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LECTURE 19

Scale Relativity

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1. INTRODUCTION

In his conference given at the St Louis congress in 1904, Henri Poincaré wrote [1]: "Comme les astres proprement dits, [les atomes] s'attirent ou se repoussent et cette attraction (...) ne dépend que de la distance. La loi suivant laquelle cette force varie en fonction de la distance n'est peut-être pas la loi de Newton, mais c'est une loi analogue; au lieu de l'exposant –2, nous avons probablement un exposant différent, et c'est de ce changement d'exposant que sort toute la diversité des phénomènes physiques. (...) Telle est la conception primitive dans toute sa pureté. (...) Néanmoins, il est arrivé un jour où la conception des forces centrales n'a plus paru suffisante. Que fit-on alors ? On renonça à pénétrer dans le détail dans la structure de l'Univers, (...) et l'on se contenta de prendre pour guides certains principes généraux: (...) [conservation de l'énergie et de la masse, principe de Carnot, égalité de l'action et de la réaction, principe de relativité et principe de moindre action]. L'application de ces cinq ou six principes généraux aux différents phénomènes physiques suffit pour nous en apprendre ce que nous pouvons raisonnablement espérer en connaître[1]."

[1] "As for stellar bodies, atoms are attracted or repelled to each other and this attraction (...) depends only on the distance. The law, according to which this force varies as a function of distance, is perhaps different from Newton's law, but is analogous to it; instead of the exponent –2, we probably have a different exponent, and it is from the change of the exponent that emerges all the diversity of physical phenomena. (...) Such is the primordial conception in all its purity. (...) Nevertheless, a day came when the concept of central forces did not appear sufficient anymore. What was done then? The

This quotation from Poincaré anticipates in a remarkable way some of the questions that are posed to us nowadays, nearly one century later, concerning scale invariance (and beyond), even though we are now concerned with understanding scale laws in more complex systems than those considered at that time. Indeed the simplest scale invariant equations have power-law solutions, and during some time a large part of the research on this subject was devoted to the calculation of the numerical values of the exponents. But we now know that we must go beyond such an approach, even if it has been a necessary step, and look for the fundamental principles that may allow us to really understand "where the physics is".

This is precisely the aim of the theory of scale relativity. As we shall see, we attempt to construct and generalize scale laws by using first principles, namely the principles of relativity and of covariance once extended to scale transformations [2,3], and the stationary action principle. Namely, we generalize Einstein's formulation of the principle of relativity, by requiring *that the laws of nature be valid in any reference system, whatever its state*. Up to now, this principle has been applied to changes of state of the coordinate system that concerned the origin, the axes orientation, and the motion (measured in terms of velocity, acceleration, ...).

In scale relativity, we assume that the space-time resolutions are not only a characteristic of the measurement apparatus, but acquire a universal status. They are considered as essential, intrinsic variables, inherent to the physical description. We define them as characterizing the "state of scale" of the reference system, in the same way as the velocity characterizes its state of motion, and we work in an extended phase space including coordinates, velocities and resolutions. The theory of scale relativity consists of applying the principle of relativity to such a scale-state. Its mathematical translation is *scale-covariance* [2,3], requiring that the equations of physics keep their simplest form under resolution transformations (dilations and contractions). The use of scale covariance for constraining scale laws has also more recently been proposed by Pocheau [4] in the context of turbulence.

The domains of application of this theory are typically the asymptotic domains of physics, small length-scales and small time-scales Δx, $\Delta t \to 0$ (microphysics), but also large length and time-scales Δx, $\Delta t \to \infty$ (cosmology).

Initially, the theory of scale relativity was mainly an attempt at refounding quantum mechanics on first principles [2,3]. We have demonstrated that the main axioms of quantum mechanics can be recovered as consequences of the principle of scale relativity, and that the quantum behavior can be recovered as the various manifestations of the non-differentiability and fractality of space-time at small scales [3,5,6]. Moreover, the theory allows to generalize standard quantum mechanics. Indeed, we have shown that the usual laws of scale (power law, self-similar, constant

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goal of unravelling the details of the structure of the Universe was abandonned and replaced by certain general principles acting as guiding lines: (...) [conservation of energy and mass, Carnot's principle, equality between action and reaction, relativity principle and least action principle]. The application of these five or six general principles to the different physical phenomena is enough to teach us what we can reasonably hope to know."

fractal dimension) have the status of "Galilean" scale laws, while a full implementation of the principle of scale relativity suggests that they could be medium scale approximations of more general laws which take a Lorentzian form [7]. In such a "special scale relativity" theory, the Planck length- and time-scale becomes a minimal, impassable scale, invariant under dilations and contractions, which replaces the zero point (since owning all its physical properties) and plays for scales the same role as played by the velocity of light for motion. A similar proposal can be made at large scales: the scale of the cosmological constant can be reinterpreted as a maximal, impassable length-scale, invariant under dilations [3,6,9]. In this new framework, several still unsolved problems of fundamental physics may find simple and natural solutions: new light is brought on the nature of the Grand Unification scale, on the origin of the electron scale and of the electroweak scale, on the scale-hierarchy problem, and on the values of coupling constants [3,6,7]; moreover, one can construct a scale-relativistic interpretation of gauge invariance that yields new insights on the nature of the electric charge, and allows to predict new mass-charge relations for elementary particles [6,8].

2. TOWARD A NONDIFFERENTIABLE SPACE-TIME

Since more than three century, physics relies on the implicit assumption that space-time coordinates are a priori differentiable. However, it was demonstrated by Feynman [10] that the typical paths of quantum mechanical particles are continuous but non-differentiable. Now, one of the most powerful avenue for reaching a genuine understanding of the laws of nature has been to construct them, not from setting additional hypotheses, but on the contrary by attempting to give up some of them, i.e., by going to increased generality. An example of such an approach is Einstein's explanation of the nature of gravitation as the various manifestations of the Riemannian, i.e. non-flat geometry of space-time.

However, in the light of the above remark, Einstein's principle of relativity is not yet fully general, since it applies to transformations that are continuous and at least two times differentiable. The aim of the theory of scale-relativity is to look for the laws and structures that would be the manifestations of still more general transformations, namely, continuous ones (that can be differentiable or not). In such a construction the standard theory will be recovered as a special case, since differentiable spaces are a particular subset of the set of all continuous spaces.

In that quest, the first step consists of realizing that a continuous but non-differentiable space-time is necessarily fractal. More precisely, one can demonstrate [3,5,11] that a continuous but nondifferentiable function is explicitly resolution-dependent, and that its length L tends to infinity when the resolution interval tends to zero, i.e. $L = L(\varepsilon)_{\varepsilon \to 0} \to \infty$. This theorem naturally leads to the proposal that the concept of *fractal space-time*, [12,13,3] is the geometric tool adapted to the research of such a new description based on non-differentiability. This leads to introduce new intrinsic scale variables in the very definition of physical quantities (among which the coordinates themselves), but also to construct the differential equations (in the "scale space") that would describe this new dependence.

We shall see that, since the new scale equations are themselves constrained by the principle of relativity, the new concepts fit well established structures. Namely, the so-called symplectic structure of most physical theories, i.e., the Poisson bracket / Euler-Lagrange / Hamilton formulation, can be also used to construct scale laws. Under such a viewpoint, scale invariance is recovered as corresponding to the "free" case (the equivalent of what inertia is for motion laws).

3. SCALE INVARIANCE AND SCALE RELATIVITY

Scaling laws have been discovered and studied at length in several domains of science. A power-law scale dependence is encountered in a lot of natural systems. It can be described geometrically in terms of fractals [14,15], and algebrically in terms of the renormalization group [16,17]. As we shall see now, such simple scale-invariant laws can be identified with a "Galilean" version of scale-relativistic laws.

In most present use and applications of fractals, the fractal dimension D is defined, following Mandelbrot, from the variation with resolution ε of the main fractal variable (e.g., the length L of a fractal curve –which plays here the role of a fractal curvilinear coordinate–, the area of a fractal surface, etc...). Namely, the scale dimension $\delta = D - D_T$ is defined as:

$$\delta = \frac{d \ln L}{d \ln(\lambda/\varepsilon)} \quad , \tag{1}$$

where D_T is the topological dimension ($D > D_T$ most of the time). When δ is constant, we obtain a *power-law* resolution dependence $L = L_0 \, (\lambda / \varepsilon)^{\delta}$. Here the scale λ appears for dimensional reasons, but it remains arbitrary because there is a strict scale invariance. We shall see in Sec. 3.2 that it takes its full meaning once a scale symmetry breaking is introduced. The Galilean structure of the group of scale transformations that corresponds to this law can be verified in a straightforward manner from the fact that it transforms in a scale transformation $\varepsilon \to \varepsilon'$ as

$$\ln \frac{L(\varepsilon')}{L_o} = \ln \frac{L(\varepsilon)}{L_o} + \delta(\varepsilon) \ln \frac{\varepsilon}{\varepsilon'} \quad ; \quad \delta(\varepsilon') = \delta(\varepsilon) \tag{2}$$

The relativity of scales is now apparent from the fact that λ has disappeared from this transformation. It has exactly the structure of the Galileo group, as confirmed by the law of composition of dilations $\varepsilon \to \varepsilon' \to \varepsilon''$, which writes

$$\ln\rho'' = \ln\rho + \ln\rho' \quad , \tag{3}$$

where $\rho = \varepsilon'/\varepsilon$, $\rho' = \varepsilon''/\varepsilon'$ and $\rho'' = \varepsilon''/\varepsilon$ are the three dilations

3.1 Least action principle for scale laws

We are then naturally led, in the scale-relativistic approach, to reverse the definition and the meaning of variables. The scale dimension δ becomes, in general, an essential, fundamental *variable*, that remains constant only in very particular situations (namely, in the case of scale invariance, that corresponds to "scale-freedom"). It plays for scale laws the same role as played by time for motion laws (at a given instant). The log-resolution is a measure of scale in the same way as velocity is a measure of motion, and it is now defined as a derived quantity in terms of the fractal coordinate $\ln L$ and of the scale dimension δ:

$$\mathbb{V} = \ln(\lambda/\varepsilon) = \frac{d \ln L}{d\delta} \quad . \tag{4}$$

Our identification of the standard fractal behavior as a Galilean scale-law can now be fully justified. We assume that, as in the case of motion laws, scale laws can be constructed from a Lagrangian approach. A scale Lagrange function $\mathbb{L}(\ln L, \mathbb{V}, \delta)$ is introduced, from which a scale-action is constructed:

$$\mathbb{S} = \int_{\delta_1}^{\delta_2} \mathbb{L}(\ln L, \mathbb{V}, \delta)\, d\delta \quad . \tag{5}$$

The action principle, applied on this action, yields a scale-Euler-Lagrange equation:

$$\frac{d}{d\delta} \frac{\partial \mathbb{L}}{\partial \mathbb{V}} = \frac{\partial \mathbb{L}}{\partial \ln L} \quad . \tag{6}$$

The simplest possible form for the Lagrange function is the equivalent for scales of what inertia is for motion, i.e., $\mathbb{L} \propto \mathbb{V}^2$ and $\partial \mathbb{L} / \partial \ln L = 0$ (no scale "force", see Sec. 4.1). The Lagrange equation writes in this case:

$$\frac{d\mathbb{V}}{d\delta} = 0 \Rightarrow \mathbb{V} = cst. \tag{7}$$

The constancy of $\mathbb{V} = \ln(\lambda/\varepsilon)$ means here that it is independent of the scale-time δ. Then Eq. 4 can be integrated in terms of the usual power law behavior, $L = L_0\, (\lambda / \varepsilon)^{\delta}$. This reversed viewpoint has several advantages:

(i) The scale dimension takes its actual status of "scale-time", and the logarithm of resolution $\mathbb{V}$ its status of "scale-velocity", $\mathbb{V} = d\ln L / d\delta$. This is in accordance with its scale-relativistic definition, since it characterizes the state of scale of the reference system, in the same way as the velocity $v = dx/dt$ characterizes its state of motion.

(ii) This leaves open the possibility of generalizing our formalism to the case of four independent space-time resolutions, $\mathbb{V}^{\mu} = \ln(\lambda^{\mu} / \varepsilon^{\mu}) = d\ln L^{\mu} / d\delta$.

(iii) Scale laws more general than the simplest self-similar ones can be derived from more general scale-Lagrangians (see Sec. 4 below).

3.2 Scale-symmetry breaking

An important point concerning the scale symmetry, which is highly relevant to the present study, is that, as is well-known from the observed scale-independence of physics at our own scales, the scale dependence is a spontaneously broken symmetry [2,3,7]. Let us recall the simple theoretical argument that leads to this result.

In the general framework of a continuous space-time (not necessarily differentiable), we expect a generalized curvilinear coordinate to be explicitly resolution-dependent, i.e. $L = L(\varepsilon)$. We assume that this new scale dependence is itself solution of a differential equation *in the scale space*. The simplest scale differential equation one can write is a renormalization-group-like, first order equation in which the scale variation of L depends on L only, i.e., $\mathrm{d}L / \mathrm{d}\ln\varepsilon = \beta(L)$. The function $\beta(L)$ is a priori unknown but, still taking the simplest case, we consider a perturbative approach and we take its Taylor expansion. We obtain the equation:

$$\frac{\mathrm{d}L}{\mathrm{d}\ln\varepsilon} = a + b\,L + c\,L^2 + ... \tag{8}$$

Disregarding for the moment the quadratic term, this equation is solved in terms of a standard power law of power $\delta = -b$, broken at some scale λ (integration constant):

$$L = L_0 \left[1 + \left(\frac{\lambda}{\varepsilon}\right)^{\delta} \right] . \tag{9}$$

Depending on the sign of δ, this solution represents either a small-scale fractal behavior (in which the scale variable is a resolution), broken at larger scales, or a large-scale fractal behavior (in which the scale variable ε would now represent a changing window for a fixed resolution λ), broken at smaller scales.

The symmetry between the microscopic and the macroscopic cases can be seen from the properties of Eq. 8. Let us indeed transform the two variables L and ε by *inversion*, i.e. $L \rightarrow L' = 1/L$ and $\varepsilon \rightarrow \varepsilon' = 1/\varepsilon$, we find that Eq. 8 becomes:

$$\frac{dL'}{d\ln\varepsilon'} = c + b\,L' + a\,L'^2 + ... \tag{10}$$

This is exactly the same equation up to the exchange of the constants a and c. In other words, Eq. 8 is covariant (i.e. form invariant) under the inversion transformation, which transforms the small scales into the large ones and reciprocally, but also the upper symmetry breaking scale into a lower one. Hence the inversion symmetry, which is clearly not achieved in nature at the level of the observed structures, may nevertheless be an exact symmetry at the level of the fundamental laws. This is confirmed by directly looking at the solutions of Eq. 8 while keeping now the quadratic term. We get a scaling behavior which is broken toward both the small and large scales, in accordance with most real fractal systems:

$$L = L_0 \left[1 + (\lambda_1 / \varepsilon)^{\delta} \right] / \left[1 + (\lambda_2 / \varepsilon)^{\delta} \right] . \tag{11}$$

There is another way to obtain the scale symmetry breaking, that allows to elucidate its origin. Consider a pure scale-invariant law for a "fractal coordinate" $L = L_0\,(\lambda / \varepsilon)^{\delta}$. We expect it to be also translation-invariant, i.e. L can be replaced by $L - L_1$. We then recover the broken law

$$L = L_0 \left[1 + (\lambda'/\varepsilon)^{\delta}\right] , \tag{12}$$

where $\lambda' = \lambda\,(L_0/L_1)^{1/\delta}$. This law becomes scale-independent for $\varepsilon >> \lambda'$ when $\delta > 0$, and for $\varepsilon << \lambda'$ when $\delta < 0$. This means that the scale symmetry breaking results from the effect on the scale symmetry of the translation symmetry.

3.3 Special scale-relativity

It is well known that the Galileo group of motion is only a degeneration of the more general Lorentz group. The same is true for scale laws. Indeed, if one looks for the general linear scale laws that come under the principle of scale relativity, one finds that they have the structure of the Lorentz group [32,7]. Namely, it has been shown by Levy-Leblond [32] that the Lorentz group can be obtained from requiring linearity, group law and reflection invariance (the 4-dimensional Euclidean solution is easily excluded since unphysical). More recently, we have demonstrated [7] that in two dimensions, only three axioms were needed (linearity, internal composition law and reflection invariance). Therefore, we have suggested to replace the Galilean laws of dilation $\ln\rho'' = \ln\rho + \ln\rho'$ by the more general Lorentzian law:

$$\ln\rho'' = \frac{\ln\rho + \ln\rho'}{1 + \ln\rho\,\ln\rho' / \mathbb{C}^2} \quad . \tag{13}$$

This expression is yet incomplete, since under this form the scale relativity symmetry remains unbroken. Such a law corresponds, at the present epoch, only to the null mass limit. It is expected to apply in a universal way during the very first instants of the universe.

In Eq. 13, there appears a universal, purely numerical constant $\mathbb{C} = \ln\mathbb{K}$, where $\mathbb{K}$ plays the role of a maximal possible *dilation*. However, the effect of the spontaneous scale symmetry breaking which arises at some scale λ_0 is to yield a new law in which the invariant is no longer a dilation $\mathbb{K}$, but becomes a *length-time scale* $\mathbb{\Lambda}$. In other words, there appears in the theory a fundamental scale that plays the role of an unpassable resolution, invariant under dilations [7]. Such a scale of length and time is an horizon for scale laws, in a way similar to the status of the velocity of light for motion laws. The new law of composition of dilations and the scale-dimension now write respectively (in the scale-dependent domains, i.e. only below the transition scale in microphysics and beyond it in the cosmological case):

$$\ln\frac{\varepsilon'}{\lambda_0} = \frac{\ln(\varepsilon/\lambda_0) + \ln\rho}{1 + \dfrac{\ln\rho\ \ln(\varepsilon/\lambda_0)}{\ln^2(\mathbb{\Lambda}/\lambda_0)}} \quad ; \quad \delta(\varepsilon) = \frac{1}{\sqrt{1 - \ln^2(\lambda_0/\varepsilon) / \ln^2(\lambda_0/\mathbb{\Lambda})}} \quad . \tag{14}$$

A fractal curvilinear coordinate becomes now scale-dependent in a covariant way, namely $L = L_0\,[1 + (\lambda_0/\varepsilon)^{\delta(\varepsilon)}]$. One of the main new feature of special scale relativity with respect to the previous scale-invariant approach is that the scale-dimension δ, which was constant, is now explicitly varying with scale, and it even diverges when resolution tends to the new invariant scales. In the microphysical domain, the invariant length-scale is naturally identified with the Planck scale, $\mathbb{\Lambda}_{\mathbb{P}} = (\hbar G/c^3)^{1/2}$, that now becomes impassable and plays the physical role that was previously devoted to the zero point [3,7]. The same is true in the cosmological domain, with once again an inversion of the scale laws. We have identified the invariant maximal scale with the scale of the cosmological constant, $\mathbb{L} = \Lambda^{-1/2}$. The consequences of this new interpretation have been studied in [3,6,9]. We have found from various estimates of the cosmological constant that the value of $\mathbb{K} = \mathbb{L} / \Lambda_{\mathbb{P}}$ is about 5×10^{60} [3,6]. Moreover, from the identification of the cosmological constant with the gravitational self-energy density of vacuum fluctuations [33], we have been able to propose an explanation for the Eddington-Dirac large number coincidence, which can be now written as $\mathbb{K} \approx (m_{\mathbb{P}} / m)^3$, where m is a typical elementary particle mass [6,9].

Special scale-relativistic laws have also recently been considered by Dubrulle [18] and Dubrulle and Graner [19] for the description of turbulence, with a different interpretation of the variables and a different choice for the required symmetries. Namely, since we are interested here in the internal fractal structure of space-time trajectories, the fractal dimension is unique while there are four space-time resolutions, so that we are naturally led to consider the couple ($\ln L$, δ) as the variables to transform and the resolution $\ln\varepsilon$ as the parameter of the transformation. On the contrary, Dubrulle and Graner are interested in the set of exponents that characterize the various moments of a statistical distribution in turbulence, so that the role of δ and $\ln\varepsilon$ is reversed in their approach. This reversal also led them not to use the axiom of reflection invariance, but to keep that of the existence of a symmetric element. In this case, the solution is a generalization of the Lorentz group defined by two constants instead of one [19].

It is also noticeable that recent developments in string theories have reached conclusions that are extraordinarily similar to those of scale relativity. One finds that there is a smallest circle in string theory (whose radius is about the Planck length), and that strings are characterized by duality symmetries. Therefore it has recently been argued by Castro [20] that scale relativity is the right framework in which the newly discovered string structures will take their full physical meaning.

4. GENERALIZED SCALE LAWS

4.1 Scale "dynamics"

The whole of our previous discussion indicates to us that the scale invariant behavior corresponds to freedom in the framework of a scale physics. However, in the same way as there exists forces in nature that imply departure from inertial, rectilinear uniform motion, we expect most natural fractal systems to also present distorsions in their scale behavior respectively to pure scale invariance. This means taking non-linearity in scale into account. Such distorsions may be, as a first step, attributed to the effect of a scale "dynamics", i.e. to a "scale-field". (Caution: this is only an *analog* of "dynamics" which acts on the scale axis, on the internal structures *of a given point* at this level of description, not in space-time. See Sec. 4.3 below for first hints about the effects of coupling with space-time displacements). In this case the Lagrange scale-equation takes the form of Newton's equation of dynamics:

$$\mathbb{F} = \mu \frac{d^2 \ln L}{d\delta^2} \quad , \tag{15}$$

where μ is a "scale-mass", which measures how the system resists to the "scale-force", and where $\Gamma = d^2\ln L \,/\, d\delta^2 = d \ln(\lambda/\varepsilon) \,/\, d\delta$ is the 'scale-acceleration'.

4.1.1 Constant scale-force

Let us first consider the case of a constant scale-force. The potential is $\varphi = \mathbb{F} \ln L$, and Eq. 15 writes

$$\frac{d^2 \ln L}{d\delta^2} = \mathbb{G} \quad , \tag{16}$$

where $\mathbb{G} = \mathbb{F} / \mu$ = cst. It is easily integrated in terms of a parabolic solution:

$$\mathbb{V} = \mathbb{V}_0 + \mathbb{G}\ \delta \quad ; \quad \ln L = \ln L_0 + \mathbb{V}_0\, \delta + \frac{1}{2} \mathbb{G}\ \delta^2 \ . \tag{17}$$

However the physical meaning of this result is not clear under this form. This is due to the fact that, while in the case of motion laws we search for the evolution of the system with time, in the case of scale laws we search for the dependence of the system on resolution, which is the directly measured observable. We find, after redefinition of the integration constants:

$$\delta = \delta_0 + \frac{1}{\mathbb{G}} \ln\left(\frac{\lambda}{\varepsilon}\right) \quad ; \quad \ln \frac{L}{L_0} = \frac{1}{2\mathbb{G}} \ln^2\left(\frac{\lambda}{\varepsilon}\right) \ . \tag{18}$$

The scale dimension δ becomes a linear function of resolution (the same being then true of the fractal dimension $1+\delta$), and the ($\log L$, $\log\varepsilon$) relation is now parabolic rather than linear as in the standard power-law case. There are several physical situations where, after careful examination of the data, the power-law models were clearly rejected since no constant slope could be defined in the ($\log L$, $\log\varepsilon$) plane. In the several cases where a clear curvature appears in this plane (e.g., turbulence, sand piles, ...), the physics could come under such a "scale-dynamical" description. In these cases it might be of the highest interest to identify and study the scale-force responsible for the scale distorsion (i.e., for the deviation to standard scaling).

4.1.2 Harmonic oscillator

Another interesting case of scale-potential is that of a repulsive harmonic oscillator, $\varphi = -(\ln L / \delta)^2 / 2$. It is solved as

$$ln \frac{L}{L_0} = \delta \sqrt{\ln^2\left(\frac{\lambda}{\varepsilon}\right) - \delta^{-2}} \quad . \tag{19}$$

For $\varepsilon \ll \lambda$ it gives the standard Galilean case $L = L_0\, (\lambda / \varepsilon)^{\delta}$, but its large-scale behavior is particularly interesting, since it does not permit the existence of resolutions larger than a scale $\lambda_{max} = \lambda\ e^{1/\delta}$. Since the gauge symmetry group of

QCD, SU(3), is the dynamical symmetry group of the 3-dimensional isotropic harmonic oscillator, and since gauge invariance can be interpreted in scale relativity as scale invariance on space-time resolutions (see Sec. 4.3), such a behavior could provide a model of confinement in QCD [L.N., in preparation].

More generally, we shall be led to look for the general non-linear scale laws that satisfy the principle of scale relativity. Such a generalized framework, in which scale covariance can be fully implemented, implies working in a five-dimensional fractal space-time (the scale dimension plays the role of the fifth variable). The development of such a "general scale-relativity" lies outside the scope of the present contribution and will be considered elsewhere.

4.2 Discrete scale invariance and log-periodic behavior

Another correction to pure scale invariance is potentially important, namely the log-periodic correction to power laws that is provided, e.g., by complex exponents or complex fractal dimensions [21]. Sornette et al. (see Sornette's contribution to this school and refs. therein) have shown that such a behavior provides a very satisfactory and possibly predictive model of some earth-quakes and market krachs. This may be a first step toward a general description of the time evolution of crises. Let us show how one can recover log-periodic corrections from requiring scale covariance. Consider a scale-dependent "field" $\Phi(\varepsilon)$. The scale variable is identified with the interval $|\, t - t_c \,|$, where t_c is the date of crisis. Assume that Φ satisfies a renormalization-group first order differential equation,

$$\frac{d\Phi}{d\ln\varepsilon} - D\ \Phi \ = \ 0\ , \tag{20}$$

whose solution is a power law $\Phi(\varepsilon) \propto \varepsilon^D$. Now looking for corrections to this law, we remark that simply jumping to a complex exponent D would lead to large log-periodic fluctuations rather than to a controlable correction to the power-law. So let us assume that the right-hand side of Eq. 20 actually differs from zero

$$\frac{d\Phi}{d\ln\varepsilon} - D\ \Phi = \chi\ \ . \tag{21}$$

We can now apply the *scale-covariance* principle and require that the new function χ be solution of an equation which keeps the same form as the initial equation

$$\frac{d\chi}{d\ln\varepsilon} - D'\ \chi = 0\ \ . \tag{22}$$

Setting $D' = D + \delta$, we find that Φ must be solution of a second-order equation

$$\frac{d^2\Phi}{(d\ln\varepsilon)^2} - (2\,D + \delta)\ \frac{d\Phi}{d\ln\varepsilon} + D\,(D+\delta)\ \Phi = 0\ \ . \tag{23}$$

It writes $\Phi(\varepsilon) = a\ \varepsilon^D\,(1 + b\ \varepsilon^\delta)$, and finally, the choice of an imaginary exponent $\delta = i\omega$ yields a solution whose real part includes a log-periodic correction:

$$\Phi(\varepsilon) = a\ \varepsilon^{D}\ [\ 1 + b\ \cos(\omega \ln\varepsilon)\ ]. \tag{24}$$

4.3 Scale-motion coupling and gauge invariance

The theory of scale relativity also allows us to get new insights about the physical meaning of gauge invariance [8,6]. In the previous sections, only scale transformations at a given point were considered. But we must also wonder about what happens to the structures in scale of a scale-dependent object when it is displaced. Consider anyone of these structures, lying at some (relative) resolution ε (such that $\varepsilon < \lambda$, where λ is the fractal /nonfractal transition) for a given position of the particle. Under a translation, the relativity of scales implies that the resolution at which this given structure appears in the new position will *a priori* be different from the initial one. In other words, ε is now a function of the space-time coordinates, $\varepsilon = \varepsilon(x,t)$, and we expect the occurrence of *dilatations of resolutions induced by translations*, which read:

$$e\ \frac{d\varepsilon}{\varepsilon} = -\,A_{\mu}\, dx^{\mu}\ , \tag{25}$$

where a four-vector A_{μ} must be introduced since dx^{μ} is itself a four-vector and $d\ln\varepsilon$ is a scalar (in the case of a global dilation). This behavior can be expressed in terms of a new scale-covariant derivative:

$$e\ D_{\mu} \ln(\lambda/\varepsilon) = e\ \partial_{\mu} \ln(\lambda/\varepsilon) + A_{\mu}\ . \tag{26}$$

However, if one wants the "field" A_{μ} to be physical, it must be defined whatever the initial scale from which we started. Starting from another scale $\varepsilon' = \rho\varepsilon$ (we consider only Galilean scale-relativity here), we get $e\ d\varepsilon' / \varepsilon' = -A'_{\mu}\, dx^{\mu}$, so that we obtain:

$$A'_{\mu} = A_{\mu} + e\ \partial_{\mu} \ln\rho\ \ , \tag{27}$$

which depends on the relative "state of scale", $\mathbb{V} = \ln\rho = \ln(\varepsilon/\varepsilon')$. If one now considers translation along two different coordinates (or, in an equivalent way, displacement on a closed loop), one may write a commutator relation which defines a tensor field $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$. This field is, contrarily to A_{μ}, independent of the initial scale. One recognizes in $F_{\mu\nu}$ the analog of an electromagnetic field, in A_{μ} that of an electromagnetic potential, in e that of the electric charge, and in Eq. 27 the property of gauge invariance which, in accordance with Weyl's initial ideas [22,23], recovers its initial status of scale invariance. However, Eq. 27 represents a progress compared with these early attempts and with the status of gauge invariance in today's physics. Indeed the gauge function, which has, up to now, been considered as arbitrary, is now identified with the logarithm of internal resolutions.

Another advantage with respect to Weyl's theory is that we are now allowed to define four different and independent dilations along the four space-time resolutions instead of only one global dilation. The above U(1) field is then expected to be embedded into a larger field (in accordance with the electroweak theory) and the charge e to be one element of a more complicated, "vectorial" charge [6,8]. Moreover, when combined with the Lorentzian structure of dilations of special scale relativity, our interpretation of gauge invariance yields new relations between the charges and the masses of elementary particles [6,8].

5. DISCUSSION AND OPEN PROBLEMS

In the present contribution, we have mainly considered the question of pure scale laws. We have shown how they can be constructed and generalized by using the standard tools of classical physics (Euler-Lagrange equations / action principle), once applied not only to displacements in space-time, but to displacements in the scale-space. In this regard our tool can be thought of as the theoretical equivalent of what are wavelets in fractal and multifractal data analysis (see e.g. [24,25]). It allows one to construct conservation laws and invariant quantities through a "scale mechanics" [3,7], quite analog to the case of motion.

There is another well-developed domain of application of scale relativity that we did not mention here for lack of place, namely, the study of the induced effect of scale laws on motion laws. These effects can be described in terms of a complex, scale-covariant time derivative [3,5,6]. When written in terms of this new derivative, the equations of classical mechanics take a quantum-like form [3,6,11,26,27]. This new approach provides a new formulation of quantum mechanics [3], allows to obtain solutions to the Schrödinger equation by numerical simulations without using it [28; see also 29], and leads to new proposals concerning the formation and evolution of gravitational structures at all scales [11,30,31].

Several problems remain open in the theory. Only the minimal effects of nondifferentiability have been up to now taken into account. One of the main study to be developed in the near future is special scale-relativity in the case of four independent space-time resolutions. This can be carried out by working in a five-dimensional fractal space-time, where the role of the fifth dimension is played by the scale exponent. In the end, through a proper covariant representation, scale transformations and space-time transformations could be treated on the same footing. This would open the road toward a generalized theory of scale relativity, including non-linear scale transformations and "scale-accelerations" in its description.

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