Exploring black hole spacetimes with computers

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based on a collaboration with

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Riemann, Einstein and geometry
94th Encounter between Mathematicians and Theoretical Physicists
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Outline

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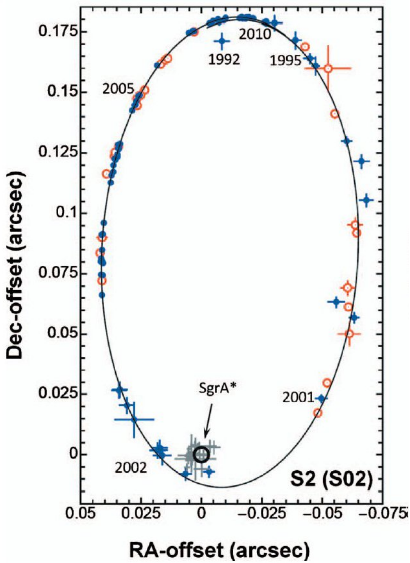
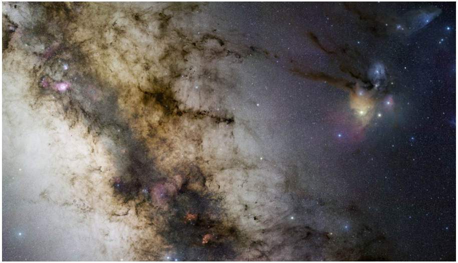
The black hole at the centre of our galaxy: Sgr A*





[ESO (2009)]

Measure of the mass of Sgr A* black hole by stellar dynamics:

$$M_{\rm BH} = 4.3 \times 10^6\, M_\odot$$

← Orbit of the star S2 around Sgr A*

$P = 16\,{\rm yr}$, $r_{\rm per} = 120\,{\rm UA} = 1400\,R_{\rm S}$, $V_{\rm per} = 0.02\,c$

[Genzel, Eisenhauer & Gillessen, RMP **82**, 3121 (2010)]

Can we see a black hole from the Earth?

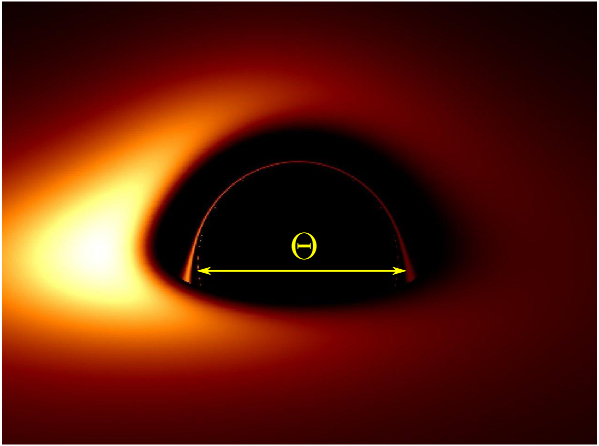


Image of a thin accretion disk around a Schwarzschild BH
[Vincent, Paumard, Gourgoulhon & Perrin, CQG **28**, 225011 (2011)]

Angular diameter of the event horizon of a Schwarzschild BH of mass M seen from a distance d:

$$\Theta = 6\sqrt{3}\,\frac{GM}{c^2 d} \simeq 2.60\,\frac{2R_S}{d}$$

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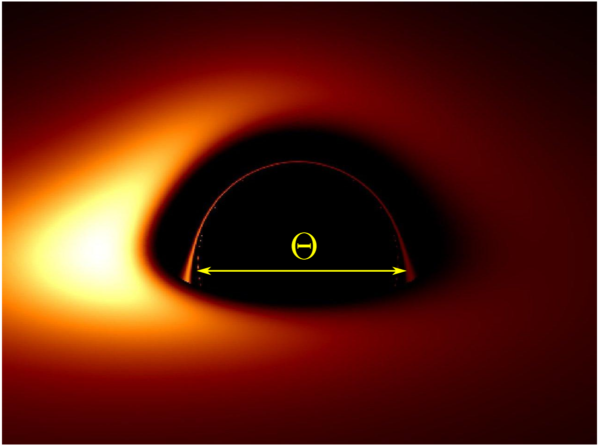

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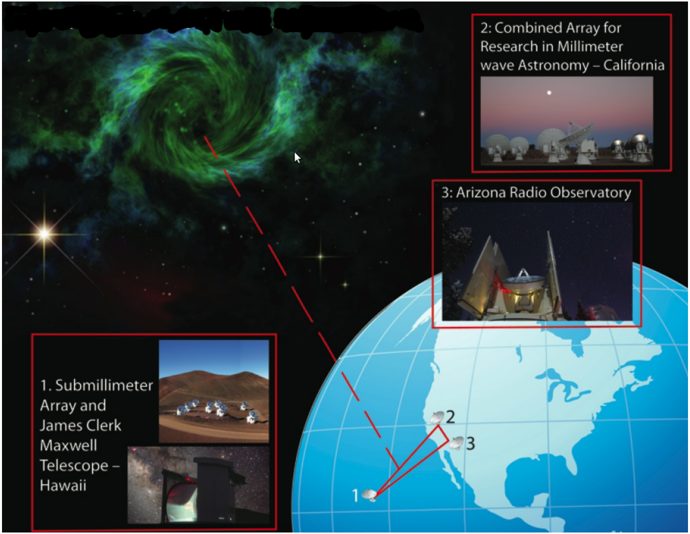
Largest black holes in the Earth's sky:

Sgr A* : $\Theta = 53\ \mu\mathrm{as}$
M87 : $\Theta = 21\ \mu\mathrm{as}$
M31 : $\Theta = 20\ \mu\mathrm{as}$

Remark: black holes in X-ray binaries are $\sim 10^5$ times smaller, for $\Theta \propto M/d$

The solution to reach the μas regime: interferometry !



Very Large Baseline Interferometry (VLBI) in (sub)millimeter waves

Existing American VLBI network [Doeleman et al. 2011]

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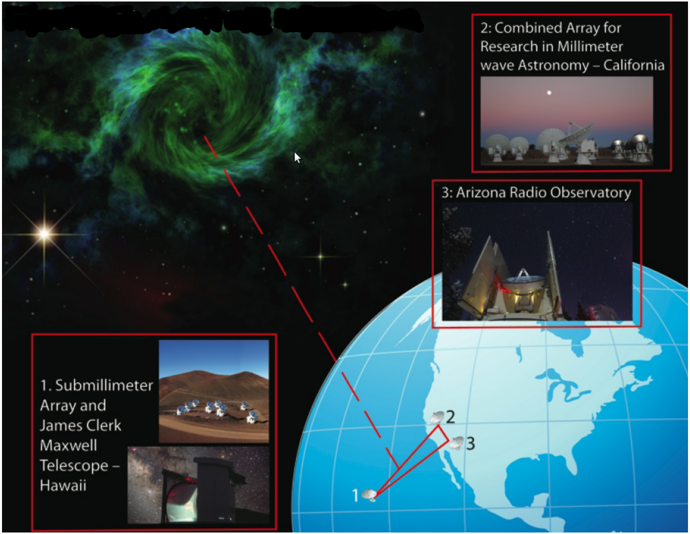


Existing American VLBI network [Doeleman et al. 2011]

Very Large Baseline Interferometry (VLBI) in (sub)millimeter waves

The best result so far: VLBI observations at 1.3 mm have shown that the size of the emitting region in Sgr A* is only $37\ \mu$as

[Doeleman et al., Nature **455**, 78 (2008)]

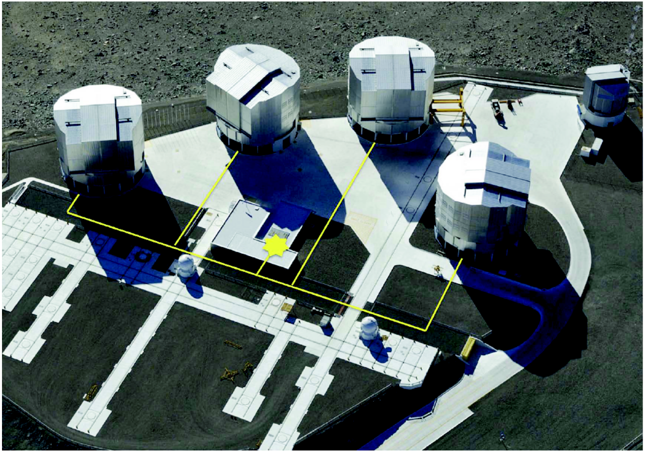
The near future: the Event Horizon Telescope

To go further:

- shorten the wavelength: $1.3\ \mathrm{mm} \to 0.8\ \mathrm{mm}$
- increase the number of stations; in particular add ALMA

Atacama Large Millimeter Array (ALMA)
part of the Event Horizon Telescope (EHT) to be completed by 2020

Near-infrared optical interferometry: GRAVITY



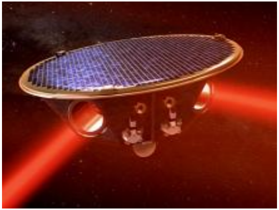
[Gillessen et al. 2010]

GRAVITY instrument at VLTI (2015)

Beam combiner (the four 8 m telescopes + four auxiliary telescopes) $\Longrightarrow$ astrometric precision of $10\ \mu\text{as}$

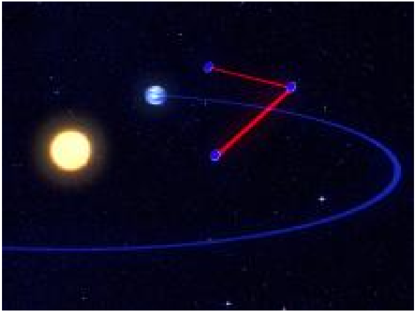
Observing BH with gravitational waves: eLISA

Interferometric gravitational wave detector in space



[eLISA (ESA)]

- Selected by ESA in November 2013 (L3 mission)
- Launch $\sim$ 2030
- LISA Pathfinder to be launched in 2015



The "no-hair" theorem

Dorochkevich, Novikov & Zel'dovich (1965), Israel (1967), Carter (1971), Hawking (1972)

Within 4-dimensional general relativity, a stationary black hole in an otherwise empty universe is necessarily a Kerr-Newman black hole, which is a vacuum solution of Einstein equation described by only three parameters:

- the total mass M
- the total angular momentum J
- the total electric charge Q

$\Longrightarrow$ *"a black hole has no hair"* (John A. Wheeler)

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Astrophysical black holes have to be electrically neutral:

- $Q = 0$: Kerr solution (1963)

Other special cases:

- $Q = 0$ and $a = 0$: Schwarzschild solution (1916)
- $a = 0$: Reisnerr-Nordström solution (1916, 1918)

Lowest order no-hair theorem: quadrupole moment

Asymptotic expansion (large r) of the metric in terms of multipole moments $(\mathcal{M}_k, \mathcal{J}_k)_{k\in\mathbb{N}}$ [Geroch (1970), Hansen (1974)]:

- $\mathcal{M}_k$: mass 2^k-pole moment
- $\mathcal{J}_k$: angular momentum 2^k-pole moment

$\Longrightarrow$ For the Kerr metric, all the multipole moments are determined by (M, a):

- $\mathcal{M}_0 = M$
- $\mathcal{J}_1 = aM = J/c$
- $\boxed{\mathcal{M}_2 = -a^2 M = -\frac{J^2}{c^2 M}}$ $(*)$ $\leftarrow$ mass quadrupole moment
- $\mathcal{J}_3 = -a^3 M$
- $\mathcal{M}_4 = a^4 M$
- $\cdots$

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Measuring the three quantities M, J, $\mathcal{M}_2$ provides a compatibility test w.r.t. the Kerr metric, by checking $(*)$

Theoretical alternatives to the Kerr black hole

Within general relativity

The compact object is not a black hole but

- a boson star
- a gravastar
- a dark star
- ...

Beyond general relativity

The compact object is a black hole but in a theory that differs from GR:

- Einstein-Gauss-Bonnet with dilaton
- Chern-Simons gravity
- Hořava-Lifshitz gravity
- Einstein-Yang-Mills
- ...

Outline

How to test the alternatives to the Kerr black hole?

Search for

- stellar orbits deviating from Kerr timelike geodesics (GRAVITY)
- accretion disk spectra different from those arising in Kerr metric (X-ray observatories)
- images of the black hole shadow different from that of a Kerr black hole (EHT)

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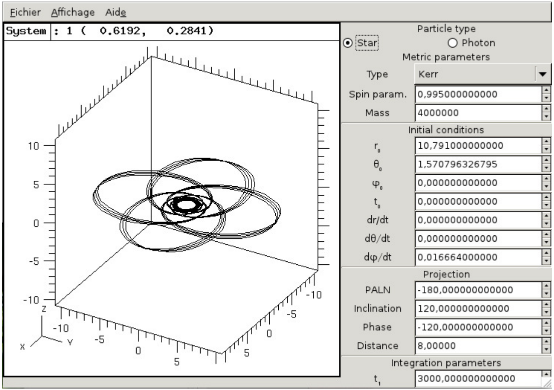
Need for a good and versatile geodesic integrator
to compute timelike geodesics (orbits) and null geodesics (ray-tracing) in any kind of metric

Gyoto code

Main developers: T. Paumard & F. Vincent



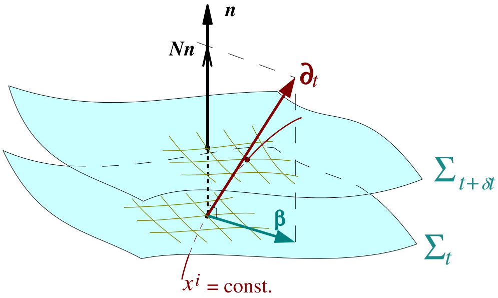
- Integration of geodesics in Kerr metric
- Integration of geodesics in any numerically computed 3+1 metric
- Radiative transfer included in optically thin media
- Very modular code (C++)
- Yorick interface
- Free software (GPL) : http://gyoto.obspm.fr/

[Vincent, Paumard, Gourgoulhon & Perrin, CQG **28**, 225011 (2011)]

[Vincent, Gourgoulhon & Novak, CQG **29**, 245005 (2012)]

3+1 decomposition of the geodesic equation (1/3)

Numerical spacetimes are generally computed within the 3+1 formalism



4-dimensional spacetime $(\mathcal{M}, \boldsymbol{g})$ foliated by spacelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$

Unit timelike normal: $\underline{\boldsymbol{n}} = -N\boldsymbol{\nabla}t$

Induced metric: $\boldsymbol{\gamma} = \boldsymbol{g} + \underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{n}}$

Shift vector of adapted coordinates (t, x^i): vector $\boldsymbol{\beta}$ tangent to Σ_t such that $\partial/\partial t = N\boldsymbol{n} + \boldsymbol{\beta}$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -N^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

3+1 decomposition of the geodesic equation (2/3)

The geodesic equation

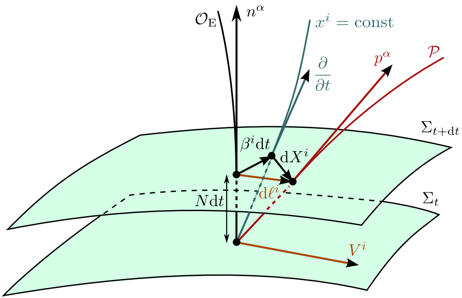
A particle $\mathcal{P}$ of 4-momentum vector $\boldsymbol{p}$ follows a geodesic iff

$$\boxed{\boldsymbol{\nabla}_{\boldsymbol{p}}\,\boldsymbol{p} = 0}$$

3+1 decomposition of $\boldsymbol{p}$: $\boxed{\boldsymbol{p} = E(\boldsymbol{n} + \boldsymbol{V})}$, with

- E : particle's energy with respect to the Eulerian observer (4-velocity $\boldsymbol{n}$)
- $\boldsymbol{V}$: vector tangent to Σ_t, representing the particle's 3-velocity with respect to the Eulerian observer

3+1 decomposition of the geodesic equation (3/3)



Equation of $\mathcal{P}$'s worldline in terms of the 3+1 coordinates : $x^i = X^i(t)$

The physical 3-velocity $\boldsymbol{V}$ is related to the coordinate velocity $\dot{X}^i := dx^i/dt$ by

$$V^i = \frac{d\ell^i}{d\tau_E} = \frac{1}{N}\frac{d\ell^i}{dt} = \frac{1}{N}\frac{\beta^i dt + dX^i}{dt}$$

$$\Longrightarrow V^i = \frac{1}{N}\left(\dot{X}^i + \beta^i\right)$$

Orth. projection of $\boldsymbol{\nabla}_{\boldsymbol{p}}\,\boldsymbol{p} = 0$ along $\boldsymbol{n}$:

$$\frac{dE}{dt} = E\left(N K_{jk} V^j V^k - V^j \partial_j N\right)$$

Orth. projection of $\boldsymbol{\nabla}_{\boldsymbol{p}}\,\boldsymbol{p} = 0$ onto Σ_t:

$$\begin{cases} \dfrac{dX^i}{dt} = N V^i - \beta^i \\ \dfrac{dV^i}{dt} = N V^j \left[V^i \left(\partial_j \ln N - K_{jk} V^k\right) + 2K^i_{\ j} - {}^3\Gamma^i_{jk} V^k \right] - \gamma^{ij}\partial_j N - V^j \partial_j \beta^i \end{cases}$$

[Vincent, Gourgoulhon & Novak, CQG **29**, 245005 (2012)]

3+1 geodesic integration in Gyoto code (1/2)

Numerical spacetime $\Longrightarrow (N, \beta^i, \gamma_{ij}, K_{ij})$

System to be integrated

$$\left\{ \begin{array}{l} \dfrac{\mathrm{d}E}{\mathrm{d}t} = E \left(N K_{jk} V^j V^k - V^j \partial_j N \right) \\ \dfrac{\mathrm{d}X^i}{\mathrm{d}t} = N V^i - \beta^i \\ \dfrac{\mathrm{d}V^i}{\mathrm{d}t} = N V^j \left[V^i \left(\partial_j \ln N - K_{jk} V^k \right) + 2 K^i_{\ j} - {}^3\Gamma^i_{jk} V^k \right] - \gamma^{ij} \partial_j N - V^j \partial_j \beta^i \end{array} \right.$$

Integration (backward) in time: Runge–Kutta algorithms of fourth to eighth order

Problem: the 3+1 quantities $(N, \beta^i, \gamma_{ij}, K_{ij})$ and their spatial derivatives have to be known at any point along the geodesic and not only at the grid points issued from the numerical relativity computation

3+1 geodesic integration in Gyoto code (2/2)

Solution within spectral methods: thanks to their spectral expansions, the fields $(N, \beta^i, \gamma_{ij}, K_{ij})$ are actually known at any point !

For instance, a scalar field, like N, is expanded as

$$N(t, r, \theta, \varphi) = \sum_{i,\ell,m} \hat{N}_{i\ell m}(t)\, T_i(r)\, Y_\ell^m(\theta, \varphi)$$

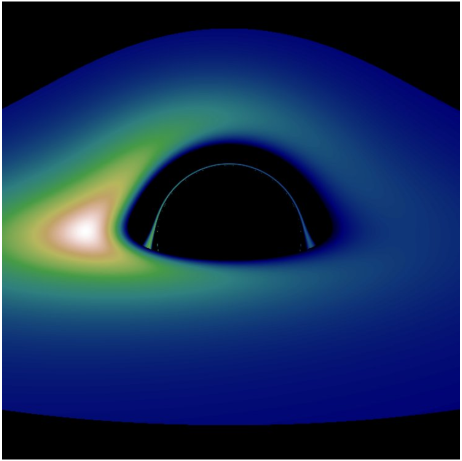
with

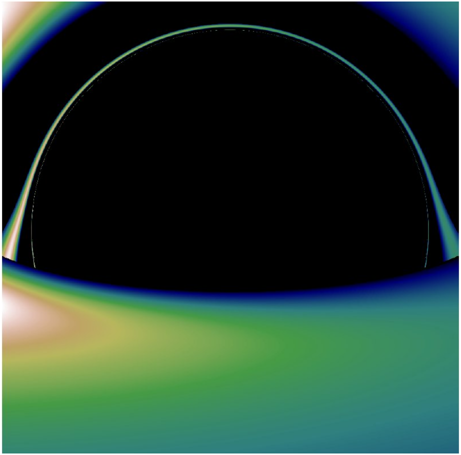
- T_i: Chebyshev polynomial of degree i
- Y_ℓ^m: spherical harmonic of index (ℓ, m)

Within spectral methods, the discretization does not occur on the values in the physical space (no grid !) but on the finite number of coefficients $\hat{N}_{i\ell m}$

The data are $(\hat{N}_{i\ell m}(t_J))$ for a finite series of time steps $(t_J)_{0\le J\le J_{\max}}$
$\Longrightarrow$ the values $(\hat{N}_{i\ell m}(t))$ at an arbitrary time t are obtained by a third order interpolation from 4 neighbouring t_J's

Gyoto code



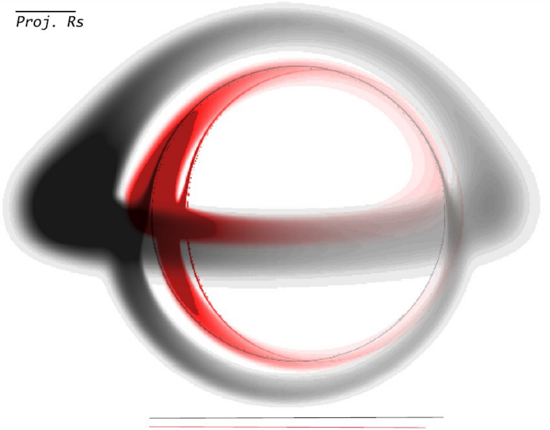


Computed images of a thin accretion disk around a Schwarzschild black hole

Measuring the spin from the black hole silhouette

Ray-tracing in the Kerr metric (spin parameter a)

Accretion structure around Sgr A* modelled as a ion torus, derived from the *polish doughnut* class [Abramowicz, Jaroszynski & Sikora (1978)]



Radiative processes included: thermal synchrotron, bremsstrahlung, inverse Compton

← Image of an ion torus computed with `Gyoto` for the inclination angle $i = 80°$:

- black: $a = 0.5M$
- red: $a = 0.9M$

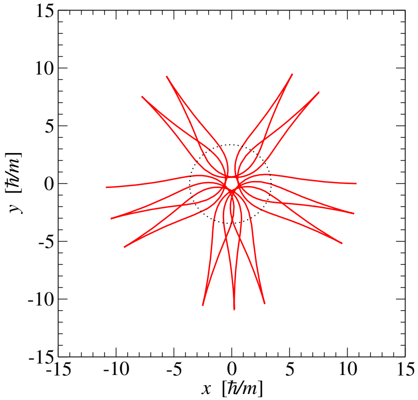
[Straub, Vincent, Abramowicz, Gourgoulhon & Paumard, A&A **543**, A83 (2012)]

Orbits around a rotating boson star

Boson star = localized configurations of a self-gravitating complex scalar field Φ ≡ *"Klein-Gordon geons"* [Bonazzola & Pacini (1966), Kaup (1968)]

Boson stars may behave as black-hole mimickers

- Solutions of the *Einstein-Klein-Gordon* system computed by means of Kadath [Grandclément, JCP **229**, 3334 (2010)]
- Timelike geodesics computed by means of Gyoto



Zero-angular-momentum orbit around a rotating boson star based on a free scalar field $\Phi = \phi(r,\theta)e^{i(\omega t + 2\varphi)}$ with $\omega = 0.75\, m/\hbar$.

[Granclément, Somé & Gourgoulhon, PRD **90**, 024068 (2014)]

Outline

Software for differential geometry

Packages for general purpose computer algebra systems:

- xAct free package for Mathematica [J.-M. Martin-Garcia]
- Ricci free package for Mathematica [J. L. Lee]
- MathTensor package for Mathematica [S. M. Christensen & L. Parker]
- DifferentialGeometry included in Maple [I. M. Anderson & E. S. Cheb-Terrab]
- Atlas 2 for Maple and Mathematica
- ...

Standalone applications:

- SHEEP, Classi, STensor, based on Lisp, developed in 1970's and 1980's (free) [R. d'Inverno, I. Frick, J. Åman, J. Skea, et al.]
- Cadabra field theory (free) [K. Peeters]
- SnapPy topology and geometry of 3-manifolds, based on Python (free) [M. Culler, N. M. Dunfield & J. R. Weeks]
- ...

cf. the complete list on `http://www.xact.es/links.html`

Sage in a few words

- Sage is a **free open-source** mathematics software system
- it is based on the Python programming language
- it makes use of **many pre-existing open-sources packages**, among which
  - Maxima (symbolic calculations, since 1968!)
  - GAP (group theory)
  - PARI/GP (number theory)
  - Singular (polynomial computations)
  - matplotlib (high quality 2D figures)

  and provides a **uniform interface** to them
- William Stein (Univ. of Washington) created Sage in 2005; since then, ∼**100 developers** (mostly mathematicians) have joined the Sage team

The mission

Create a viable free open source alternative to Magma, Maple, Mathematica and Matlab.

Some advantages of Sage

Sage is free

Freedom means

1. everybody can use it, by downloading the software from http://sagemath.org
2. everybody can examine the source code and improve it

Sage is based on Python

- no need to learn any specific syntax to use it
- easy access for students
- Python is a very powerful *object oriented language*, with a neat syntax

Sage is developing and spreading fast

...sustained by an important community of developers

Sage approach to computer mathematics

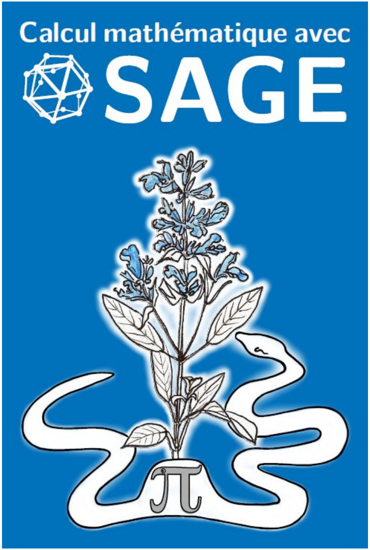
Sage relies on a `Parent` / `Element` scheme: each object x on which some calculus is performed has a "parent", which is another Sage object X representing the set to which x belongs.
The calculus rules on x are determined by the *algebraic structure* of X.
Conversion rules prior to an operation, e.g. $x + y$ with x and y having different parents, are defined at the level of the parents

Example

```
sage: x = 4 ; x.parent()
Integer Ring
sage: y = 4/3 ; y.parent()
Rational Field
sage: s = x + y ; s.parent()
Rational Field
sage: y.parent().has_coerce_map_from(x.parent())
True
```

This approach is similar to that of Magma and different from that of Mathematica, in which everything is a tree of symbols

The Sage book



by Paul Zimmermann et al. (2013)

Released under *Creative Commons* license:

- freely downloadable from http://sagebook.gforge.inria.fr/
- printed copies can be ordered at moderate price (10 €)

English translation in progress...

Differential geometry in Sage

Sage is well developed in many domains of mathematics:
number theory, group theory, linear algebra, combinatorics, etc.

...but not too much in the area of differential geometry:

Already in Sage

- differential forms on an open subset of Euclidean space *(with a fixed set of coordinates)* (J. Vankerschaver)
- parametrized 2-surfaces in 3-dim. Euclidean space (M. Malakhaltsev, J. Vankerschaver, V. Delecroix)

Proposed extensions (Sage Trac)

- 2-D hyperbolic geometry (V. Delecroix, M. Raum, G. Laun, trac ticket #9439)

The SageManifolds project

http://sagemanifolds.obspm.fr/

Aim

Implement the concept of real smooth manifolds of arbitrary dimension in Sage and tensor calculus on them, in a coordinate/frame-independent manner

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In practice, this amounts to introducing new Python classes in Sage, basically one class per mathematical concept, for instance:

- `Manifold`: differentiable manifolds over $\mathbb{R}$, of arbitrary dimension
- `Chart`: coordinate charts
- `Point`: points on a manifold
- `DiffMapping`: differential mappings between manifolds
- `ScalarField`, `VectorField`, `TensorField`: tensor fields on a manifold
- `DiffForm`: p-forms
- `AffConnection`, `LeviCivitaConnection`: affine connections
- `Metric`: pseudo-Riemannian metrics

Implementing coordinate charts

Given a manifold $\mathcal{M}$ of dimension n, a coordinate chart on an open subset $U \subset \mathcal{M}$ is implemented in SageManifolds via the class `Chart`, whose main data is a n-uple of *Sage symbolic variables* `x`, `y`, ..., each of them representing a coordinate

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In general, more than one (regular) chart may be required to cover the entire manifold:

Examples:

- at least 2 charts are necessary to cover the circle $\mathbb{S}^1$, the sphere $\mathbb{S}^2$, and more generally the n-dimensional sphere $\mathbb{S}^n$
- at least 3 charts are necessary to cover the real projective plane $\mathbb{RP}^2$

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In SageManifolds, an arbitrary number of charts can be introduced

To fully specify the manifold, one shall also provide the *transition maps* on overlapping chart domains (SageManifolds class `CoordChange`)

Implementing scalar fields

A **scalar field** on manifold $\mathcal{M}$ is a smooth mapping

$$\begin{array}{llll} f: & U \subset \mathcal{M} & \longrightarrow & \mathbb{R} \\ & p & \longmapsto & f(p) \end{array}$$

where U is an open subset of $\mathcal{M}$

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The various coordinate representations F, $\hat{F}$, ... of f are stored as a *Python dictionary* whose keys are the charts C, $\hat{C}$, ...:

$$f.\_\mathtt{express} = \left\{ C : F,\ \hat{C} : \hat{F}, \ldots \right\}$$

with $\underbrace{f(\,p\,)}_{\text{point}} = F(\underbrace{x^1, \ldots, x^n}_{\text{coord. of } p \text{ in chart } C}) = \hat{F}(\underbrace{\hat{x}^1, \ldots, \hat{x}^n}_{\text{coord. of } p \text{ in chart } \hat{C}}) = \ldots$

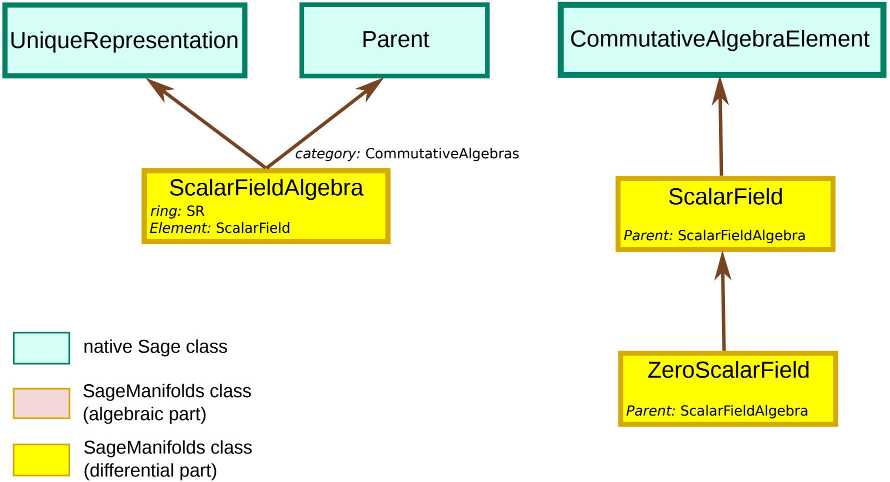
The scalar field algebra

Given an open subset $U \subset \mathcal{M}$, the set $C^\infty(U)$ of scalar fields defined on U has naturally the structure of a **commutative algebra over** $\mathbb{R}$: it is clearly a vector space over $\mathbb{R}$ and it is endowed with a commutative ring structure by pointwise multiplication:

$$\forall f, g \in C^\infty(U), \quad \forall p \in U, \quad (f.g)(p) := f(p)g(p)$$

The algebra $C^\infty(U)$ is implemented in SageManifolds via the class `ScalarFieldAlgebra`.

Classes for scalar fields



Vector fields

Given an open subset $U \subset \mathcal{M}$, the set $\mathcal{X}(U)$ of smooth vector fields defined on U has naturally the structure of a **module over the scalar field algebra** $C^\infty(U)$.

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Modules vs. vector spaces

A **module** is $\sim$ **vector space**, except that it is based on a **ring** (here $C^\infty(U)$) instead of a **field** (usually $\mathbb{R}$ or $\mathbb{C}$ in physics)

An importance difference: a vector space always has a **basis***, while a module does not necessarily have any*

$\rightarrow$ A module with a basis is called a **free module**

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$\rightarrow$ A module with a basis is called a **free module**

When $\mathcal{X}(U)$ is a free module, a basis is a **vector frame** $(\boldsymbol{e}_a)_{1\leq a\leq n}$ on U:

$$\forall \boldsymbol{v} \in \mathcal{X}(U), \quad \boldsymbol{v} = v^a \boldsymbol{e}_a, \quad \text{with } v^a \in C^\infty(U)$$

At a point $p \in U$, the above translates into an identity in the *tangent vector space* $T_p\mathcal{M}$:

$$\boldsymbol{v}(p) = v^a(p)\ \boldsymbol{e}_a(p), \quad \text{with } v^a(p) \in \mathbb{R}$$

Vector fields

A manifold $\mathcal{M}$ that admits a global vector frame (or equivalently, such that $\mathcal{X}(\mathcal{M})$ is a free module) is called a **parallelizable manifold**

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Examples of parallelizable manifolds

- $\mathbb{R}^n$ (global coordinate charts $\Rightarrow$ global vector frames)
- the circle $\mathbb{S}^1$ (NB: no global coordinate chart)
- the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$
- the 3-sphere $\mathbb{S}^3 \simeq \mathrm{SU}(2)$, as any Lie group
- the 7-sphere $\mathbb{S}^7$

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Examples of non-parallelizable manifolds

- the sphere $\mathbb{S}^2$ (hairy ball theorem!) and any n-sphere $\mathbb{S}^n$ with $n \notin \{1, 3, 7\}$
- the real projective plane $\mathbb{RP}^2$
- most manifolds...

Implementing vector fields

Ultimately, in SageManifolds, vector fields are to be described by their components w.r.t. various vector frames.

If the manifold $\mathcal{M}$ is not parallelizable, one has to decompose it in parallelizable open subsets U_i $(1 \leq i \leq N)$ and consider **restrictions** of vector fields to these domains.

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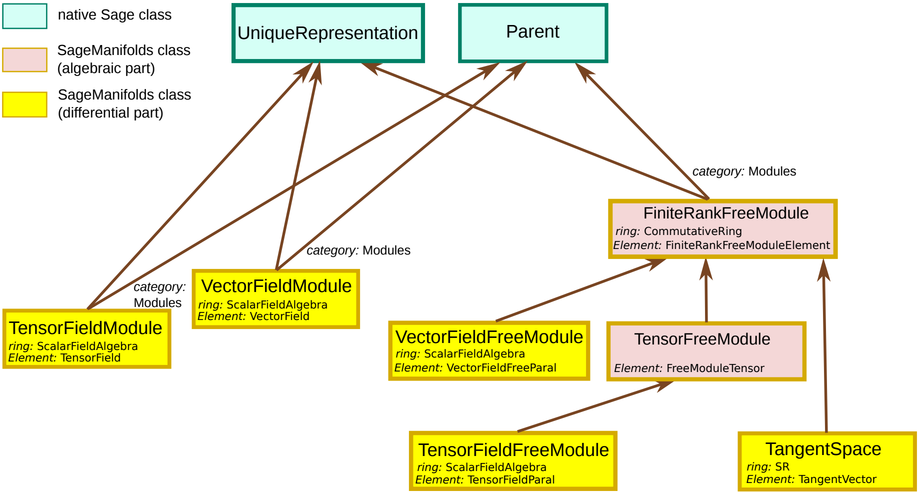
If the manifold $\mathcal{M}$ is not parallelizable, one has to decompose it in parallelizable open subsets U_i $(1 \leq i \leq N)$ and consider **restrictions** of vector fields to these domains.

For each i, $\mathcal{X}(U_i)$ is a free module of rank $n = \dim \mathcal{M}$ and is implemented in SageManifolds as an instance of `VectorFieldFreeModule`, which is a subclass of `FiniteRankFreeModule`.

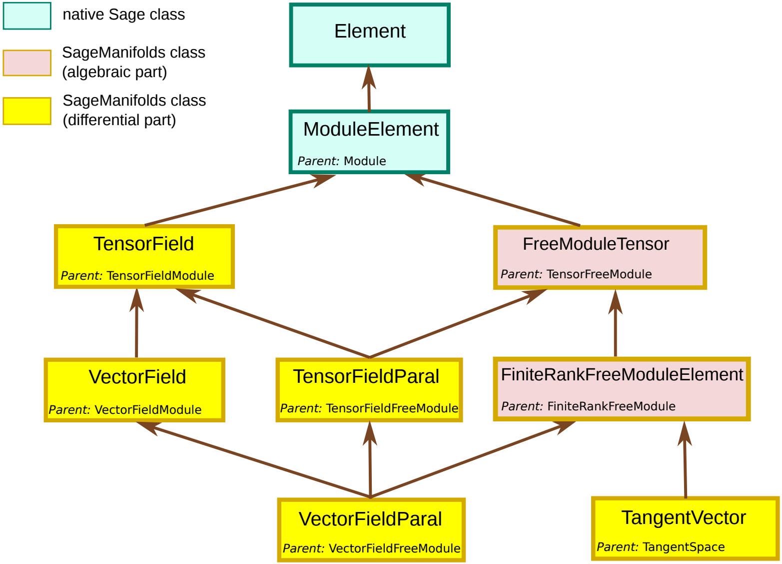
Each vector field $\boldsymbol{v} \in \mathcal{X}(U_i)$ has different set of components $(v^a)_{1\leq a\leq n}$ in different vector frames $(\boldsymbol{e}_a)_{1\leq a\leq n}$ introduced on U_i. They are stored as a *Python dictionary* whose keys are the vector frames:

$$\boldsymbol{v}.\texttt{\_components} = \{(\boldsymbol{e}) : (v^a),\ (\hat{\boldsymbol{e}}) : (\hat{v}^a), \ldots\}$$

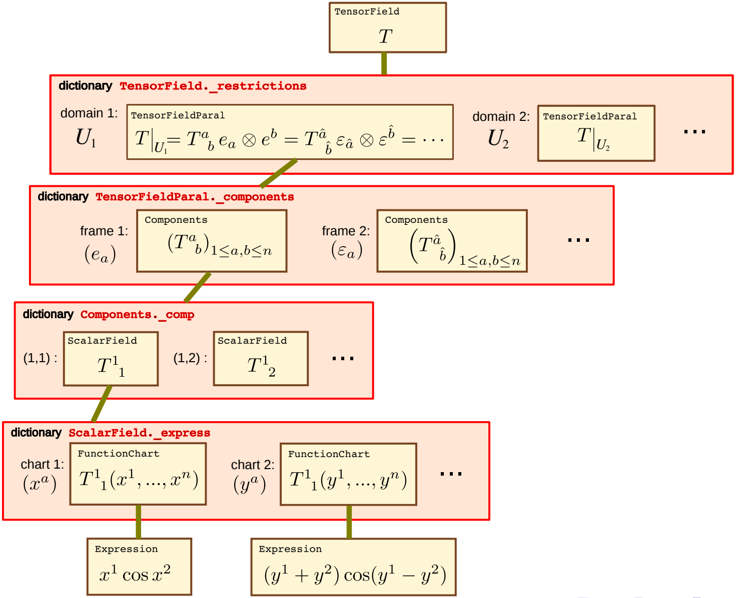
Module classes in SageManifolds



Tensor field classes in SageManifolds



Tensor field storage



SageManifolds at work: the Mars-Simon tensor example

Definition [M. Mars, CQG **16**, 2507 (1999)]

Given a 4-dimensional spacetime $(\mathcal{M}, \boldsymbol{g})$ endowed with a Killing vector field ξ, the **Mars-Simon tensor w.r.t.** ξ is the type-(0,3) tensor $\boldsymbol{S}$ defined by

$$S_{\alpha\beta\gamma} := 4\mathcal{C}_{\mu\alpha\nu[\beta}\,\xi^\mu\xi^\nu\,\sigma_{\gamma]} + \gamma_{\alpha[\beta}\,\mathcal{C}_{\gamma]\rho\mu\nu}\,\xi^\rho\,\mathcal{F}^{\mu\nu}$$

where

- $\gamma_{\alpha\beta} := \lambda\, g_{\alpha\beta} + \xi_\alpha\xi_\beta$, with $\lambda := -\xi_\mu\xi^\mu$
- $\mathcal{C}_{\alpha\beta\mu\nu} := C_{\alpha\beta\mu\nu} + \frac{i}{2}\epsilon^{\rho\sigma}{}_{\mu\nu}\,C_{\alpha\beta\rho\sigma}$, with $C^\alpha{}_{\beta\mu\nu}$ being the Weyl tensor and $\epsilon_{\alpha\beta\mu\nu}$ the Levi-Civita volume form
- $\mathcal{F}_{\alpha\beta} := F_{\alpha\beta} + i\,{}^*F_{\alpha\beta}$, with $F_{\alpha\beta} := \nabla_\alpha\xi_\beta$ (Killing 2-form) and ${}^*F_{\alpha\beta} := \frac{1}{2}\epsilon^{\mu\nu}{}_{\alpha\beta}F_{\mu\nu}$ (Hodge dual of $F_{\alpha\beta}$)
- $\sigma_\alpha := 2\mathcal{F}_{\mu\alpha}\xi^\mu$ (Ernst 1-form)

Mars-Simon tensor

The Mars-Simon tensor provides a nice characterization of Kerr spacetime:

Theorem (Mars, 1999)

If $\boldsymbol{g}$ satisfies the vacuum Einstein equation and $(\mathcal{M}, \boldsymbol{g})$ contains a stationary asymptotically flat end $\mathcal{M}^\infty$ such that $\boldsymbol{\xi}$ tends to a time translation at infinity in $\mathcal{M}^\infty$ and the Komar mass of $\boldsymbol{\xi}$ in $\mathcal{M}^\infty$ is non-zero, then

$$\boldsymbol{S} = 0 \iff (\mathcal{M}, \boldsymbol{g}) \text{ is locally isometric to a Kerr spacetime}$$

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$$\boldsymbol{S} = 0 \iff (\mathcal{M}, \boldsymbol{g}) \text{ is locally isometric to a Kerr spacetime}$$

Let us use SageManifolds...

...to check the $\Leftarrow$ part of the theorem, namely that the Mars-Simon tensor is identically zero in Kerr spacetime.

NB: what follows illustrates only certain features of SageManifolds; other ones, like the multi-chart and multi-frame capabilities on non-parallelizable manifolds, are not considered in this example. $\Longrightarrow$ *More examples are provided at*
http://sagemanifolds.obspm.fr/examples.html

Object-oriented notation

To understand what follows, be aware that

as an object-oriented language, Python (and hence Sage) makes use of the following postfix notation:

```
result = object.function(arguments)
```

In a functional language, this would be written as

```
result = function(object,arguments)
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```
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Examples

```
riem = g.riemann()
lie_t_v = t.lie_der(v)
```

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```
M = Manifold(4, 'M', latex_name=r'\mathcal{M}')
print M
```

```
4-dimensional manifold 'M'
```

We introduce the standard **Boyer-Lindquist coordinates** as follows:

```
X.<t,r,th,ph> = M.chart(r't r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')
print X ; X
```

```
chart (M, (t, r, th, ph))
```

$(\mathcal{M}, (t, r, \theta, \phi))$

Metric tensor

The 2 parameters m and a of the Kerr spacetime are declared as symbolic variables:

```
var('m, a')
```

(m, a)

Let us introduce the spacetime metric g and set its components in the coordinate frame associated with Boyer-Lindquist coordinates, which is the current manifold's default frame:

```
g = M.lorentz_metric('g')
rho2 = r^2 + (a*cos(th))^2
Delta = r^2 -2*m*r + a^2
g[0,0] = -(1-2*m*r/rho2)
g[0,3] = -2*a*m*r*sin(th)^2/rho2
g[1,1], g[2,2] = rho2/Delta, rho2
g[3,3] = (r^2+a^2+2*m*r*(a*sin(th))^2/rho2)*sin(th)^2
g.view()
```

$$g = \left(-\frac{a^2\cos(\theta)^2 - 2\,mr + r^2}{a^2\cos(\theta)^2 + r^2}\right)\mathrm{d}t\otimes\mathrm{d}t + \left(-\frac{2\,amr\sin(\theta)^2}{a^2\cos(\theta)^2 + r^2}\right)\mathrm{d}t\otimes\mathrm{d}\phi + \left(\frac{a^2\cos(\theta)^2 + r^2}{a^2 - 2\,mr + r^2}\right)\mathrm{d}r\otimes\mathrm{d}r + \left(a^2\cos(\theta)^2 + r^2\right)\mathrm{d}\theta\otimes\mathrm{d}\theta + \left(-\frac{2\,amr\sin}{a^2\cos(\theta}\right.$$

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```
g[:]
```

$$\begin{pmatrix} -\frac{a^2\cos(\theta)^2 - 2mr + r^2}{a^2\cos(\theta)^2 + r^2} & 0 & 0 & -\frac{2amr\sin(\theta)^2}{a^2\cos(\theta)^2 + r^2} \\ 0 & \frac{a^2\cos(\theta)^2 + r^2}{a^2 - 2mr + r^2} & 0 & 0 \\ 0 & 0 & a^2\cos(\theta)^2 + r^2 & 0 \\ -\frac{2amr\sin(\theta)^2}{a^2\cos(\theta)^2 + r^2} & 0 & 0 & \frac{2a^2mr\sin(\theta)^4 + \left(a^2r^2 + r^4 + \left(a^4 + a^2r^2\right)\cos(\theta)^2\right)\sin(\theta)^2}{a^2\cos(\theta)^2 + r^2} \end{pmatrix}$$

The Levi-Civita connection ∇ associated with g:

```
nab = g.connection() ; print nab
```

```
Levi-Civita connection 'nabla_g' associated with the Lorentzian metric
'g' on the 4-dimensional manifold 'M'
```

As a check, we verify that the covariant derivative of g with respect to ∇ vanishes identically:

```
nab(g).view()
```

$$\nabla_g g = 0$$

Killing vector

The default vector frame on the spacetime manifold is the coordinate basis associated with Boyer-Lindquist coordinates:

```
M.default_frame() is X.frame()
```

True

```
X.frame()
```

$$\left(\mathcal{M}, \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right)\right)$$

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Let us consider the first vector field of this frame:

```
xi = X.frame()[0] ; xi
```

$$\frac{\partial}{\partial t}$$

```
print xi
```

```
vector field 'd/dt' on the 4-dimensional manifold 'M'
```

The 1-form associated to it by metric duality is

```
xi_form = xi.down(g)
xi_form.set_name('xi_form', r'\underline{\xi}')
print xi_form ; xi_form.view()
```

```
1-form 'xi_form' on the 4-dimensional manifold 'M'
```

$$\underline{\xi} = \left(-\frac{a^2\cos(\theta)^2 - 2\,mr + r^2}{a^2\cos(\theta)^2 + r^2}\right)\mathrm{d}t + \left(-\frac{2\,amr\sin(\theta)^2}{a^2\cos(\theta)^2 + r^2}\right)\mathrm{d}\phi$$

Its covariant derivative is

```
nab_xi = nab(xi_form)
print nab_xi ; nab_xi.view()
```

```
tensor field 'nabla_g xi_form' of type (0,2) on the 4-dimensional
manifold 'M'
```

$$\nabla_g\underline{\xi} = \left(\frac{a^2m\cos(\theta)^2 - mr^2}{a^4\cos(\theta)^4 + 2\,a^2r^2\cos(\theta)^2 + r^4}\right)\mathrm{d}t\otimes\mathrm{d}r + \left(\frac{2\,a^2mr\cos(\theta)\sin(\theta)}{a^4\cos(\theta)^4 + 2\,a^2r^2\cos(\theta)^2 + r^4}\right)\mathrm{d}t\otimes\mathrm{d}\theta + \left(-\frac{a^2m\cos(\theta)^2 - mr^2}{a^4\cos(\theta)^4 + 2\,a^2r^2\cos(\theta)^2 + r^4}\right)\mathrm{d}r\otimes\mathrm{d}t + \left(\frac{(a^3m\cos}{a^4\cos(\theta)}\right.$$

Let us check that the Killing equation is satisfied:

```
nab_xi.symmetrize().view()
```

$$0$$

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Equivalently, we check that the Lie derivative of the metric along ξ vanishes:

```
g.lie_der(xi).view()
```

0

Thank to Killing equation, $\nabla_g \xi$ is antisymmetric. We may therefore define a 2-form by $F := -\nabla_g \xi$. Here we enforce the antisymmetry by calling the function `antisymmetrize()` on `nab_xi`:

```
F = - nab_xi.antisymmetrize()
F.set_name('F')
print F
F.view()
```

```
2-form 'F' on the 4-dimensional manifold 'M'
```

$$F = \left(-\frac{a^2 m \cos(\theta)^2 - m r^2}{a^4 \cos(\theta)^4 + 2\,a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}t \wedge \mathrm{d}r + \left(-\frac{2\,a^2 m r \cos(\theta)\sin(\theta)}{a^4 \cos(\theta)^4 + 2\,a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}t \wedge \mathrm{d}\theta + \left(-\frac{\left(a^3 m \cos(\theta)^2 - a m r^2\right)\sin(\theta)^2}{a^4 \cos(\theta)^4 + 2\,a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}r \wedge \mathrm{d}\phi + \left(-\frac{2\,(a}{a^4 \mathrm{c}}\right.$$

We check that

```
F == - nab_xi
```

True

The squared norm of the Killing vector is:

```
lamb = - g(xi,xi)
lamb.set_name('lambda', r'\lambda')
print lamb
lamb.view()
```

```
scalar field 'lambda' on the 4-dimensional manifold 'M'
```

$$\begin{array}{llcl} \lambda: & \mathcal{M} & \longrightarrow & \mathbb{R} \\ & (t, r, \theta, \phi) & \longmapsto & \frac{a^2 \cos(\theta)^2 - 2\,mr + r^2}{a^2 \cos(\theta)^2 + r^2} \end{array}$$

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Instead of invoking $g(\xi,\xi)$, we could have evaluated λ by means of the 1-form $\underline{\xi}$ acting on the vector field ξ:

```
lamb == - xi_form(xi)
```

True

or, in index notation,

```
lamb == - ( xi_form['_a']*xi['^a'] )
```

True

Curvature

The Riemann curvature tensor associated with g is:

```
Riem = g.riemann()
print Riem
```

```
tensor field 'Riem(g)' of type (1,3) on the 4-dimensional manifold 'M'
```

The component $R^0{}_{123}$ is

```
Riem[0,1,2,3]
```

$$-\frac{\left(a^7m-2\,a^5m^2r+a^5mr^2\right)\cos(\theta)\sin(\theta)^5+\left(a^7m+2\,a^5m^2r+6\,a^5mr^2-6\,a^3m^2r^3+5\,a^3mr^4\right)\cos(\theta)\sin(\theta)^3-2\left(a^7m-a^5mr^2-5\,a^3mr^4-3\,amr^6\right)\cos(\theta)\sin(\theta)}{a^2r^6-2\,mr^7+r^8+\left(a^8-2\,a^6mr+a^6r^2\right)\cos(\theta)^6+3\left(a^6r^2-2\,a^4mr^3+a^4r^4\right)\cos(\theta)^4+3\left(a^4r^4-2\,a^2mr^5+a^2r^6\right)\cos(\theta)^2}$$

The Ricci tensor:

```
Ric = g.ricci()
print Ric
```

```
field of symmetric bilinear forms 'Ric(g)' on the 4-dimensional manifold
'M'
```

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Let us check that we are dealing with a solution of Einstein equation:

```
Ric.view()
```

$$\mathrm{Ric}(g) = 0$$

The Weyl conformal curvature tensor is

```
C = g.weyl()
print C
```

```
tensor field 'C(g)' of type (1,3) on the 4-dimensional manifold 'M'
```

Let us exhibit two of its components $C^0{}_{123}$ and $C^0{}_{101}$:

```
C[0,1,2,3]
```

$$-\frac{\left(a^7m-2\,a^5m^2r+a^5mr^2\right)\cos(\theta)\,\sin(\theta)^5+\left(a^7m+2\,a^5m^2r+6\,a^5mr^2-6\,a^3m^2r^3+5\,a^3mr^4\right)\cos(\theta)\,\sin(\theta)^3-2\left(a^7m-a^5mr^2-5\,a^3mr^4-3\,amr^6\right)\cos(\theta)\,\sin(\theta)}{a^2r^6-2\,mr^7+r^8+\left(a^8-2\,a^6mr+a^6r^2\right)\cos(\theta)^6+3\left(a^6r^2-2\,a^4mr^3+a^4r^4\right)\cos(\theta)^4+3\left(a^4r^4-2\,a^2mr^5+a^2r^6\right)\cos(\theta)^2}$$

```
C[0,1,0,1]
```

$$\frac{3\,a^4mr\cos(\theta)^4+3\,a^2mr^3+2\,mr^5-\left(9\,a^4mr+7\,a^2mr^3\right)\cos(\theta)^2}{a^2r^6-2\,mr^7+r^8+\left(a^8-2\,a^6mr+a^6r^2\right)\cos(\theta)^6+3\left(a^6r^2-2\,a^4mr^3+a^4r^4\right)\cos(\theta)^4+3\left(a^4r^4-2\,a^2mr^5+a^2r^6\right)\cos(\theta)^2}$$

To form the Mars-Simon tensor, we need the fully covariant (type-(0,4) tensor) form of the Weyl tensor (i.e. $C_{\alpha\beta\mu\nu} = g_{\alpha\sigma}C^{\sigma}{}_{\beta\mu\nu}$); we get it by lowering the first index with the metric:

```
Cd = C.down(g)
print Cd
```

```
tensor field of type (0,4) on the 4-dimensional manifold 'M'
```

The (monoterm) symmetries of this tensor are those inherited from the Weyl tensor, i.e. the antisymmetry on the last two indices (position 2 and 3, the first index being at position 0):

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The (monoterm) symmetries of this tensor are those inherited from the Weyl tensor, i.e. the antisymmetry on the last two indices (position 2 and 3, the first index being at position 0):

```
Cd.symmetries()
```

```
no symmetry;  antisymmetry: (2, 3)
```

Actually, Cd is also antisymmetric with respect to the first two indices, as we can check:

```
Cd == Cd.antisymmetrize((0,1))
```

True

To take this symmetry into account explicitely, we set

```
Cd = Cd.antisymmetrize((0,1))
```

Hence we have now

```
Cd.symmetries()
```

```
no symmetry;  antisymmetries: [(0, 1), (2, 3)]
```

Mars-Simon tensor

The Mars-Simon tensor with respect to the Killing vector ξ is a rank-3 tensor introduced by Marc Mars in 1999 (Class. Quantum Grav. **16**, 2507). It has the remarkable property to vanish identically if, and only if, the spacetime $(\mathcal{M}, g)$ is locally isometric to a Kerr spacetime.

Let us evaluate the Mars-Simon tensor by following the formulas given in Mars' article. The starting point is the self-dual complex 2-form associated with the Killing 2-form F, i.e. the object $\mathcal{F} := F + i\,{}^*F$, where ${}^*F$ is the Hodge dual of F:

```
FF = F + I * F.hodge_star(g)
FF.set_name('FF', r'\mathcal{F}') ; print FF
```

```
2-form 'FF' on the 4-dimensional manifold 'M'
```

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```
FF.view()
```

$$\mathcal{F} = \left(-\frac{a^2 m \cos(\theta)^2 + 2i\, amr \cos(\theta) - mr^2}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4} \right) dt \wedge dr + \left(\frac{\left(i\, a^3 m \cos(\theta)^2 - 2\, a^2 mr \cos(\theta) - i\, amr^2\right) \sin(\theta)}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4} \right) dt \wedge d\theta + \left(\frac{-4i\, a^4 m^2 r^2 \cos(\theta) \sin(\theta)^4 + \left(a^3 mr^4 - 2\, am^2 \cdots\right.}{a^2 r^6 - 2\, m \cdots} \right.$$

Let us check that $\mathcal{F}$ is self-dual, i.e. that it obeys ${}^*\mathcal{F} = -i\mathcal{F}$:

```
FF.hodge_star(g) == - I * FF
```

True

Let us form the right self-dual of the Weyl tensor as follows

$$\mathcal{C}_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu} + \frac{i}{2}\, \epsilon^{\rho\sigma}{}_{\mu\nu}\, C_{\alpha\beta\rho\sigma}$$

where $\epsilon^{\rho\sigma}{}_{\mu\nu}$ is associated to the Levi-Civita tensor $\epsilon_{\rho\sigma\mu\nu}$ and is obtained by

```
eps = g.volume_form(2)  # 2 = the first 2 indices are contravariant
print eps
eps.symmetries()
```

```
tensor field of type (2,2) on the 4-dimensional manifold 'M'
no symmetry;  antisymmetries: [(0, 1), (2, 3)]
```

The right self-dual Weyl tensor is then:

```
CC = Cd + I/2*( eps['^rs_..']*Cd['_..rs'] )
CC.set_name('CC', r'\mathcal{C}') ; print CC
```

```
tensor field 'CC' of type (0,4) on the 4-dimensional manifold 'M'
```

```
CC.symmetries()
```

```
no symmetry;  antisymmetries: [(0, 1), (2, 3)]
```

```
CC[0,1,2,3]
```

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```
CC[0,1,2,3]
```

$$\frac{\left(a^5 m \cos(\theta)^5 + 3i\, a^4 mr \cos(\theta)^4 + 3i\, a^2 mr^3 + 2i\, mr^5 - \left(3\, a^5 m + 5\, a^3 mr^2\right) \cos(\theta)^3 + \left(-9i\, a^4 mr - 7i\, a^2 mr^3\right) \cos(\theta)^2 + 3\left(3\, a^3 mr^2 + 2\, amr^4\right) \cos(\theta)\right) \sin(\theta)}{a^6 \cos(\theta)^6 + 3\, a^4 r^2 \cos(\theta)^4 + 3\, a^2 r^4 \cos(\theta)^2 + r^6}$$

The Ernst 1-form $\sigma_\alpha = 2\mathcal{F}_{\mu\alpha}\,\xi^\mu$ (0 = contraction on the first index of $\mathcal{F}$):

```
sigma = 2*FF.contract(0, xi)
```

Instead of invoking the function `contract()`, we could have used the index notation to denote the contraction:

```
sigma == 2*( FF['_ma']*xi['^m'] )
```

True

```
sigma.set_name('sigma', r'\sigma') ; print sigma
sigma.view()
```

```
1-form 'sigma' on the 4-dimensional manifold 'M'
```

$$\sigma = \left(-\frac{2\, a^2 m \cos(\theta)^2 + 4i\, amr \cos(\theta) - 2\, mr^2}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}r + \left(\frac{\left(2i\, a^3 m \cos(\theta)^2 - 4\, a^2 mr \cos(\theta) - 2i\, amr^2\right) \sin(\theta)}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}\theta$$

The symmetric bilinear form $\gamma = \lambda g + \underline{\xi} \otimes \underline{\xi}$:

```
gamma = lamb*g + xi_form * xi_form
gamma.set_name('gamma', r'\gamma') ; print gamma
gamma.view()
```

```
field of symmetric bilinear forms 'gamma' on the 4-dimensional manifold
'M'
```

$$\gamma = \left(\frac{a^2 \cos(\theta)^2 - 2\, mr + r^2}{a^2 - 2\, mr + r^2}\right) \mathrm{d}r \otimes \mathrm{d}r + \left(a^2 \cos(\theta)^2 - 2\, mr + r^2\right) \mathrm{d}\theta \otimes \mathrm{d}\theta + \left(\frac{2\, a^2 mr \sin(\theta)^4 - \left(2\, a^2 mr - a^2 r^2 + 2\, mr^3 - r^4 - \left(a^4 + a^2 r^2\right) \cos(\theta)^2\right) \sin(\theta)^2}{a^2 \cos(\theta)^2 + r^2}\right) \mathrm{d}\phi$$

SM_Mars-Simon -- Sage - Mozilla Firefox

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Final computation leading to the Mars-Simon tensor:

First, we evaluate

$$S^{(1)}_{\alpha\beta\gamma} = 4\,\mathcal{C}_{\mu\alpha\nu\beta}\,\xi^\mu\xi^\nu\,\sigma_\gamma$$

```
S1 = 4*( CC.contract(0,xi).contract(1,xi) ) * sigma
print S1
```

```
tensor field of type (0,3) on the 4-dimensional manifold 'M'
```

Then we form the tensor

$$S^{(2)}_{\alpha\beta\gamma} = \gamma_{\alpha\beta}\,\mathcal{C}_{\gamma\rho\mu\nu}\,\xi^\rho\,\mathcal{F}^{\mu\nu}$$

by first computing $\mathcal{C}_{\gamma\rho\mu\nu}\,\xi^\rho$:

```
xiCC = CC['_.r..']*xi['^r']
print xiCC
```

```
tensor field of type (0,3) on the 4-dimensional manifold 'M'
```

and evaluating $\mathcal{F}^{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}\mathcal{F}_{\mu\nu}$:

```
FFuu = FF.up(g)
```

We use the index notation to perform the double contraction $\mathcal{C}_{\gamma\rho\mu\nu}\mathcal{F}^{\mu\nu}$:

```
S2 = gamma * ( xiCC['_.mn']*FFuu['^mn'] )
print S2
S2.symmetries()
```

```
tensor field of type (0,3) on the 4-dimensional manifold 'M'
symmetry: (0, 1);  no antisymmetry
```

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The Mars-Simon tensor with respect to ξ is obtained by antisymmetrizing $S^{(1)}$ and $S^{(2)}$ on their last two indices and adding them:

$$S_{\alpha\beta\gamma} = S^{(1)}_{\alpha[\beta\gamma]} + S^{(2)}_{\alpha[\beta\gamma]}$$

We use the index notation for the antisymmetrization:

```
S1A = S1['_a[bc]']
S2A = S2['_a[bc]']
```

An equivalent writing would have been (the last two indices being in position 1 and 2):

```
# S1A = S1.antisymmetrize((1,2))
# S2A = S2.antisymmetrize((1,2))
```

The Mars-Simon tensor is

```
S = S1A + S2A
S.set_name('S') ; print S
S.symmetries()
```

```
tensor field 'S' of type (0,3) on the 4-dimensional manifold 'M'
no symmetry;  antisymmetry: (1, 2)
```

```
S.view()
```

$S = 0$

We thus recover the fact that the Mars-Simon tensor vanishes identically in Kerr spacetime.

To check that the above computation was not trival, here is the component 112=$rr\theta$ for each of the two parts of the Mars-Simon tensor:

```
S1A[1,1,2]
```

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The Mars-Simon tensor is

```
S = S1A + S2A
S.set_name('S') ; print S
S.symmetries()
```

```
tensor field 'S' of type (0,3) on the 4-dimensional manifold 'M'
no symmetry;  antisymmetry: (1, 2)
```

```
S.view()
```

$$S = 0$$

We thus recover the fact that the Mars-Simon tensor vanishes identically in Kerr spacetime.

To check that the above computation was not trival, here is the component $112 = rr\theta$ for each of the two parts of the Mars-Simon tensor:

```
S1A[1,1,2]
```

$$-\frac{\left(4\,a^8m^2\cos(\theta)^7+20i\,a^7m^2r\cos(\theta)^6-8i\,am^3r^6+4i\,am^2r^7-4\left(2\,a^6m^3r+9\,a^6m^2r^2\right)\cos(\theta)^5+\left(-40i\,a^5m^3r^2-20i\,a^5m^2r^3\right)\cos(\theta)^4+20\left(4\,a^4m^3r^3-a^4m^2r^4\right)\cos(\theta)^3+\left(80i\,a^3m^3r^4-\right.}{a^2r^{10}-2\,mr^{11}+r^{12}+\left(a^{12}-2\,a^{10}mr+a^{10}r^2\right)\cos(\theta)^{10}+5\left(a^{10}r^2-2\,a^8mr^3+a^8r^4\right)\cos(\theta)^8+10\left(a^8r^4-2\,a^6mr^5+a^6r^6\right)\cos(\theta)^6+10\left(a^6r^6-2\,a^4mr^7+a^4r^8\right)\cos\left(\right.}$$

```
S2A[1,1,2]
```

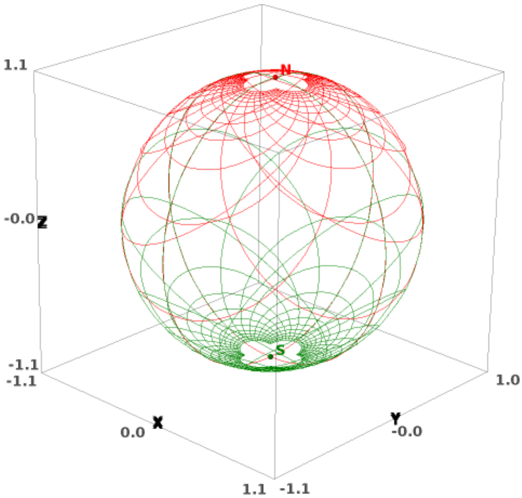
$$\frac{\left(4\,a^8m^2\cos(\theta)^7+20i\,a^7m^2r\cos(\theta)^6-8i\,am^3r^6+4i\,am^2r^7-4\left(2\,a^6m^3r+9\,a^6m^2r^2\right)\cos(\theta)^5+\left(-40i\,a^5m^3r^2-20i\,a^5m^2r^3\right)\cos(\theta)^4+20\left(4\,a^4m^3r^3-a^4m^2r^4\right)\cos(\theta)^3+\left(80i\,a^3m^3r^4-36\right.}{a^2r^{10}-2\,mr^{11}+r^{12}+\left(a^{12}-2\,a^{10}mr+a^{10}r^2\right)\cos(\theta)^{10}+5\left(a^{10}r^2-2\,a^8mr^3+a^8r^4\right)\cos(\theta)^8+10\left(a^8r^4-2\,a^6mr^5+a^6r^6\right)\cos(\theta)^6+10\left(a^6r^6-2\,a^4mr^7+a^4r^8\right)\cos(\theta)^4}$$

```
S1A[1,1,2] + S2A[1,1,2]
```

$$0$$

evaluate

Another feature of SageManifolds: display of chart grids



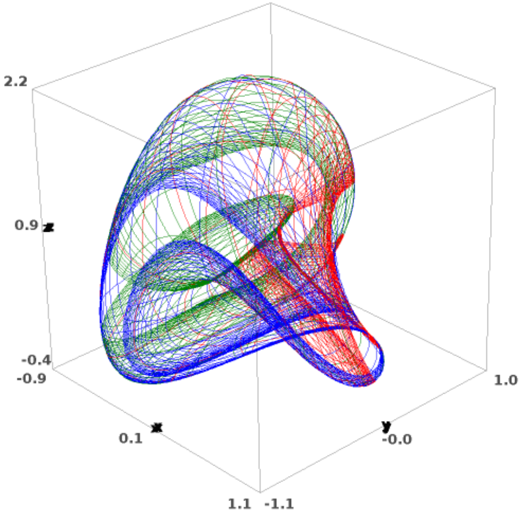
Function `Chart.plot()`

Stereographic coordinates on the 2-sphere

Two charts:

- X_1: $\mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$
- X_2: $\mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^2$

Another feature of SageManifolds: display of chart grids



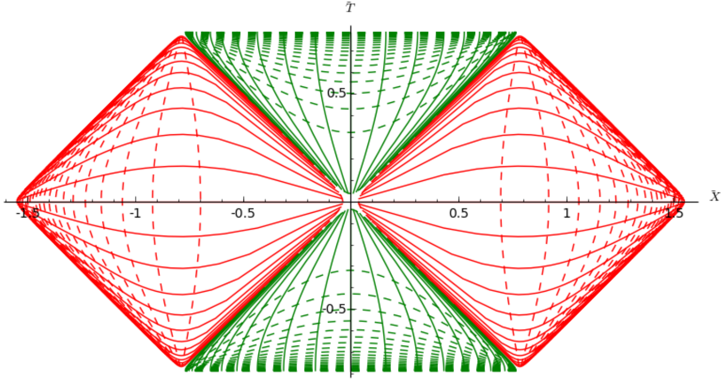
Three charts X_1, X_2, X_3 covering the real projective plane $\mathbb{RP}^2$, displayed via the Apéry immersion of $\mathbb{RP}^2$ into $\mathbb{R}^3$ *(Boy surface)*

Identifying $\mathbb{RP}^2$ with the set of lines Δ through the origin of $\mathbb{R}^3$, we have

- X_1: $\Delta \mapsto (x_1, y_1)$ such that $\Delta \cap \Pi_{z=1} = (x_1, y_1, 1)$
- X_2: $\Delta \mapsto (x_2, y_2)$ such that $\Delta \cap \Pi_{x=1} = (1, x_2, y_2)$
- X_3: $\Delta \mapsto (x_3, y_3)$ such that $\Delta \cap \Pi_{y=1} = (y_3, 1, x_3)$

Another feature of SageManifolds: display of chart grids

Carter-Penrose diagram of Schwarzschild spacetime



Plot of the standard Schwarzschild-Droste coordinates (t, r) in terms of the conformal Kruskal-Szekeres coordinates $(\tilde{T}, \tilde{X})$.

Outline

Conclusion and perspectives

- **SageManifolds** is a work in progress
  $\sim$ 34,000 lines of Python code up to now (including comments and doctests)
- A preliminary version (v0.5) is freely available (GPL) at
  http://sagemanifolds.obspm.fr/
  and the development version (to become v0.6 soon) is available from the Git repository https://github.com/sagemanifolds/sage
- *Already present:*
  - maps between manifolds, pullback operator
  - submanifolds, pushforward operator
  - standard tensor calculus (tensor product, contraction, symmetrization, etc.), even on non-parallelizable manifolds
  - all monoterm tensor symmetries
  - exterior calculus, Hodge duality
  - Lie derivatives
  - affine connections, curvature, torsion
  - pseudo-Riemannian metrics, Weyl tensor

Conclusion and perspectives

- *Not implemented yet (but should be soon):*
  - extrinsic geometry of pseudo-Riemannian submanifolds
  - computation of geodesics (numerical integration via Sage/GSL or `Gyoto`)
  - integrals on submanifolds
- *To do:*
  - add more graphical outputs
  - add more functionalities: symplectic forms, fibre bundles, spinors, variational calculus, etc.
  - connection with `Lorene`, `CoCoNuT`, ...

Want to join the project or simply to stay tuned?

visit `http://sagemanifolds.obspm.fr/`
(download page, documentation, example worksheets, mailing list)