativistic case, the wave function is introduced as being nothing but a reexpression of the action:

$$\psi = e^{i\mathcal{S}/mc\lambda} \;\Rightarrow\; \mathcal{V}_\mu = i\lambda\,\partial_\mu(\ln\psi), \tag{60}$$

so that the equations of motion become:

$$\acute{d}\,\mathcal{V}_\alpha/ds = i\lambda\left(\mathcal{V}^\mu + \frac{1}{2}i\lambda\partial^\mu\right)\partial_\mu\mathcal{V}_\alpha = 0 \Rightarrow \left(\partial^\mu\ln\psi + \frac{1}{2}\partial^\mu\right)\partial_\mu\partial_\alpha\ln\psi = 0. \tag{61}$$

Now, by using the remarkable identity (32) established in [4], it reads:

$$\partial_\alpha(\partial_\mu\partial^\mu\ln\psi + \partial_\mu\ln\psi\,\partial^\mu\ln\psi) = \partial_\alpha\left(\frac{\partial_\mu\partial^\mu\psi}{\psi}\right) = 0. \tag{62}$$

So the equation of motion can finally be integrated in terms of the Klein-Gordon equation for a free particle:

$$\lambda^2\partial^\mu\partial_\mu\psi = \psi, \tag{63}$$

where $\lambda = \hbar/mc$ is the Compton length of the particle. The integration constant is chosen so as to ensure the identification of $\varrho = \psi\psi^\dagger$ with a probability density for the particle.

From the results of Zastawniak [63] and as can be easily recovered from the definition (55), the quadratic invariant of special motion-relativity, $v^\mu v_\mu = 1$, is naturally generalized as

$$\mathcal{V}^\mu\mathcal{V}_\mu^\dagger = 1, \tag{64}$$

where $\mathcal{V}_\mu^\dagger$ is the complex conjugate of $\mathcal{V}_\mu$. This ensures the covariance (i.e. the invariance of the form of equations) of the theory at this level.

5.1.2. Quadratic invariant, Leibniz rule and complex velocity operator

However, it has been recalled by Pissondes [60] that the square of the complex four-velocity is no longer equal to unity, since it is now complex. It can be derived directly from (62) after accounting for the Klein-Gordon equation. One obtains the generalized energy (or quadratic invariant) equation:

$$\mathcal{V}_\mu\mathcal{V}^\mu + i\lambda\partial_\mu\mathcal{V}^\mu = 1. \tag{65}$$

Now taking the gradient of this equation, one obtains:

$$\partial_\alpha(\mathcal{V}_\mu\mathcal{V}^\mu + i\lambda\partial_\mu\mathcal{V}^\mu) = 0 \Rightarrow \left(\mathcal{V}^\mu + \frac{1}{2}i\lambda\partial^\mu\right)\partial_\alpha\mathcal{V}_\mu = 0, \tag{66}$$

which is equivalent to Eq. (61) in the case of free motion, since in the absence of external field $\partial_\alpha\mathcal{V}_\mu = \partial_\mu\mathcal{V}_\alpha$.

Clearly, the new form of the quadratic invariant comes only under 'weak covariance'. Pissondes has therefore addressed the problem of implementing the strong covariance (i.e., keeping the free, Galilean form of the equations of physics even in the new, more complicated situation) at all levels of the description.

The additional terms in the various equations find their origin in the very definition of the 'quantum-covariant' total derivative operator. Indeed, it contains derivatives of first order (namely, $\partial/\partial s + \mathcal{V}^\mu \partial_\mu$), but also derivatives of second order ($i(\lambda/2)\partial^\mu\partial_\mu$). Therefore, when one is led to compute quantities like $\acute{d}(fg)/dt = 0$ the Leibniz rule that must be used becomes a linear combination of the first order and second order Leibniz rules. There is no problem provided one always come back to the definition of the covariant total derivative. (Some inconsistency would appear only if one, in contradiction with this definition, wanted to use the first order Leibniz rule $d(fg) = fdg + gdf$). Indeed, one finds:

$$\frac{\acute{d}}{ds}(fg) = f\,\frac{\acute{d}\,g}{ds} + g\,\frac{\acute{d}\,f}{ds} + i\lambda\,\partial^\mu f\,\partial_\mu g. \tag{67}$$

Pissondes attempted to find a formal tool in terms of which the form of the first order Leibniz rule would be preserved. He introduced the following 'symmetric product':

$$f \circ \frac{\acute{d}\,g}{ds} = f\,\frac{\acute{d}\,g}{ds} + i\,\frac{\lambda}{2}\,\partial^\mu f\,\partial_\mu g, \tag{68}$$

and he showed that, using this product, the covariance can be fully implemented. In particular, one recovers the form of the derivative of a product, $\acute{d}(fg) = f \circ \acute{d}\,g + g \circ \acute{d}\,f$, and the standard decomposition, $\acute{d}\,f = \partial_\mu f \circ \acute{d}\,x^\mu$.

However, this tool has the inconvenience of depending on two functions f and g. We shall therefore use another equivalent tool, which has the advantage to depend only on one function. We define a complex velocity operator:

$$\widehat{\mathcal{V}^\mu} = \mathcal{V}^\mu + i\,\frac{\lambda}{2}\,\partial^\mu, \tag{69}$$

so that the covariant derivative is now written in terms of an operator product that keeps the standard, first order form:

$$\frac{\acute{d}}{ds} = \widehat{\mathcal{V}^\mu}\,\partial_\mu\,. \tag{70}$$

More generally, one defines the operator:

$$\widehat{\frac{\acute{d}\,g}{ds}} = \frac{\acute{d}\,g}{ds} + i\frac{\lambda}{2}\,\partial^\mu g\,\partial_\mu \tag{71}$$

which has the advantage to be defined only in terms of g. The covariant derivative of a product now writes

$$\frac{\acute{d}\,(fg)}{ds} = \widehat{\frac{\acute{d}\,f}{ds}}\,g + \widehat{\frac{\acute{d}\,g}{ds}}\,f \tag{72}$$

i.e., one recovers the form of the first order Leibniz rule. Since $\widehat{f}g \neq \widehat{g}f$, one is led to define a symmetrized product, following Pissondes [60]. One defines $\dot{f} = \acute{d}\,f/ds$, then

$$\dot{f} \otimes \dot{g} = \widehat{\dot{f}}\dot{g} + \widehat{\dot{g}}\dot{f} - \dot{f}\dot{g}. \tag{73}$$

This product is now commutative, $\dot{f} \otimes \dot{g} = \dot{g} \otimes \dot{f}$, and in its terms the standard expression for the square of the velocity is recovered, namely, $\mathcal{V}^\mu \otimes \mathcal{V}_\mu = 1$.

The introduction of such a tool, that may appear formal in the case of free motion, becomes particularly useful in the presence of an electromagnetic field. Indeed, the introduction of a new level of complexity in the description of a relativistic fractal space-time, namely, the account of resolutions that become functions of coordinates, leads to a new geometric theory of gauge fields, in particular of the U(1) electromagnetic field [26, 23, 73, 38]. We find that the complex velocity is given in this case by:

$$\mathcal{V}^\mu = i\lambda\,D^\mu \ln\psi = i\lambda\,\partial^\mu \ln\psi - \frac{e}{mc^2}A^\mu, \tag{74}$$

where A^μ is a field of dilations of internal resolutions that can be identified with an electromagnetic field. Inserting this expression in Eq. (66) yields the standard Klein-Gordon equation with electromagnetic field, $[i\hbar\partial_\mu - (e/c)A_\mu][i\hbar\partial^\mu - (e/c)A^\mu]\psi = m^2c^2\psi$.

5.2. Dirac Equation

5.2.1. Reflection symmetry breaking of spatial differential element

All the new structures implied by the non-differentiability have not yet been considered. The total derivative of a physical quantity also involves partial derivatives with respect to the space variables, $\partial/\partial x^\mu$. From the very definition of derivatives, the discrete symmetry under the reflection $dx^\mu \leftrightarrow -dx^\mu$ should also broken at a more profound level of description. Therefore, we expect the appearance of a new two-valuedness of the generalized velocity.

At this level one should also account for parity violation. Finally, we have suggested that the three discrete symmetry breakings, $\{ds \leftrightarrow -ds, \qquad dx^\mu \leftrightarrow -dx^\mu, \qquad x^\mu \leftrightarrow -x^\mu\}$, can be accounted for by the introduction of a bi-quaternionic velocity. It has been subsequently shown [56, 24] that one can derive in this way the Dirac equation, namely as an integral of a geodesics equation: this demonstration is summarized in what follows. In other words, this means that this new two-valuedness is at the origin of the bi-spinor nature of the electron wave function.

5.2.2. Spinors as bi-quaternionic wave-function

Since $\mathcal{V}^\mu$ is now bi-quaternionic, the Lagrange function is also bi-quaternionic and, therefore, the same is true of the action. Moreover, it has been shown [56] that, for s-stationary processes, the bi-quaternionic generalisation of the quantum-covariant derivative keeps the same form as in the complex number case, namely,

$$\frac{\widehat{d}}{ds} = \mathcal{V}^\nu \partial_\nu + i\frac{\lambda}{2}\partial^\nu \partial_\nu. \tag{75}$$

The generalized equivalence principle, as well as the strong covariance principle, allows us to write the equation of motion under a free-motion form, i.e., under the form of a differential geodesics equation

$$\frac{\widehat{d}\,\mathcal{V}_\mu}{ds} = 0, \tag{76}$$

where $\mathcal{V}_\mu$ is the bi-quaternionic four-velocity, e.g., the covariant counterpart of $\mathcal{V}^\mu$. The elementary variation of the action, considered as a functional of the coordinates, keeps the usual form $\delta\mathcal{S} = -mc\,\mathcal{V}_\mu\,\delta x^\mu$. We thus obtain the bi-quaternionic four-momentum, as $\mathcal{P}_\mu = mc\mathcal{V}_\mu = -\partial_\mu\mathcal{S}$.

We are now able to introduce the wave function. We define it as a re-expression of the bi-quaternionic action by

$$\psi^{-1}\partial_\mu\psi = \frac{i}{cS_0}\partial_\mu\mathcal{S}, \tag{77}$$

using, in the left-hand side, the quaternionic product (we can no longer use logarithms in this case). Therefore the bi-quaternionic four-velocity reads

$$\mathcal{V}_\mu = i\frac{S_0}{m}\psi^{-1}\partial_\mu\psi. \tag{78}$$

This is the bi-quaternionic generalization of the definition used in the Schrödinger case, $\psi = e^{iS/S_0}$. Note that the non-commutativity is at a more profound level here than that already involved on the quantum operatorial tool by the fractal and non-differentiable description [60]. It is now at the level of the fractal space-time itself, which therefore fundamentally comes under Connes's noncommutative geometry [66]. Finally, the isomorphism which can be established between the quaternionic and spinorial algebrae through the multiplication rules applying to the Pauli spin matrices allows us to identify the wave function ψ to a Dirac bispinor. Indeed, spinors and quaternions are both a representation of the SL(2,C). This identification is reinforced by the result [56, 24] that follows, according to which the geodesics equation written in terms of bi-quaternions is naturally integrated under the form of the Dirac equation.

5.2.3. Free-particle bi-quaternionic Klein-Gordon equation

The whole results obtained for the complex Klein-Gordon equation can be recovered in the bi-quaternionic case, after accounting for the necessary re-formulation of the equations due to non-commutativity. The equation of motion, Eq. (76), still writes

$$\left(\mathcal{V}^\nu \partial_\nu + i\frac{\lambda}{2}\partial^\nu \partial_\nu\right)\mathcal{V}_\mu = 0. \tag{79}$$

After change of variables using Eq. 78, we obtain, after some calculations [24],

$$\partial_\mu[(\partial^\nu \partial_\nu \psi)\psi^{-1}] = 0. \tag{80}$$

It is integrated as $(\partial^\nu \partial_\nu \psi)\psi^{-1} + C = 0$, of which we take the right product by ψ to obtain

$$\partial^\nu \partial_\nu \psi + C\psi = 0. \tag{81}$$

We therefore recognize the Klein-Gordon equation for a free particle with a mass m, after the identification $C = m^2c^2/\hbar^2 = 1/\lambda^2$. But in this equation ψ is now a biquaternion, i.e. a Dirac bispinor.

5.2.4. Derivation of the Dirac and Pauli equations

We now use a long-known property of the quaternionic formalism, which allows one to obtain the Dirac equation for a free particle as a mere square root of the Klein-Gordon operator (see Refs. in [56, 24]). We first develop the Klein-Gordon equation as

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{m^2c^2}{\hbar^2}\psi. \tag{82}$$

Thanks to the property of the quaternionic and complex imaginary units $e_1^2 = e_2^2 = e_3^2 = i^2 = -1$, we can write Eq. (82) under the form

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = e_3^2\frac{\partial^2\psi}{\partial x^2}e_2^2 + ie_1^2\frac{\partial^2\psi}{\partial y^2}i + e_3^2\frac{\partial^2\psi}{\partial z^2}e_1^2 + i^2\frac{m^2c^2}{\hbar^2}e_3^2\psi e_3^2. \tag{83}$$

This expression can be transformed using the Conway matrices, which are themselves equivalent to the Dirac β and α matrices. One can therefore finally write Eq. (83) as the non-covariant Dirac equation for a free fermion (see the detailed calculation in Refs. [56, 24]),

$$\frac{1}{c}\frac{\partial\psi}{\partial t} = -\alpha^k\frac{\partial\psi}{\partial x^k} - i\frac{mc}{\hbar}\beta\psi. \tag{84}$$

from which the covariant form can be recovered.

Finally it is easy to derive the Pauli equation, since it is known that it can be obtained as a non-relativistic approximation of the Dirac equation [69]. Two of the components of the Dirac bi-spinor become negligible when $v \ll c$, so that they become Pauli spinors (i.e., in our representation the bi-quaternions are reduced to quaternions) and the Dirac equation is transformed in a Schödinger equation for these spinors with a magnetic dipole additional term. Such an equation is but the Pauli equation. Therefore the Pauli equation is understood in the scale-relativity framework as a manifestation of the fractality of space (but not time), while the symmetry breaking of space differential elements is nevertheless at work.

6. 'THIRD QUANTIZATION': QUANTUM MECHANICS IN SCALE-SPACE

6.1. Motivation

Let us now consider a new tentative development of the scale relativity theory. Recall that this theory is founded on the giving up of the hypothesis of differentiability of space-time coordinates. We reached the conclusion that the problem of dealing with non-derivable coordinates could be circumvented by replacing them by fractal functions of the resolutions. These functions are defined in a space of resolutions, or 'scale-space'. The advantage of this approach is that it sends the problem of non-differentiability to infinity in the scale space ($\ln(\lambda/\varepsilon) \to \infty$). In such a framework, standard physics should be completed by scale laws allowing to determine the physically relevant functions of resolution. We have suggested that these fundamental scale laws be written in terms of differential equations (which amounts to define a differential fractal 'generator'). Then we have found that the simplest possible scale laws that are consistent (i) with the principle of scale relativity and (ii) with the standard laws of motion and displacements, lead to a quantum-like mechanics in space-time.

However, the choice to write the transformation laws of the scale space in terms of differential equations, even though it allows non-differentiability in standard space-time, implicitly assumes differentiability in the scale space. This is once again a mere hypothesis that can be given up. We may therefore use the method that has been built for dealing with non-differentiability in space-time and explore a new level of structures that may be its manifestation. As we shall now see, this results in the obtention of scale laws that take quantum-like forms instead of a classical ones.

6.2. Schrödinger equation in scale-space

6.2.1. Lagrangian approach to scale laws

The Lagrangian approach can be used in the scale space in order to obtain physically relevant generalizations of the simplest (scale-invariant) laws [27, 28]. When going beyond strict self-similarity, the scale dimension $\delta = D_F - D_T$ (where D_F is the fractal dimension and D_T the topological dimension) is no longer a constant. It can be defined in a local way as $\delta = d\ln\mathcal{L}/d\ln\varepsilon$. We have suggested to call 'djinn' this variable.

Now, when the djinn varies monotonically in terms of the resolution (on a given interval of the scale-space) we can reverse the definition and the meaning of the variables. The resolution, ε, can be defined as a derived quantity in terms of the length of a fractal curve $\mathcal{L}$ and of the djinn, δ:

$$\mathbb{V} = \ln\left(\frac{\lambda}{\varepsilon}\right) = \frac{d\ln\mathcal{L}}{d\delta}\,. \tag{85}$$

This means that the djinn becomes a primary variable that plays, for scale laws, the same role as played by time in motion laws; in the special scale relativity theory [15], it becomes a fifth dimension (namely, the four-dimensional scale-space becomes a fifth-dimenstional "space-djinn"). A scale Lagrange function $\tilde{L}(\ln\mathcal{L}, \mathbb{V}, \delta)$ is introduced, from which a scale action is constructed

$$\tilde{S} = \int_{\delta_1}^{\delta_2} \tilde{L}(\ln\mathcal{L}, \mathbb{V}, \delta)\; d\delta. \tag{86}$$

The application of the action principle yields a scale Euler-Lagrange equation that writes

$$\frac{d}{d\delta}\frac{\partial\tilde{L}}{\partial\mathbb{V}} = \frac{\partial\tilde{L}}{\partial\ln\mathcal{L}}\,. \tag{87}$$

The right-hand part of this equation, $\partial\tilde{L}/\partial\ln\mathcal{L} = 0$, can be interpreted as a 'scale-force' ; however, this concept is a mere practical way to describe what is actually pure geometry in the scale space, in analogy with general relativity where one can recover Newton's gravitational force as a simplified re-expression of the effects of curvature. In analogy with the physics of motion, in the absence of any "scale-force" the Euler-Lagrange equation becomes

$$\partial\tilde{L}/\partial\mathbb{V} = \text{const} \Rightarrow \mathbb{V} = \text{const}. \tag{88}$$

The simplest possible form for the Lagrange function is a quadratic dependence on the "scale velocity", (i.e., $\tilde{L} \propto \mathbb{V}^2$). The constancy of $\mathbb{V} = \ln(\lambda/\varepsilon)$ means that it is independent of the djinn δ. Equation (85) can therefore be integrated to give the usual power law behavior, $\mathcal{L} = \mathcal{L}_0(\lambda/\varepsilon)^\delta$.

6.2.2. Third Quantization

The general Euler-Lagrange equation that describes scale laws in Sec. 6.2.1 after introduction of the djinn δ becomes in the simplified Newtonian case (in which it takes the form of Newton's equation of dynamics, but now in scale space)

$$\frac{d^2 \ln \mathcal{L}}{d\delta^2} = -\frac{\partial \Phi_S}{\partial \ln \mathcal{L}}. \tag{89}$$

Since the scale-space is now assumed to be itself non-differentiable and fractal, the various elements of the new description can be used again in this case, namely:

(i) Infinity of trajectories, leading to introduce a scale-velocity field $I\!V = I\!V(\ln \mathcal{L}(\delta), \delta)$;

(ii) Decomposition of $d \ln \mathcal{L}$ in terms of a 'classical part' and a 'fractal part' such that $< d\xi_S^2 >= 2\mathcal{D}_S d\delta$, and two-valuedness because of the symmetry breaking of the reflection invariance under the exchange $(d\delta \leftrightarrow -d\delta)$;

(iv) Introduction of a complex scale-velocity $\tilde{\mathcal{V}}$ based on this two-valuedness;

(v) Construction of a new total covariant derivative with respect to the djinn, that reads:

$$\frac{\acute{d}}{d\delta} = \frac{\partial}{\partial \delta} + \tilde{\mathcal{V}} \frac{\partial}{\partial \ln \mathcal{L}} - i \mathcal{D}_s \frac{\partial^2}{(\partial \ln \mathcal{L})^2} \; . \tag{90}$$

(vi) Introduction of a wave function as a re-expression of the action (which is now complex), $\Psi_s(\ln \mathcal{L}) = \exp(i\mathcal{S}_s/2\mathcal{D}_s)$;

(vii) Transformation and integration of the above Newtonian scale-dynamics equation under the form of a Schrödinger equation now acting on scale variables:

$$\mathcal{D}_s^2 \frac{\partial^2 \Psi_s}{(\partial \ln \mathcal{L})^2} + i \mathcal{D}_s \frac{\partial \Psi_s}{\partial \delta} - \frac{1}{2} \Phi_s \Psi_s = 0. \tag{91}$$

6.3. Complexergy

In order to understand the meaning of this new Schrödinger equation, let us review the various levels of evolution of the concept of physical fractals adapted to a geometric description of a non-derivable space-time.

The first level in the definition of fractals is Mandelbrot's concept of 'fractal objects' [2].

The second step has consisted to jump from the concept of fractal objects to scale-relativistic fractals. Namely, the scales at which the fractal structures appear are no longer defined in an absolute way: only scale ratios do have a physical meaning, not absolute scales.

The third step, that is achieved in the new interpretation of gauge transformations performed in scale relativity [26, 23, 67] (which has not been considered in the present contribution), considers fractal structures (still defined in a relative way) that are no longer static. Namely, the scale ratios between structures become a field that may vary from place to place and with time.

The final level (in the present state of the theory) is given by the solutions of the above scale-Schrödinger equation. The Fourier transform of these solutions will provide probability amplitudes for the possible values of the logarithms of scale ratios, $\Psi_s(\ln \varrho)$. Then $|\Psi_s|^2(\ln \varrho)$ gives the probability density of these values. Depending on the scale-field and on the boundary conditions (in the scale-space), peaks of probability density will be obtained, this meaning that some specific scale ratios become more probable than others. Therefore, such solutions now describe quantum probabilistic fractal structures. The statement about these fractals is no longer that they own given structures at some (relative) scales, but that there is a given probability for two structures to be related by a given scale ratio.

With regards to the solutions of the scale-Schrödinger equation, they provide probability densities for the position on the fractal coordinate (or fractal length) $\ln \mathcal{L}$. This means that, instead of having a unique and determined $\mathcal{L}(\ln \varepsilon)$ dependence (e.g., the length of the Britain coast), an infinite family of possible behaviors is defined, which self-organize in such a way that some values of $\ln \mathcal{L}$ become more probable than others.

A more complete understanding of the meaning of this new description can be reached by considering an explicit example, e.g., the case of a scale-harmonic oscillator potential well. This is the quantum equivalent of the scale force considered previously, but now in the attractive case. The stationary Schrödinger equation reads in this case:

$$2\mathcal{D}_s^2 \, \frac{\partial^2 \Psi_s}{(\partial \ln \mathcal{L})^2} + \left[I\!E - \frac{1}{2}\omega^2 (\ln \mathcal{L})^2 \right] \Psi_s = 0. \tag{92}$$

The stationarity of this equation means that it does no longer depend on the djinn δ.

A new important quantity, denoted here $I\!E$, appears in this equation. It is the conservative quantity which, according to Noether's theorem, must emerge from the uniformity of the new djinn variable. It is defined, in terms of the scale-Lagrange function $\tilde{L}$ and of the resolution $I\!V = \ln(\lambda/\varepsilon)$, as:

$$I\!E = I\!V \frac{\partial \tilde{L}}{\partial I\!V} - \tilde{L}. \tag{93}$$

This new fundamental prime integral had already been introduced in Refs. [15, 4], but we are now able to be more specific about its physical meaning .

As we shall now see, the behavior of the above equation suggests an interpretation for this conservative quantity and allows one to link it to the complexity of the system under consideration. For this reason, and because it is linked to the djinn in the same way as energy is linked to the time, we have suggested to call this new fundamental quantity 'complexergy'.

Indeed, let us consider the momentum solutions $a[\ln(\lambda/\varepsilon)]$ of the above scale-Schrödinger equation. Recall that the main variable is now $\ln \mathcal{L}$ and that the scale-momentum is the resolution, $\ln \rho = \ln(\lambda/\varepsilon) = d \ln \mathcal{L}/d\delta$ (since we take here a scale-mass $\mu = 1$). The squared modulus of the wave function yields the probability density of the possible values of resolution ratios:

$$|a_n (\ln \rho)|^2 = \frac{1}{2^n n! \sqrt{2\pi \mathcal{D}_s \omega}} \, e^{-(\ln \rho)^2 / 2\mathcal{D}_s \omega} \, H_n^2 \left(\frac{\ln \rho}{\sqrt{2\mathcal{D}_s \omega}} \right), \tag{94}$$

where the H_n's are the Hermite polynomials (see Fig. 6).

The complexergy is quantized, in terms of the quantum number n, according to the well-known relation for the harmonic oscillator:

$$I\!E_n = 2\mathcal{D}_s \, \omega \left(n + \frac{1}{2} \right). \tag{95}$$

As can be seen in Fig. 6, the solution of minimal complexergy shows a unique peak in the probability distribution of the $\ln(\lambda/\varepsilon)$ values. This can be interpreted as describing a system characterized by a single, more probable relative scale. Now, when the complexergy increases, the number of probability peaks $(n+1)$ increases. Since these peaks are nearly regularly distributed in terms of $\ln \varepsilon$ (i.e., probabilistic log-periodicity), it can be interpreted as describing a system characterized by a hierarchy of imbricated levels of organization. Such a hierarchy of organization levels is one of the criterions that define complexity. Therefore increasing complexergy corresponds to increasing complexity, which justifies the chosen name for the new conservative quantity.

More generally, one can remark that the djinn is universally limited from below ($\delta > 0$), which implies that the complexergy is universally quantized, and that we expect the existence of discretized levels of hierarchy of organisation in nature (as actually observed) instead of a continuous hierarchy.

6.4. Applications

6.4.1. Elementary particle physics

Let us give very shortly some hints of possible applications of these new concepts in the physics of elementary particles. There is an experimentally observed hierarchy of elementary particles,which are organized in terms of three known families, with mass increasing with the family quantum number. For example, there is a (e, μ, τ) universality among leptons, namely these particles have exactly the same properties except for their mass and their family number. However, there is, in the present standard model, no understanding of the nature of the families and no prediction of the values of the masses.

Hence, the experimental masses of charged leptons and of the 'current' quark masses are ([72]:

- $m_e = 0.510998902(21)$ MeV; $m_\mu = 105.658357(5)$ MeV; $m_\tau = 1776.99(28)$ MeV;

- $m_u = 0.003$ GeV; $m_c = 1.25$ GeV; $m_t = 174$ GeV;
- $m_d = 0.006$ GeV; $m_s = 0.125$ GeV; $m_b = 4.2$ GeV.

Basing ourselves on the above definition of complexergy and on this mass hierarchy, we suggest that the existence of particle families are a manifestation of increasing complexergy, i.e., that the family quantum number is nothing but a complexergy quantum number. This would explain why the electron, muon and tau numbers are conserved in particle collisions, since such a fondamental conservative quantity (like energy) can be neither created nor destroyed.

One also expects the energy (i.e. the mass) of systems which are described by the above scale-Schrödinger equation to be themselves organized in a hierarchical way. Indeed, in the case where $\ln(\mathcal{L}/\mathcal{L}_0)$ is mainly a time variable (as for example in motion-relativistic high energy physics), the associated conservative quantity is $\mathcal{E} = \ln(E/E_0)$ (see [4], p.242). Because of the fractal-nonfractal transition (precisely at the Compton scale, given by the mass of the particle), $\ln(\mathcal{L}/\mathcal{L}_0) > 0$ is also limited from below, so that we expect the energy to be generally quantized, but now in exponential form. In other words, it describes a hierarchy of energies.

Although a full treatment of the problem must await a more advanced level of development of the theory, that would mix the 'third quantization' description with the gauge field one, some existing structures of the particle mass hierarchy already support such a view:

(i) The above values of quark and lepton masses are clearly organized in a hierarchical way. This suggests that their understanding is indeed to be searched in terms of structures of the scale-space, for example as manifestation of internal structures of iterated fractals [4].

(ii) With regards to the e, μ, τ leptons, we had already remarked in [4], in the framework of a fractal sef-similar model, that their mass ratios followed a power-law sequence, namely, $m_\mu \approx 3 \times 4.1^3 \times m_e = 105.656$ MeV and $m_\tau \approx 3 \times 4.1^5 \times m_e = 1776.1$ MeV. While the mass derived for the τ from this empirical formula was in disagreement with its known value at that time (1784 MeV), more recent experimental determinations have given a mass of 1777 MeV [72], very close to the 'predicted' value.

(iii) We have suggested [73] that QCD is linked with a 3D harmonic oscillator scale-potential (since its symmetry group is precisely SU(3)). In such a framework, the energy ratios are expected to be quantized as $\ln E \propto (3+2n)$. It may therefore be significant in this regard that the s/d mass ratio, which is far more precisely known that the individual masses since it can be directly determined from the pion and kaon masses, is found to be $m_s/m_d = 20.1$ [72], to be compared with $e^3 = 20.086$, which is the fundamental ($n = 0$) level predicted by the above formula.

6.4.2. Biology: nature of first evolutionary leaps

Another tentative application of the complexergy concept concerns biology. Several lineages of the tree of life, including the first events of species evolution, have been described in terms of a log-periodic acceleration or deceleration law [32, 33, 34]. In this model, we had voluntarily limited ourselves to an analysis of only the chronology of events, independently of the nature of the major evolutionary leaps. But we have now at our disposal a tool that allows us to reconsider the question.

We indeed suggest that life evolution proceeds in terms of increasing quantized complexergy. This would account for the existence of punctuated evolution [75], and for the log-periodic behavior of the leap dates, which can now be interpreted in terms of probability density of the events, $P = \psi\psi^\dagger \propto \sin^2[\omega \ln(T - T_c)]$. Moreover, one may contemplate the possibility of an understanding of the nature of the events, even though in a rough way as a first step.

Indeed, let us consider the free Schrödinger equation in scale-space. Its solutions are determined by the limiting conditions, in particular by the minimal and maximal possible scales. One expects the formation of a structure at the fundamental level (lowest complexergy) characterized by one length-scale (Fig. 6). Moreover, the most probable value for this scale of formation is predicted to be the 'middle' of the logarithmic scale-space. The universal boundary conditions are the Planck-length l_P in the microscopic domain and the cosmic scale $\boldsymbol{L} = \Lambda^{-1/2}$ given by the cosmological constant Λ in the macroscopic domain [4, 23, 30]. From the predicted value of the cosmological constant [4, 23] (which is now supported by observational results [74] , one finds $\boldsymbol{L}/l_P = 5.3 \times 10^{60}$, so that the mid scale is at $2.3 \times 10^{30}\, l_P = 40\ \mu$m. Taking the scale boundaries of living systems (0.5 Angströms - 30 m) instead of the universal scale boundaries yields the same result. This scale of 40 μm is indeed a typical scale of living cells. Moreover, the 'prokaryot' cells, which are the first living systems appeared more than three Gyrs ago, are precisely characterized by this typical size and by having only one hierarchy level (no nucleus).

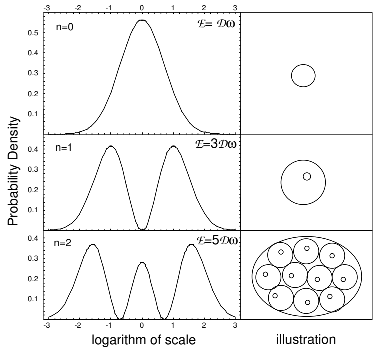

FIGURE 6. Solutions of increasing complexergy of the scale-Schrödinger equation for an harmonic oscillator scale-potential. These solutions can be interpreted as describing systems characterized by an increasing number of hierarchical levels, as illustrated in the right hand side of the figure. For example, living systems such as procaryots, eucaryots and simple multicellular organisms have respectively one (cell size), two (nucleus and cell) and three (nucleus, cell and organism) characteristic scales.

In this framework, a further increase of complexergy can occur only in a quantized way. The second level describes a system with two levels of organization, in agreement with the second step of evolution leading to eukaryots about 1.7 Gyrs ago. One expects (in this very simplified model), that the scale of nuclei be smaller than the scale of prokaryots, itself smaller than the scale of eucaryots: this is indeed what is observed.

The following expected major evolutionary leap is a three organization level system, in agreement with the apparition of multicellular forms (animals, plants and fungi) about 1 Gyr ago. It is also predicted that the multicellular stage can be built only from eukaryots, in agreement with what is observed. Namely, the cells of multicellulars do have nuclei; more generally, evolved organisms keep in their internal structure the organization levels of the preceeding stages.

The following major leaps correspond to more complicated structures then functions (supporting structures such as exoskeletons, tetrapody, homeothermy, viviparity), but they are still characterized by fundamental changes in the number of organization levels. The above model (based on spherical symmetry) is clearly too simple to account for these events. But the theoretical biology approach outlined here for the first time is still in the infancy: future attempts of description using the scale-relativity methods will have the possibility to take into account more complicated symmetries, boundary conditions and constraints, so that such a new field of research (which may become a predictive theoretical biology) seems to be wide open to investigation.

7. CONCLUSION

We have attempted, in the present review paper, to give an extended discussion of some of the developments of the theory of scale-relativity (in particular as concerns the re-foundation of the qunatum theory on non-differentiable geometry), including some new proposals concerning in particular the quantization in the scale-space and tentative applications to the sciences of life.

The aim of this theory is to describe space-time as a continuous manifold (either derivable or not) that would be constrained by the principle of relativity (of motion and of scale). Such an attempt is a natural extension of general relativity, since the two-times differentiable continuous manifolds of Einstein's theory, that are constrained b the principle of relativity of motion, are particular sub-cases of the new geometry to be built.

Now, giving up the differentiability hypothesis involves an extremely large number of new possible structures to be investigated and described. In view of the immensity of the task, we have chosen to proceed by adding self-imposed structures in a progressive way, using presently known physics as a guide. Such an approach is rendered possible by the result according to which small-scale structures issued from non-differentiability are smoothed out beyond some transitions at large scale. Moreover, these transitions have profound physical meaning, since they are themselves linked to fundamental mass scales.

This means that the program that consists of developing a full scale-relativistic physics is still widely open. Much work remains to be done, in order (i) to describe the effect on motion laws of the various levels of scale laws that have been considered, and of their generalizations still to come (general scale-relativity); (ii) to take into account the various new symmetries, as well continuous as discrete, of the new variables that must be introduced for the full description of a fractal space-time, and of the conservative quantities constructed from them (including their quantum counterparts).

Let us conclude by a final remark: one of the main interest of the new approach is that, being based on the universality of fractal geometry already unveiled by Mandelbrot, it allows one to go beyond the frontiers between sciences. In particular, it opens the hope of a future refoundation on first principles of sciences of life and of some human sciences.

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