Black holes: myths and facts

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Outline

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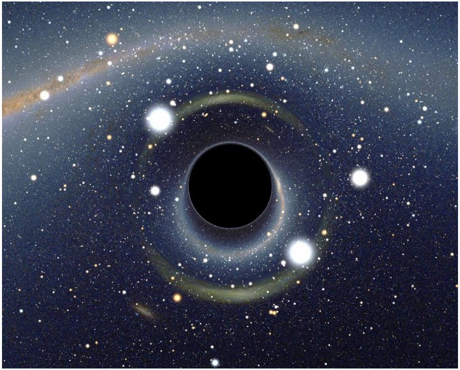
What is a black hole?

A layperson (loose) definition

A **black hole** is a localized region of spacetime from which no particle, be it massive or massless (photon), can escape.



[A. Riazuelo, IJMPD **28**, 1950042 (2019)]

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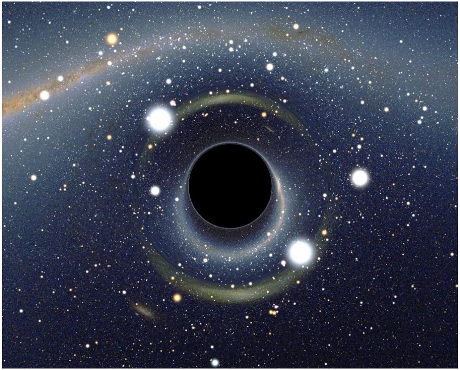

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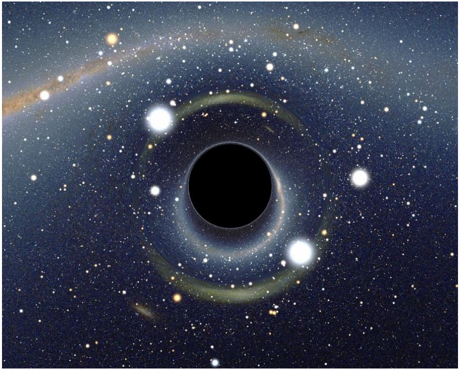

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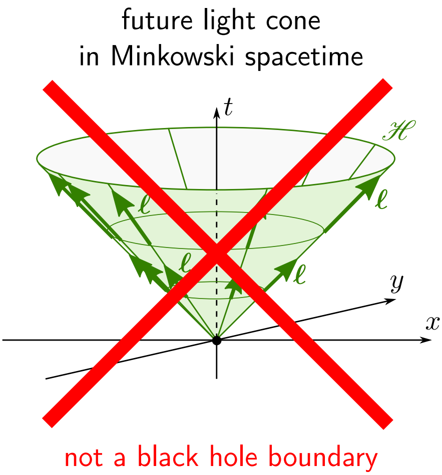
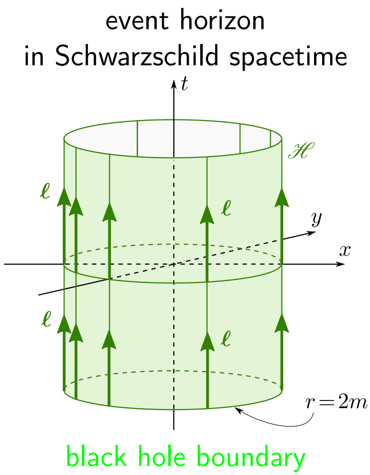
Two aspects:

- localization
- impassable boundary (to the exterior), called the **event horizon**

Importance of the localized aspect in the BH definition





One cannot escape from the interior of a future light cone, but one can travel arbitrary far from the central region $\Longrightarrow$ this is not a black hole.

Towards a precise definition

1. The framework

Relativistic spacetime

spacetime = $(\mathscr{M}, \boldsymbol{g})$

- $\mathscr{M}$: 4-dimensional smooth manifold
- $\boldsymbol{g}$: Lorentzian metric on $\mathscr{M}$ $\Longrightarrow$ causal structure
- $(\mathscr{M}, \boldsymbol{g})$ is time-orientable $\Longrightarrow$ future and past directions

NB: Einstein's equation $\boldsymbol{R} - \frac{1}{2}R\boldsymbol{g} = 8\pi\boldsymbol{T}$ *not* assumed at this stage

$\Longrightarrow$ BH definition will be valid in any metric theory of gravity, not necessarily general relativity.

Towards a precise definition

2. Coordinate-free concept of "infinitely far region"

Penrose's idea of "conformal compactification"

Assume that there exists a positive scalar field Ω on $\mathscr{M}$ such that the conformal metric

$$\tilde{\boldsymbol{g}} := \Omega^2 \boldsymbol{g}$$

admits a regular limit when $\Omega \to 0$. Then the points "located at $\Omega = 0$" are "infinitely far": the spacetime metric $\boldsymbol{g} = \Omega^{-2}\tilde{\boldsymbol{g}}$ yields an infinite distance between these points and those located inside $\mathscr{M}$.

Towards a precise definition

3. Conformal completion and null infinity

Definition

A spacetime $(\mathscr{M}, \boldsymbol{g})$ admits a **conformal completion at null infinity** iff there exists a Lorentzian manifold with boundary $(\tilde{\mathscr{M}}, \tilde{\boldsymbol{g}})$ equipped with a smooth non-negative scalar field $\Omega : \tilde{\mathscr{M}} \to \mathbb{R}^+$ such that

- $\tilde{\mathscr{M}} = \mathscr{M} \cup \mathscr{I}$, with $\mathscr{I} := \partial\tilde{\mathscr{M}}$ *($\mathscr{I}$ is the boundary of $\tilde{\mathscr{M}}$)*
- on $\mathscr{M}$, $\tilde{\boldsymbol{g}} = \Omega^2 \boldsymbol{g}$
- on $\mathscr{I}$, $\Omega = 0$ *(makes $\mathscr{I}$ infinitely remote)*
- on $\mathscr{I}$, $\mathbf{d}\Omega \neq 0$ *(makes $\mathscr{I}$ a regular hypersurface of $\tilde{\mathscr{M}}$)*
- $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$, with $\mathscr{I}^+$ (resp. $\mathscr{I}^-$) being never intersected by any past-directed (resp. future-directed) causal curve originating in $\mathscr{M}$

$\mathscr{I}^+$ is called the **future null infinity** and $\mathscr{I}^-$ the **past null infinity** of $(\mathscr{M}, \boldsymbol{g})$.

Remark: $\mathscr{I}^+$ and $\mathscr{I}^-$ are part of $\tilde{\mathscr{M}}$ but not of $\mathscr{M}$.

Example: conformal completion of Minkowski spacetime

Spacetime metric $\boldsymbol{g}$ in spherical coordinates (t, r, θ, φ):

$$\boxed{\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \, \mathrm{d}\varphi^2}$$

Move to coord. $(\tau, \chi, \theta, \varphi)$ def. by $\left\{ \begin{array}{l} \tau = \arctan(t+r) + \arctan(t-r) \\ \chi = \arctan(t+r) - \arctan(t-r) \end{array} \right.$

$$\Longrightarrow \boxed{\mathrm{d}s^2 = (\cos\tau + \cos\chi)^{-2} \left[-\mathrm{d}\tau^2 + \mathrm{d}\chi^2 + \sin^2\chi \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2 \right) \right]}$$

with $0 \leq \chi < \pi$ and $\chi - \pi < \tau < \pi - \chi$ $\leftarrow$ finite range of coord. (τ, χ)

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with $0 \leq \chi < \pi$ and $\chi - \pi < \tau < \pi - \chi$ ← finite range of coord. (τ, χ)

Hence $\boxed{\boldsymbol{g} = \Omega^{-2}\tilde{\boldsymbol{g}}}$ with

- $\Omega := \cos\tau + \cos\chi = \dfrac{2}{\sqrt{(t-r)^2+1}\sqrt{(t+r)^2+1}}$
- $\tilde{\boldsymbol{g}}$ is the metric defined by

$$\mathrm{d}\tilde{s}^2 = -\mathrm{d}\tau^2 + \underbrace{\mathrm{d}\chi^2 + \sin^2\chi\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)}_{\text{standard (round) metric on } \mathbb{S}^3}$$

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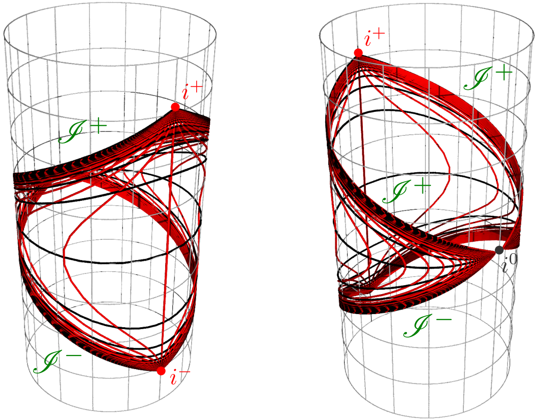


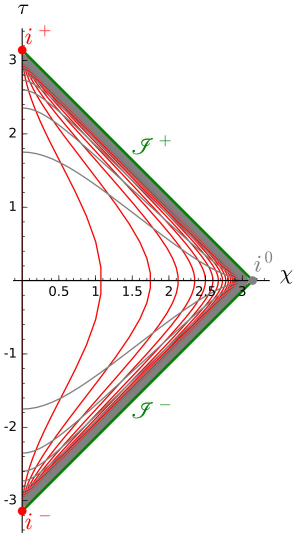
red: $r = \text{const}$
black: $t = \text{const}$
grey: $\mathbb{R} \times \mathbb{S}^3$ *(Einstein cylinder)*

- on $\mathscr{M}$:
  $0 \leq \chi < \pi$
  $\chi - \pi < \tau < \pi - \chi$
- $\mathscr{I}^+$ = hypersurface $\{\tau = \pi - \chi,\ 0 < \tau < \pi\}$
- $\mathscr{I}^-$ = hypersurface $\{\tau = \chi - \pi,\ -\pi < \tau < 0\}$

For an interactive 3D view, cf. https://nbviewer.org/github/egourgoulhon/BHLectures/blob/master/sage/conformal_Minkowski.ipynb

Example: conformal completion of Minkowski spacetime

Conformal diagram



View in the (τ, χ) coordinate plane

$0 \leq \chi < \pi$
$\chi - \pi < \tau < \pi - \chi$

red: $r = \mathrm{const}$
grey: $t = \mathrm{const}$

Example: conformal completion of Minkowski spacetime

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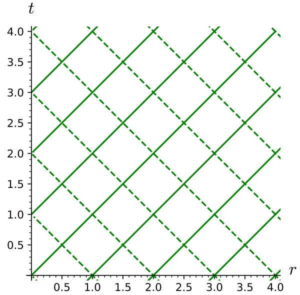
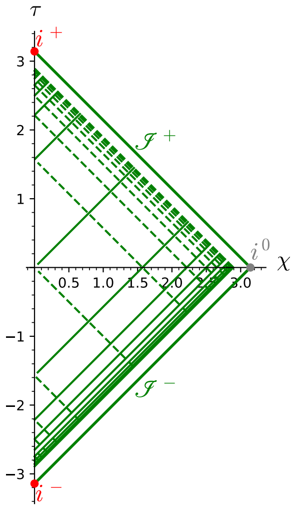


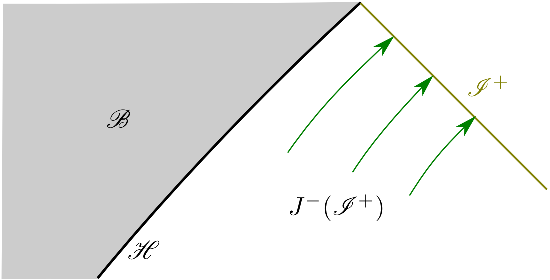


Radial null geodesics:
solid:
$u := t - r = \text{const}$
dashed:
$v := t + r = \text{const}$

Radial null geodesics appear as straight lines with $\pm 45°$ slope

$\Longrightarrow$ conformal diag. also called Penrose diagram or Carter-Penrose diagram

General definition of a black hole, at last!



Causal past $J^-(\mathscr{I}^+)$: set of points of $\tilde{\mathscr{M}}$ that can be reached from a point of $\mathscr{I}^+$ by a past-directed causal (i.e. null or timelike) curve.

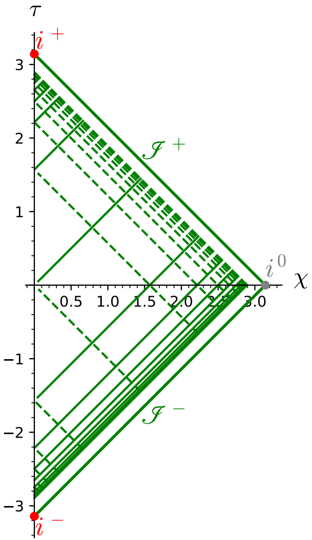
Definition

Let $(\mathscr{M}, \boldsymbol{g})$ be a spacetime with a conformal completion at null infinity such that $\mathscr{I}^+$ is complete; the **black hole region**, or simply **black hole**, is the set of points of $\mathscr{M}$ that are not in the causal past of the future null infinity:

$$\mathscr{B} := \mathscr{M} \setminus (J^-(\mathscr{I}^+) \cap \mathscr{M})$$

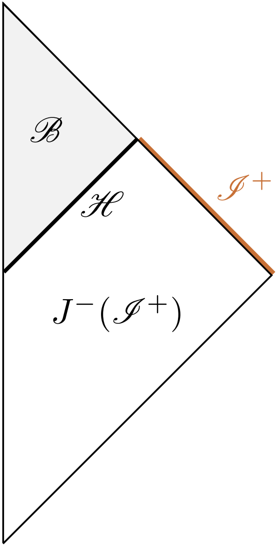
The boundary of $\mathscr{B}$ is called the **(future) event horizon**: $\mathscr{H} = \partial \mathscr{B}$

No black hole in Minkowski spacetime



$$J^-(\mathscr{I}^+) \cap \mathscr{M} = \mathscr{M} \Longrightarrow \mathscr{B} = \varnothing$$

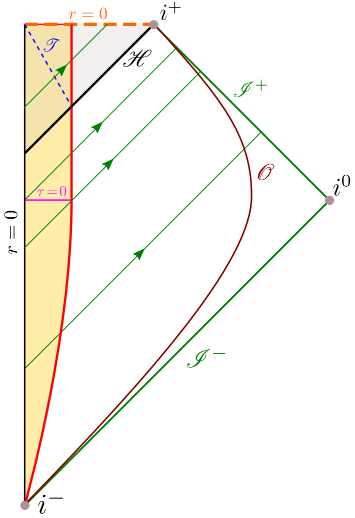
Completeness of $\mathscr{I}^+$ to avoid spurious BH



If $\mathscr{I}^+$ is a null hypersurface, $\mathscr{I}^+$ complete $\iff$ $\mathscr{I}^+$ generated by complete null geodesics.

$\leftarrow$ Spurious black hole region $\mathscr{B}$ in Minkowski spacetime resulting from a conformal completion with a non-complete $\mathscr{I}^+$.

Example: black hole formed by gravitational collapse



Carter-Penrose diagram of gravitational collapse of a spherically symmetric star

Outside the star, $\boldsymbol{g}$ is Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

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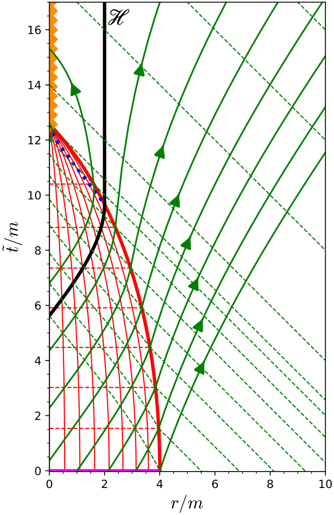

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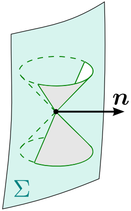
Outline

The event horizon is a null hypersurface

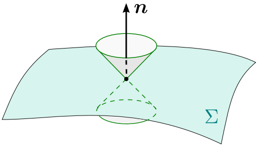
Wherever it is smooth, the black hole event horizon $\mathscr{H}$ is a null hypersurface.

Recall: **hypersurface** = 3-dimensional submanifold of $\mathscr{M}$

Locally, a hypersurface Σ (normal $\boldsymbol{n}$) can be of one of 3 types:



Σ timelike
$\boldsymbol{g}|_\Sigma$ Lorentzian
$\boldsymbol{n}$ spacelike



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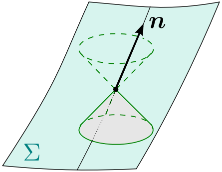

Σ null
$\boldsymbol{g}|_\Sigma$ degenerate
$\boldsymbol{n}$ null (and tangent to Σ)

Spacelike and null hypersurfaces are 1-way membranes.

The no-hair theorem: all black holes are Kerr

Dorochkevitch, Novikov & Zeldovitch (1965), Israel (1967), Carter (1971), Hawking (1972)

Within 4-dimensional general relativity, a *stationary* black hole in an otherwise empty universe is necessarily a Kerr-Newmann black hole, which is an electro-vacuum solution of Einstein equation parametrized by

- the total mass M
- the total specific angular momentum $a = J/M$
- the total electric charge Q

Only 3 numbers $(M, a, Q) \Longrightarrow$ *"a black hole has no hair"* (J. A. Wheeler)

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Astrophysical black holes have to be electrically neutral:

- $Q = 0$: Kerr solution (1963)
- $a = 0$ and $Q = 0$: Schwarzschild solution (1916)

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Other special case:

- $a = 0$: Reissner-Nordström solution (1916, 1918)

The no-hair theorem: a precise mathematical statement

Any spacetime $(\mathscr{M}, \boldsymbol{g})$ that

- is 4-dimensional
- is asymptotically flat
- is pseudo-stationary
- is a solution of the vacuum Einstein's equation: $\boldsymbol{R} = 0$
- contains a black hole with a connected regular horizon
- has no closed timelike curve in the domain of outer communications (DOC) (= black hole exterior)
- is analytic

has a DOC that is isometric to the DOC of Kerr spacetime.

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Possible improvements: remove the hypotheses of analyticity and non-existence of closed timelike curves (analyticity removed but only for slow rotation [Alexakis, Ionescu & Klainerman, Duke Math. J. **163**, 2603 (2014)])

The Kerr solution (1963)

Roy Kerr (1963)

Expression in Boyer-Lindquist coordinates (t, r, θ, φ):

$$\begin{aligned} ds^2 = & -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}\, dt\, d\varphi + \frac{\rho^2}{\Delta}\, dr^2 \\ & + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2\theta}{\rho^2}\right) \sin^2\theta\, d\varphi^2 \end{aligned}$$

where $\rho^2 := r^2 + a^2 \cos^2\theta$, $\Delta := r^2 - 2Mr + a^2$ and $\boxed{r \in (-\infty, \infty)}$

→ spacetime manifold: $\mathscr{M} = \mathbb{R}^2 \times \mathbb{S}^2 \setminus \{r = 0 \;\&\; \theta = \pi/2\}$

→ describes a rotating black hole

→ 2 parameters: M: gravitational mass; $a := J/M$ reduced angular momentum

Physical meaning of the parameters M and J

- mass M: *not* a measure of the "amount of matter" inside the black hole, but rather a *characteristic of the external gravitational field* $\rightarrow$ measurable from the orbital period of a test particle in far circular orbit around the black hole *(Kepler's third law)*

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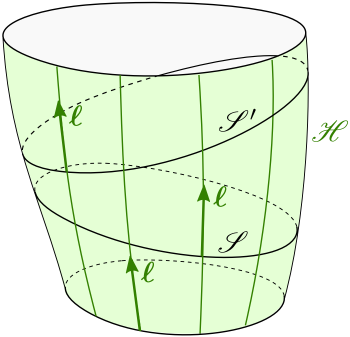
Remark: the radius of a black hole is not a well defined concept: it *does not* correspond to a distance between some black hole "centre" and the event horizon. A well defined quantity is the area of the event horizon, A. The areal radius R can be defined from it by setting $A =: 4\pi R^2$

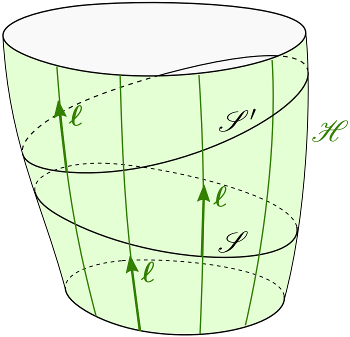
$\Longrightarrow$ for a Schwarzschild black hole: $R := \sqrt{\frac{A}{4\pi}} = \frac{2GM}{c^2} \simeq 3 \left(\frac{M}{M_\odot}\right)$ km

Area of a black hole



Cross-section of a BH event horizon $\mathscr{H}$: *spacelike* 2-surface $\mathscr{S}$ intersecting each null geodesic generator of $\mathscr{H}$ exactly once.

Area of a black hole



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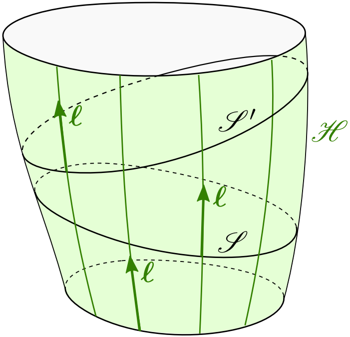

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The **area of** $\mathscr{S}$ is $\boxed{A = \int_{\mathscr{S}} \sqrt{q}\, \mathrm{d}y^1 \mathrm{d}y^2}$

where $y^a = (y^1, y^2)$ are coordinates on $\mathscr{S}$ and $q := \det(q_{ab})$

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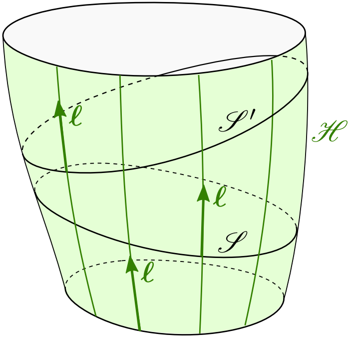

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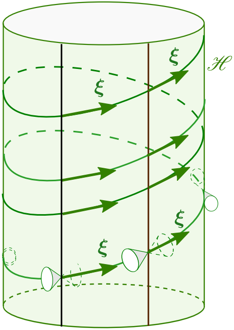
In general A depends on the choice of $\mathscr{S}$.
For a BH *in equilibrium*, A does not depend on $\mathscr{S}$ $\Longrightarrow$ **black hole area**

Kerr: $A = 8\pi M(M + \sqrt{M^2 - a^2})$; Schwarzschild: $A = 16\pi M^2$

Surface gravity of a black hole in equilibrium



Equilibrium in GR: inv. under translations $t \mapsto t + c$
$\Longrightarrow$ symmetry generator $\partial/\partial t$ (**Killing vector**)

Hawking rigidity theorem: $\exists$ another Killing vector $\partial/\partial\varphi$ generating rotations around some axis and a constant Ω_H such that the Killing vector

$$\boldsymbol{\xi} = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \varphi}$$

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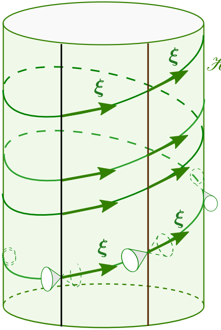

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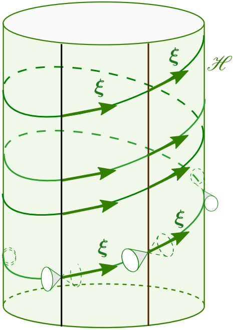
$\boldsymbol{\xi}$ is pregeodesic on $\mathscr{H}$: $\boldsymbol{\nabla}_{\boldsymbol{\xi}} \boldsymbol{\xi} \overset{\mathscr{H}}{=} \kappa\, \boldsymbol{\xi}$, κ called **surface gravity**

If a is the acceleration felt by the observer $\mathscr{O}$ of 4-veloticity $\boldsymbol{u} = \boldsymbol{\xi}/V$ just outside $\mathscr{H}$ ($V := \sqrt{-\boldsymbol{g}(\boldsymbol{\xi},\boldsymbol{\xi})}$), then $\lim\limits_{\mathscr{O}\to\mathscr{H}} a = +\infty$ but $\lim\limits_{\mathscr{O}\to\mathscr{H}} V a = \kappa$

Surface gravity of a black hole in equilibrium



Equilibrium in GR: inv. under translations $t \mapsto t + c$
$\Longrightarrow$ symmetry generator $\partial/\partial t$ (**Killing vector**)

Hawking rigidity theorem: $\exists$ another Killing vector $\partial/\partial\varphi$ generating rotations around some axis and a constant Ω_H such that the Killing vector

$$\boldsymbol{\xi} = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \varphi}$$

is tangent to the null geodesics generating $\mathscr{H}$

$\boldsymbol{\xi}$ is pregeodesic on $\mathscr{H}$: $\boxed{\boldsymbol{\nabla}_{\boldsymbol{\xi}}\boldsymbol{\xi} \overset{\mathscr{H}}{=} \kappa\, \boldsymbol{\xi}}$, κ called **surface gravity**

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Kerr: $\kappa = \dfrac{\sqrt{M^2 - a^2}}{2M\left(M + \sqrt{M^2 - a^2}\right)}$; Schwarzschild: $\kappa = \dfrac{1}{4M}$

The four laws of black hole (thermo)dynamics

Bardeen, Carter & Hawking (1973), Israel (1986)

EE = Einstein's equation, NEC = null EC (energy condition), WEC = weak EC, NDEC = null dominant EC ($-T^{\alpha}{}_{\mu}\ell^{\mu}$ future causal for any future null vector $\boldsymbol{\ell}$)

Zeroth law: assuming EE + NDEC, the surface gravity κ of a black hole in equilibrium is uniform over $\mathscr{H}$

First law: assuming EE + NDEC, two nearby black hole equilibrium configurations are related by $\boxed{\mathrm{d}M = \frac{\kappa}{8\pi}\,\mathrm{d}A + \Omega_H\,\mathrm{d}J}$

Second law: assuming EE + NEC, the area A of cross-sections of a black hole event horizon can only increase towards the future: $\boxed{\frac{\mathrm{d}A}{\mathrm{d}t} \geq 0}$

Third law: assuming EE + WEC, a nonzero surface gravity κ of a black hole in equilibrium cannot be reduced to zero by accretion of matter.

Outline

Myth: black holes are extremely dense objects

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— Well,

- for the Milky Way central black hole (Sgr A*): $\bar{\rho} \sim 10^6 \ \mathrm{kg\,m^{-3}} = 2\ 10^{-4} \times$ the density of a white dwarf
- for the central black hole of the galaxy M 87 (M 87*): $\bar{\rho} \sim 2 \ \mathrm{kg\,m^{-3}} = 1/500 \times$ the density of water!

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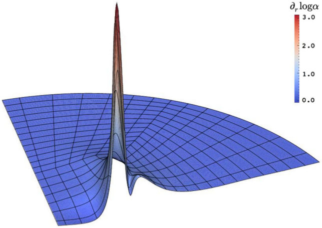
Actually black holes are **compact objects**: they have a large compactness M/R, not necessarily a large mean density M/R^3.

Fact: formation of a black hole in a totally empty universe



A black hole can be formed from the (nonlinear) evolution of gravitational waves of large amplitude

No matter involved in the process!

[Abrahams & Evans, PRL **70**, 2980 (1993)]

[Hilditch, Weyhausen & Brügmann, PRD **96**, 104051 (2017)]

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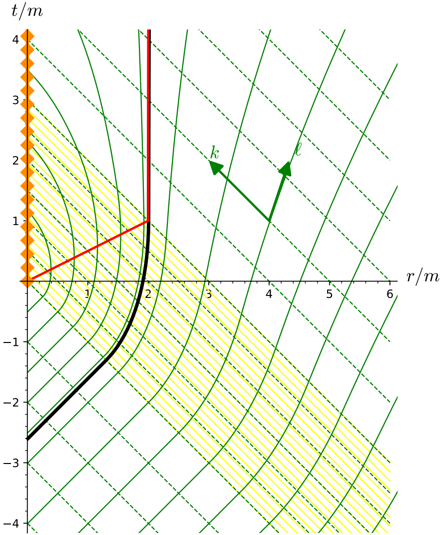
Moreover, for some non-stationary black holes, the spacetime curvature at the horizon can even be zero...

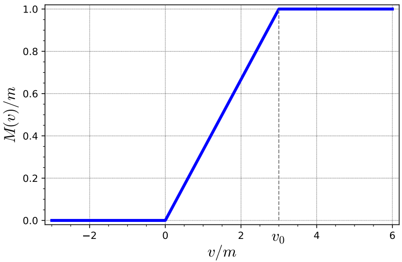
Fact: a black hole event horizon can appear in a flat region



Vaidya solution

Infalling spherical shell of electromagnetic radiation ($v = t + r$)

$$ds^2 = -\left(1 - \frac{2M(v)}{r}\right) dv^2 + 2\, dv\, dr + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$



https://nbviewer.org/github/egourgoulhon/BHLectures/blob/master/sage/Vaidya.ipynb

Fact: event horizon is not locally detectable

The Vaidya example leads us to the following conclusion:

No local physical experiment whatsoever can detect the crossing of a black hole event horizon.

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— But the Schwarzschild and Kerr black holes do harbor a curvature singularity
— True

— Moreover, doesn't Penrose's singularity theorem state that the occurence of a singularity is inevitable in any gravitational collapse that has reached a certain stage?
— True, *provided that* the hypotheses of the theorem are fulfilled...

Penrose's singularity theorem

The Nobel Prize in Physics 2020 has been awarded to Roger Penrose for *"the discovery that black hole formation is a robust prediction of the general theory of relativity"*.

Theorem (Penrose, 1965)

Let $(\mathscr{M}, \boldsymbol{g})$ be a spacetime such that

1. the *null convergence condition* holds: $\boldsymbol{R}(\boldsymbol{\ell}, \boldsymbol{\ell}) \geq 0$ for any null vector $\boldsymbol{\ell}$;
2. there exists a non-compact Cauchy hypersurface;
3. there exists a trapped surface.

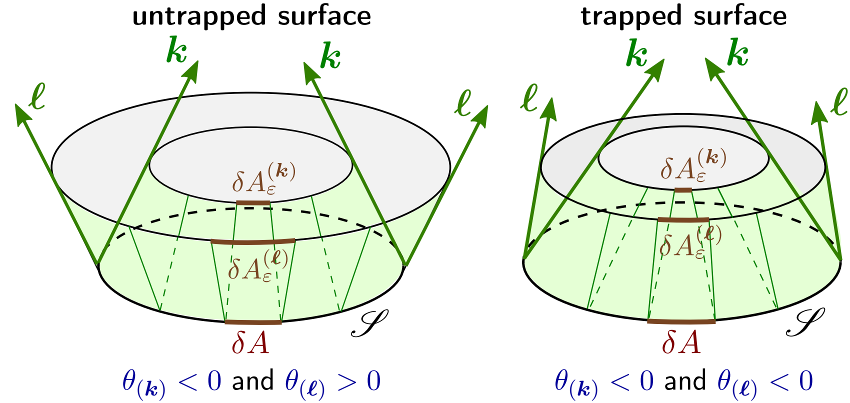
Then, there exists a future incomplete null geodesic.

Remark: if Einstein's equation holds, (1) is implied by the *null energy condition*: $\boldsymbol{T}(\boldsymbol{\ell}, \boldsymbol{\ell}) \geq 0$ for any null vector $\boldsymbol{\ell}$, which is itself implied by the *weak energy condition*: $\boldsymbol{T}(\boldsymbol{u}, \boldsymbol{u}) \geq 0$ for any timelike vector $\boldsymbol{u}$

Trapped surfaces

$\mathscr{S}$: closed spacelike 2-dimensional surface

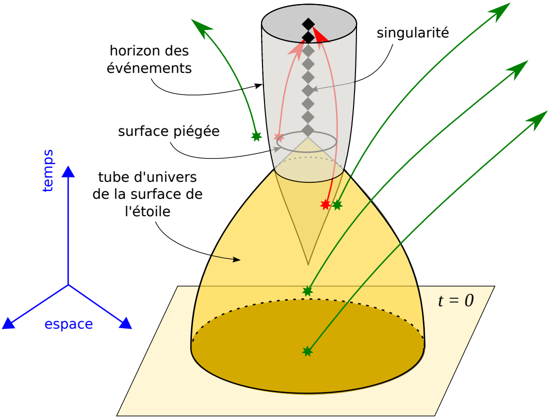
$\boldsymbol{k}, \boldsymbol{\ell}$: the two null directions normal to $\mathscr{S}$ (inward and outward)



No trapped surface in Minkowski spacetime

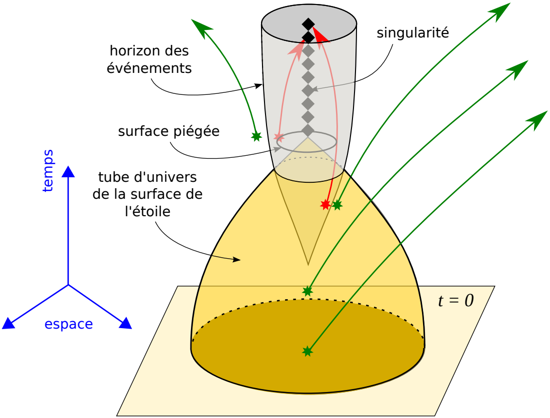
$\Longrightarrow$ *trapped surface* = **local** concept characterizing strong gravity

The singularity in Penrose's theorem: incomplete geodesic



Penrose's theorem does not say that the singularity is a curvature singularity, but only an incomplete null geodesic, i.e. a null geodesic that abruptly stops, namely ends at a finite value of its affine parameter.

The singularity in Penrose's theorem: incomplete geodesic



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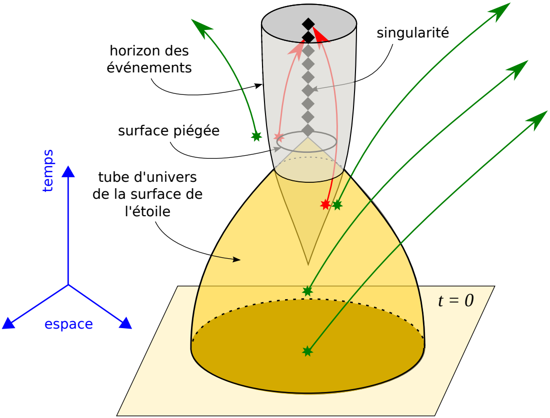

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Now, a good reason for a geodesic to stop is to hit a curvature singularity.

Remark: Penrose's theorem does not stipulate that a black hole must form; the singularity could be naked.

Hawking & Penrose's variant of the singularity theorem

Basically, this variant gets rid of the requirement of a Cauchy hypersurface (*global hyperbolicity* hypothesis) at the price of a stronger energy condition:

Theorem (Hawking & Penrose, 1970)

Let $(\mathscr{M}, \boldsymbol{g})$ be a spacetime such that

1. the *timelike convergence condition* holds: $\boldsymbol{R}(\boldsymbol{u}, \boldsymbol{u}) \geq 0$ for any timelike vector $\boldsymbol{u}$;
2. there exists no closed timelike curve;
3. the *generic condition* holds: every causal geodesic (tangent vector $\boldsymbol{u}$) contains at least one point at which $u_{[\alpha} R_{\beta]\mu\nu[\gamma} u_{\delta]} u^\mu u^\nu \neq 0$
4. there exists a trapped surface.

Then, there exists an incomplete causal geodesic.

Remark: if Einstein's equation holds, (1) is implied by the *strong energy condition*: $\boldsymbol{T}(\boldsymbol{u}, \boldsymbol{u}) + T/2 - \Lambda/(8\pi) \geq 0$ for any 4-velocity vector $\boldsymbol{u}$

Fact: singularity-free black hole solutions exist

Bardeen regular black hole (1968)

$$ds^2 = -\left(1 - \frac{2Mr^2}{(r^2+\mathfrak{g}^2)^{3/2}}\right) dt^2 + \left(1 - \frac{2Mr^2}{(r^2+\mathfrak{g}^2)^{3/2}}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

Solution of Einstein's equation $\boldsymbol{R} - \frac{1}{2}R\boldsymbol{g} = 8\pi\boldsymbol{T}_{\rm em}$, with $\boldsymbol{T}_{\rm em}$ = energy-momentum tensor of the electromagnetic field $\boldsymbol{F} = \mathfrak{g}\sin\theta\,\mathbf{d}\theta\wedge\mathbf{d}\varphi$ generated by a magnetic monopole of magnetic charge $\mathfrak{g}$ in the nonlinear electrodynamics governed by the Lagrangian

$$L = \frac{3M}{|\mathfrak{g}|^3}\left(\frac{\sqrt{2\mathfrak{g}^2\mathcal{F}}}{1+\sqrt{2\mathfrak{g}^2\mathcal{F}}}\right)^{5/2} \quad \text{with } \mathcal{F} := \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

[Ayón-Beato & García, Phys. Lett. B **493**, 149 (2000)]

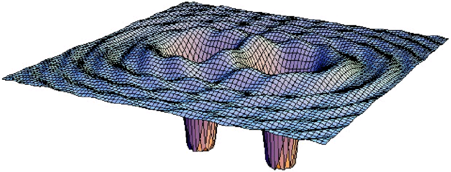
Fact: singularity-free black hole solutions exist

Bardeen spacetime $(\mathscr{M}, \boldsymbol{g})$

- is fully regular: the curvature is finite everywhere, including at $r = 0$
- contains a static black hole
- is a non-vacuum solution of Einstein's equation with $\boldsymbol{T}_{\rm em}$ obeying the weak energy condition
- contains trapped surfaces (inside the black hole region)
- evades Penrose's singularity theorem because it does not contain any Cauchy hypersurface (lack of global hyperbolicity)
- evades Hawking & Penrose's singularity theorem because $\boldsymbol{T}_{\rm em}$ violates the strong energy condition

Outline

Black holes and gravitational waves



Kerr black holes and gravitational waves are both solutions of the vacuum Einstein's equation:

$$\boldsymbol{R} = 0$$

with $\boldsymbol{R} \sim \boldsymbol{g}^{-1}\boldsymbol{\partial\partial g} + \boldsymbol{g}^{-2}(\boldsymbol{\partial g})^2$

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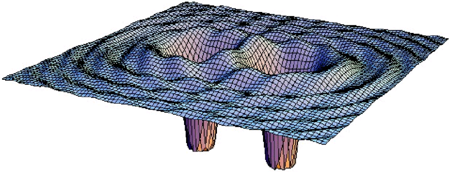

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with $\boldsymbol{R} \sim \boldsymbol{g}^{-1}\boldsymbol{\partial}\boldsymbol{\partial}\boldsymbol{g} + \boldsymbol{g}^{-2}(\boldsymbol{\partial}\boldsymbol{g})^2$

- black holes are solutions of the full nonlinear equation
  (*Remark:* there could not be vacuum black holes in Newtonian gravity for Poisson's equation is linear)
- gravitational waves are solutions of the linearized equation around the Minkowski metric $\boldsymbol{\eta}$: $\boxed{\boldsymbol{g} = \boldsymbol{\eta} + \boldsymbol{h}}$, with $|h_{\alpha\beta}| \ll 1$ in Minkowskian coordinates (t, x, y, z):
  in Lorenz gauge, $\boldsymbol{R} = -\frac{1}{2}\square_\eta \bar{\boldsymbol{h}} + O\left(|\bar{\boldsymbol{h}}|^2\right)$, with $\bar{\boldsymbol{h}} := \boldsymbol{h} - \frac{1}{2}h\boldsymbol{\eta}$

Can gravitational waves escape from a black hole?

- in a highly dynamical process (e.g. binary BH merger), the answer is NO, since gravitational waves do not even exist in the strong field region: no unique way to split $\boldsymbol{g} = \boldsymbol{g}_0 + \boldsymbol{h}$ and consider $\boldsymbol{h}$ as a perturbation of the background $\boldsymbol{g}_0$
- in situations where the split $\boldsymbol{g} = \boldsymbol{g}_0 + \boldsymbol{h}$ is meaningful near the horizon (e.g. $\boldsymbol{h}$ is high frequency, for instance generated by a small binary system falling into a supermassive black hole), the answer is still NO: the waves $\boldsymbol{h}$ will propagate along the light cones of $\boldsymbol{g}_0$

Upper bound on gravitational radiation from a BH merger (Hawking, 1971)

Consider a binary black hole merger:

- initial stage: two far apart Kerr BHs: (m_1, a_1) and (m_2, a_2)
- final stage: a single Kerr BH: (m_3, a_3)

The total amount of energy radiated via gravitational waves is $\Delta E = m_1 + m_2 - m_3$

$\Longrightarrow$ **efficiency of gravitational radiation**: $\boxed{\epsilon := \frac{m_1 + m_2 - m_3}{m_1 + m_2}}$

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Second law $\Longrightarrow A_3 \geq A_1 + A_2$, i.e.

$$m_3 \left(m_3 + \sqrt{m_3^2 - a_3^2} \right) \geq m_1 \left(m_1 + \sqrt{m_1^2 - a_1^2} \right) + m_2 \left(m_2 + \sqrt{m_2^2 - a_2^2} \right)$$

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ϵ is maximal if m_3 is minimal; given the above inequality, this is achieved for $a_1 = m_1$, $a_2 = m_2$ and $a_3 = 0$

$\Longrightarrow 2m_3^2 \geq m_1^2 + m_2^2 \Longrightarrow m_3 \geq \sqrt{(m_1^2+m_2^2)/2}$

$\Longrightarrow \epsilon \leq 1 - \dfrac{\sqrt{m_1^2+m_2^2}}{\sqrt{2}(m_1+m_2)}$

The maximum of the r.h.s. is achieved for $m_1 = m_2$ and is $1/2$, hence the upper bound:

$$\boxed{\epsilon \leq \frac{1}{2}}$$

Upper bound on gravitational radiation from a BH merger

Case of initially non-spinning equal-mass BH (Hawking, 1971)

Initially non-spinning equal-mass BH: $a_1 = a_2 = 0$ and $m_1 = m_2$
The second law yields

$$m_3 \left(m_3 + \sqrt{m_3^2 - a_3^2} \right) \geq 4m_1^2$$

Again, ϵ is maximal if m_3 is minimal; given the above inequality, this is achieved for $a_3 = 0 \Longrightarrow 2m_3^2 \geq 4m_1^2 \Longrightarrow m_3 \geq \sqrt{2}m_1$
Hence the upper bound:

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The GW efficiency for inspiralling binaries is actually much lower

Inspiralling binary BH merger with $m_1 = m_2$ and $a_1 = a_2 = 0$:
numerical relativity $\Longrightarrow a_3 = 0.68\, m_3$ and $\epsilon = 0.048$

[Scheel et al., PRD **79**, 024003 (2009)]

Outline

It all started with Hawking radiation...

Hawking radiation (1975):

black-body radiation at $\boxed{T = \frac{\hbar}{2\pi k}\,\kappa}$ **(Hawking temperature)**

with k = Boltzmann constant

$\frac{\kappa}{8\pi}\mathrm{d}A = T\mathrm{d}S \Longrightarrow \boxed{S = \frac{k}{4}\frac{A}{\ell_{\mathrm{P}}^2}}$ **(Bekenstein-Hawking entropy)**

with $\ell_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^3}}$ = Planck length $\simeq 1.6\ 10^{-35}\ \mathrm{m}$

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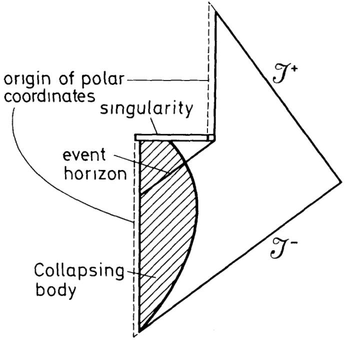
For a Schwarzschild black hole of mass M: $\kappa = (4M)^{-1}$ and $A = 16\pi M^2$

$\Longrightarrow T = 6\ 10^{-8}\left(\frac{M_\odot}{M}\right)\ \mathrm{K}$ and $S = 1.1\ 10^{77}\left(\frac{M}{M_\odot}\right)^2 k$!!!

Black hole evaporation



[Hawking, Com. Math. Phys. **43**, 199 (1975)]

Energy loss by Hawking radiation
$\Longrightarrow$ BH mass M decreases
$\Longrightarrow$ BH area A decreases
(*NB:* no contradiction with the second law of BH dynamics here since the effective energy-momentum tensor of Hawking radiation violates the null energy condition)
$\Longrightarrow$ the BH eventually disappears

The information paradox

The quantum state of Hawking radiation retains no information about the quantum state of matter that has fallen into the black hole

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In the absence of any theory of quantum gravity or of gravitational quantum mechanics, this is not truely a paradox...

> *I am not a supporter of the contention that* ***U*** *(unitarity) must be true at all levels and that, indeed, its violation (which in any case has to take place in most circumstances during measurement) will occur when gravitation gets involved. (...) I here take the strong view that information loss* does *take place at a black hole's singularity.*

R. Penrose, in *Fashion, Faith, and Fantasy in the New Physics of the Universe* (2016)