Black holes and tests of gravitation

Éric Gourgoulhon

Laboratoire Univers et Théories (LUTH)
CNRS / Observatoire de Paris / Université Paris Diderot
92190 Meudon, France

eric.gourgoulhon@obspm.fr

http://luth.obspm.fr/~luthier/gourgoulhon/

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Outline

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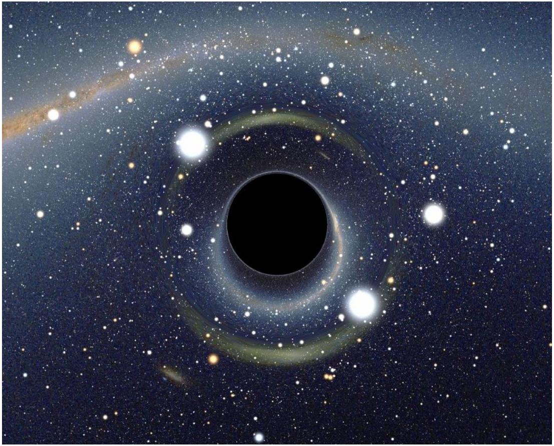
What is a black hole?



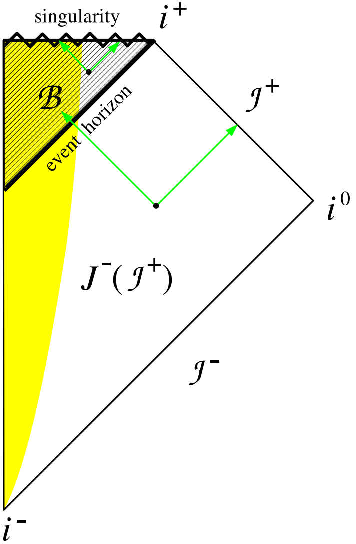
[Alain Riazuelo, 2007]

... for the layman:

A **black hole** is a region of spacetime from which nothing, not even light, can escape.

The (immaterial) boundary between the black hole interior and the rest of the Universe is called the **event horizon**.

What is a black hole?



... for the mathematical physicist:

black hole: $\mathcal{B} := \mathscr{M} - J^-(\mathscr{I}^+)$

i.e. the region of spacetime where light rays cannot escape to infinity

- $(\mathscr{M}, \boldsymbol{g})$ = asymptotically flat manifold
- $\mathscr{I}^+$ = future null infinity
- $J^-(\mathscr{I}^+)$ = causal past of $\mathscr{I}^+$

event horizon: $\mathcal{H} := \partial J^-(\mathscr{I}^+)$
(boundary of $J^-(\mathscr{I}^+)$)

$\mathcal{H}$ smooth $\Longrightarrow$ $\mathcal{H}$ null hypersurface

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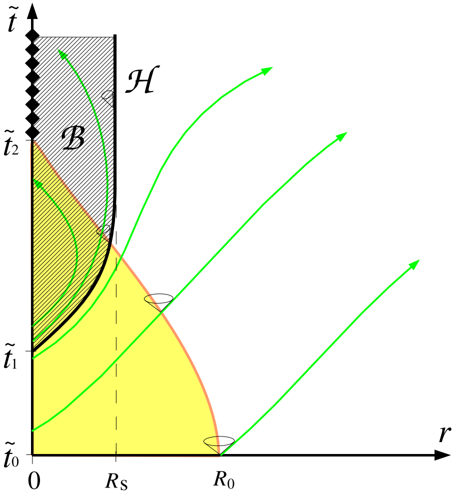

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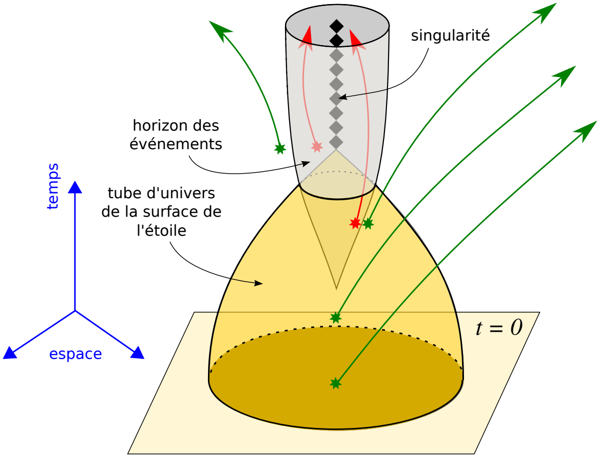


spacetime diagram depicting the formation of a black hole from the gravitational collapse of the core of a massive star (*supernova* phenomenon)

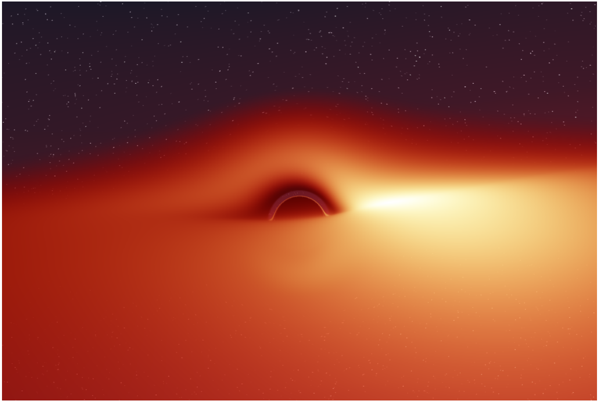
singularity: curvature $\longrightarrow \infty$

What is a black hole?

... for the astrophysicist: a very deep gravitational potential well

Release of potential gravitational energy by **accretion** on a black hole: up to 42% of the mass-energy mc^2 of accreted matter !

NB: thermonuclear reactions release less than 1% mc^2



Matter falling in a black hole forms an **accretion disk** [Lynden-Bell (1969), Shakura & Sunayev (1973)]

[J.-A. Marck (1996)]

Main properties of black holes (1/3)

- In general relativity, a black hole contains a region where the spacetime curvature diverges: the singularity *(NB: this is not the primary definition of a black hole)*. The singularity is inaccessible to observations, being hidden by the event horizon.

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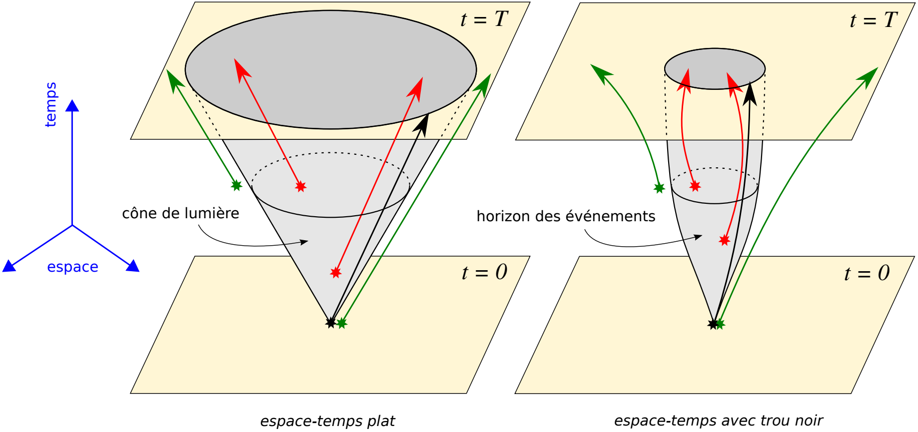
- The singularity marks the limit of validity of general relativity: to describe it, a quantum theory of gravitation would be required.
- The event horizon $\mathcal{H}$ is a global structure of spacetime: no physical experiment whatsoever can detect the crossing of $\mathcal{H}$.

Main properties of black holes (2/3)

The event horizon as a null cone



Main properties of black holes (3/3)

- Viewed by a distant observer, the horizon approach is perceived with an infinite redshift, or equivalently, by an infinite time dilation
- A black hole is not an infinitely dense object: on the contrary it is made of vacuum (except maybe at the singularity); if one defines its "mean density" by $\bar{\rho} = M/(4/3\pi R^3)$, then
  - for the Galatic center BH (Sgr A*): $\bar{\rho} \sim 10^6 \ \mathrm{kg\,m^{-3}} \sim 2\ 10^{-4}\ \rho_{\text{white dwarf}}$
  - for the BH at the center of M87: $\bar{\rho} \sim 2\ \mathrm{kg\,m^{-3}} \sim 2\ 10^{-3}\ \rho_{\text{water}}$!

  $\Longrightarrow$ a black hole is a compact object: $\dfrac{M}{R}$ large, not $\dfrac{M}{R^3}$!
- Due to the non-linearity of general relativity, black holes can form in spacetimes empty of any matter, by collapse of gravitational wave packets.

The "no-hair" theorem

Dorochkevitch, Novikov & Zeldovitch (1965), Israel (1967), Carter (1971), Hawking (1972)

Within 4-dimensional general relativity, a black hole in equilibrium in an otherwise empty universe is necessarily a Kerr-Newmann black hole, which is a vacuum solution of Einstein described by only three parameters:

- the total mass M
- the total angular momentum J
- the total electric charge Q

$\Longrightarrow$ *"a black hole has no hair"* (John A. Wheeler)

Astrophysical black holes have to be electrically neutral:

- $Q = 0$: Kerr solution (1963)

The Kerr solution

Roy Kerr (1963)

$$g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = -\left(1-\frac{2GMr}{c^2\rho^2}\right)c^2\mathrm{d}t^2 - \frac{4GMar\sin^2\theta}{c^2\rho^2}\,c\,\mathrm{d}t\,\mathrm{d}\varphi + \frac{\rho^2}{\Delta}\,\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{2GMa^2r\sin^2\theta}{c^2\rho^2}\right)\sin^2\theta\,\mathrm{d}\varphi^2$$

where

$$\rho^2 := r^2 + a^2\cos^2\theta, \qquad \Delta := r^2 - \frac{2GM}{c^2}r + a^2, \qquad a := \frac{J}{cM}$$

$$\exists \text{ event horizon (black hole)} \iff |a| \leq \frac{GM}{c^2}$$

Schwarzschild subcase ($a = 0$):

$$g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = -\left(1-\frac{2GM}{c^2r}\right)c^2\mathrm{d}t^2 + \left(1-\frac{2GM}{c^2r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)$$

The black hole parameters

- The mass M is not some measure of the "amount of matter" inside the black hole, but rather a parameter characterizing the external gravitational field; it is measurable from the orbital period of a test particle in circular orbit around the black hole and far from it *(Kepler's third law)*.

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Remark: the radius of a black hole is not a well defined concept: it *does not* correspond to some distance between the black hole "centre" (the singularity) and the event horizon. A well defined quantity is the area of the event horizon, A. The radius can be then defined from it: for a Schwarzschild black hole:

$$R := \sqrt{\frac{A}{4\pi}} = \frac{2GM}{c^2} \simeq 3 \left(\frac{M}{M_\odot} \right) \ \mathrm{km}$$

Why is the Kerr metric special?

Spherically symmetric (non-rotating) case:

Birkhoff theorem

Within 4-dimensional general relativity, the spacetime outside any spherically symmetric body is described by Schwarzschild metric

$\Longrightarrow$ No possibility to distinguish a non-rotating black hole from a non-rotating dark star by monitoring orbital motion or fitting accretion disk spectra

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Rotating axisymmetric case:

No Birkhoff theorem

Moreover, no "reasonable" matter source has ever been found for the Kerr metric (the only known source consists of two counter-rotating thin disks of collisionless particles [Bicak & Ledvinka, PRL **71**, 1669 (1993)])

$\Longrightarrow$ The Kerr metric is specific to rotating black holes (in 4-dimensional general relativity)

Lowest order no-hair theorem: quadrupole moment

Asymptotic expansion (large r) of the metric in terms of multipole moments $(\mathcal{M}_k, \mathcal{J}_k)_{k\in\mathbb{N}}$ [Geroch (1970), Hansen (1974)]:

- $\mathcal{M}_k$: mass 2^k-pole moment
- $\mathcal{J}_k$: angular momentum 2^k-pole moment

$\Longrightarrow$ For the Kerr metric, all the multipole moments are determined by (M, a):

- $\mathcal{M}_0 = M$
- $\mathcal{J}_1 = aM = J/c$
- $\boxed{\mathcal{M}_2 = -a^2 M = -\frac{J^2}{c^2 M}}$ $(*)$ $\leftarrow$ mass quadrupole moment
- $\mathcal{J}_3 = -a^3 M$
- $\mathcal{M}_4 = a^4 M$
- $\cdots$

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Measuring the three quantities M, J, $\mathcal{M}_2$ provides a compatibility test w.r.t. the Kerr metric, by checking $(*)$

Other theoretical aspects

- The four laws of black hole dynamics
- Quantum properties (Bekenstein entropy, Hawking radiation)
- Black holes in higher dimensions

Outline

Known black holes

Three kinds of black holes are known in the Universe:

- **Stellar black holes**: supernova remnants:
  $M \sim 10 - 30\ M_\odot$ and $R \sim 30 - 90\ \mathrm{km}$

  example: Cyg X-1 : $M = 15\ M_\odot$ and $R = 45\ \mathrm{km}$

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- **Supermassive black holes**, in galactic nuclei:
  $M \sim 10^5 - 10^{10}\ M_\odot$ and $R \sim 3 \times 10^5\ \mathrm{km} - 200\ \mathrm{UA}$

  example: Sgr A* : $M = 4.3 \times 10^6\ M_\odot$ and
  $R = 13 \times 10^6\ \mathrm{km} = 18\, R_\odot = 0.09\ \mathrm{UA} = \frac{1}{4}$ $\times$ radius of Mercury's orbit

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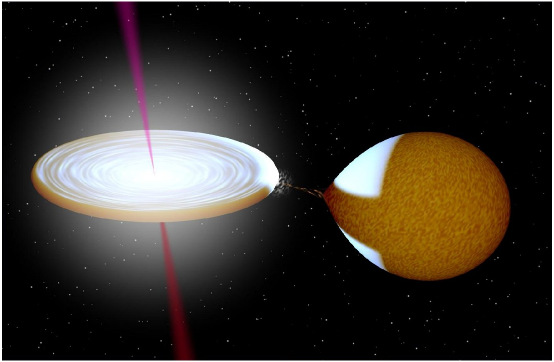
- **Intermediate mass black holes**, as ultra-luminous X-ray sources (?):
  $M \sim 10^2 - 10^4\ M_\odot$ and $R \sim 300\ \mathrm{km} - 3 \times 10^4\ \mathrm{km}$

  example: ESO 243-49 HLX-1 : $M > 500\ M_\odot$ and $R > 1500\ \mathrm{km}$

Stellar black holes in X-ray binaries



$\sim$ 20 identified stellar black holes in our galaxy

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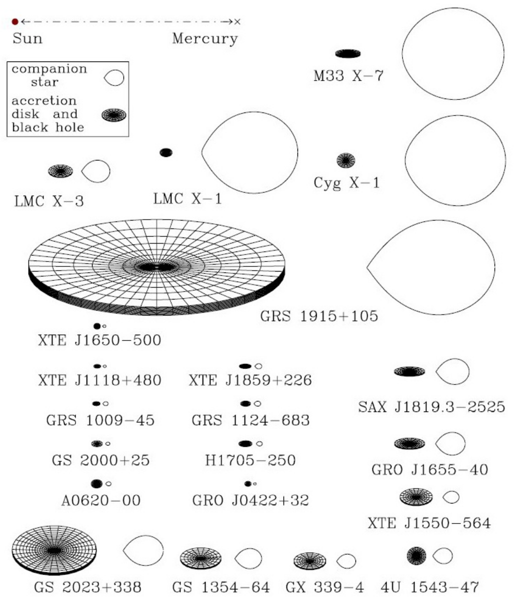
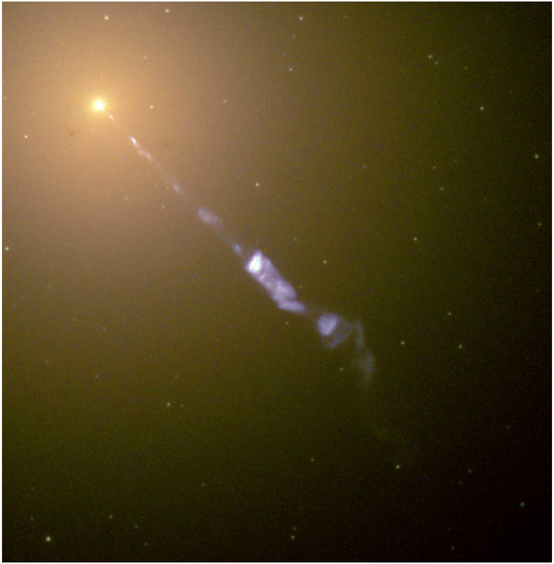


[McClintock et al. (2011)]

Supermassive black holes in active galactic nuclei (AGN)

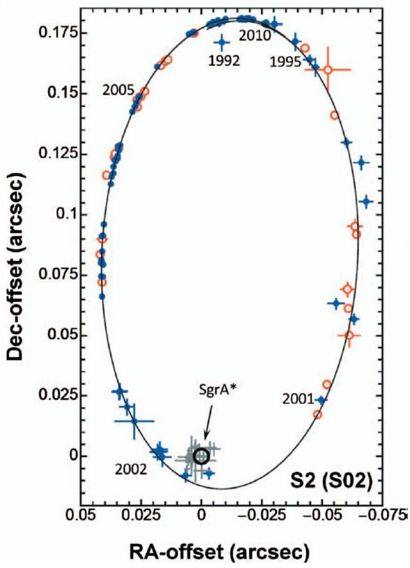


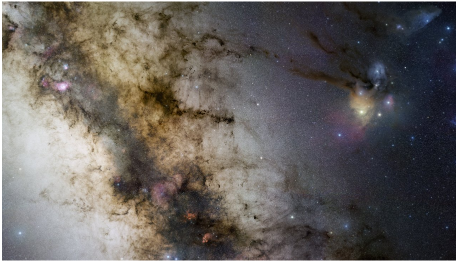
Jet emitted by the nucleus of the giant elliptic galaxy M87, at the centre of Virgo cluster [HST]

$M_{\rm BH} = 3 \times 10^9 \, M_\odot$

$V_{\rm jet} \simeq 0.99 \, c$

The black hole at the centre of our galaxy: Sgr A*





[ESO (2009)]

Measure of the mass of Sgr A* black hole by stellar dynamics:

$$M_{\rm BH} = 4.3 \times 10^6 \, M_\odot$$

← Orbit of the star S2 around Sgr A*
$P = 16\,\mathrm{yr}$, $r_{\rm per} = 120\,\mathrm{UA} = 1400\,R_{\rm S}$, $V_{\rm per} = 0.02\,c$

[Genzel, Eisenhauer & Gillessen, RMP **82**, 3121 (2010)]

Outline

Can we see a black hole from the Earth?

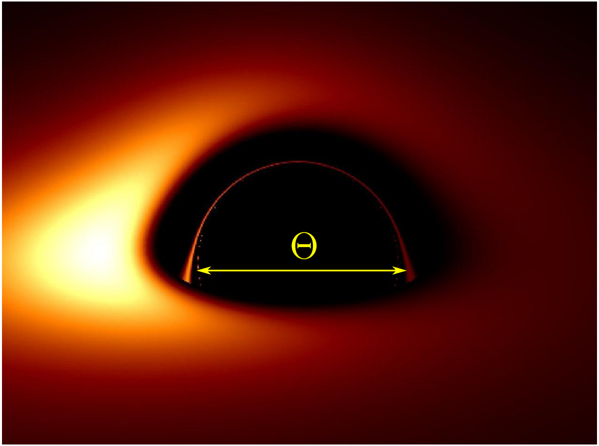


Image of a thin accretion disk around a Schwarzschild BH

[Vincent, Paumard, Gourgoulhon & Perrin, CQG **28**, 225011 (2011)]

Angular diameter of the event horizon of a Schwarzschild BH of mass M seen from a distance d:

$$\Theta = 6\sqrt{3}\,\frac{GM}{c^2 d} \simeq 2.60\,\frac{2R_{\rm S}}{d}$$

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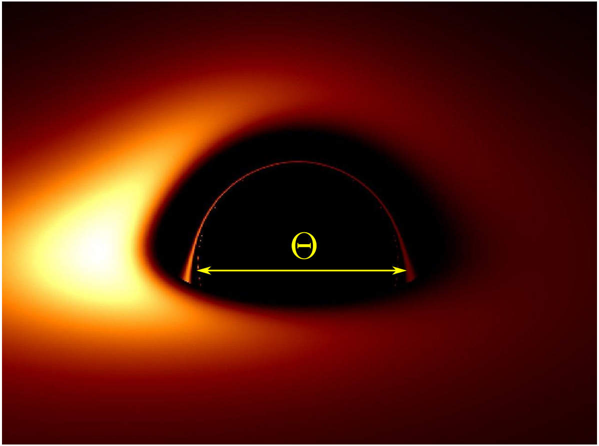

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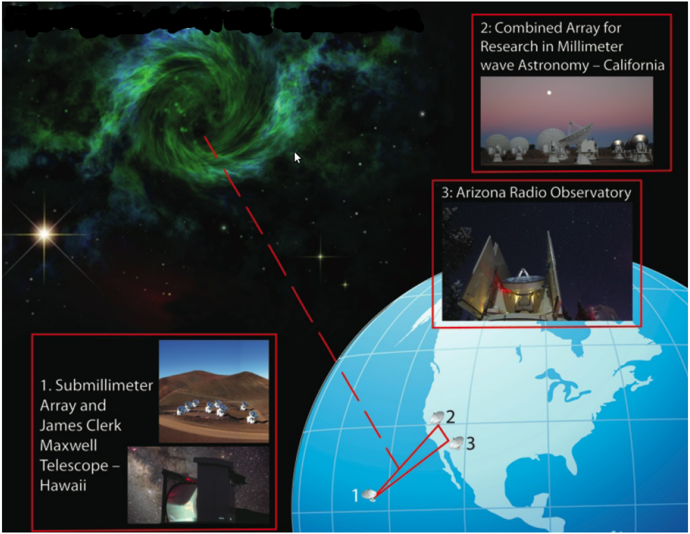
Largest black holes in the Earth's sky:

Sgr A* : $\Theta = 53\ \mu\mathrm{as}$
M87 : $\Theta = 21\ \mu\mathrm{as}$
M31 : $\Theta = 20\ \mu\mathrm{as}$

Remark: black holes in X-ray binaries are $\sim 10^5$ times smaller, for $\Theta \propto M/d$

The solution to reach the μas regime: interferometry !



Very Large Baseline Interferometry (VLBI) in (sub)millimeter waves

Existing American VLBI network [Doeleman et al. 2011]

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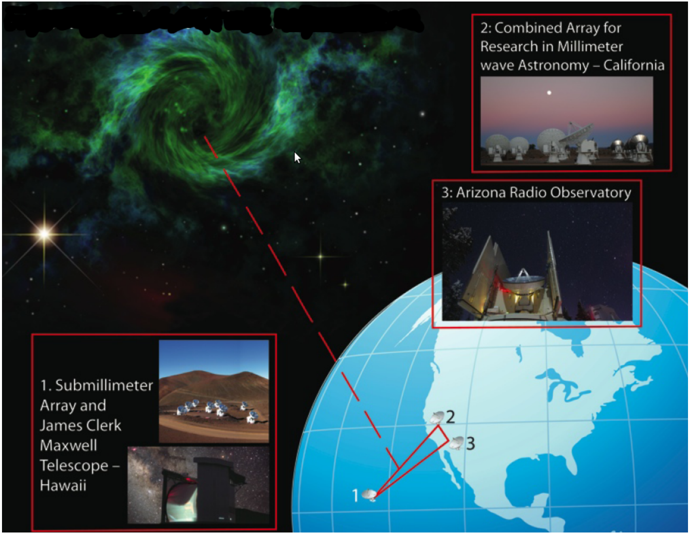

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Very Large Baseline Interferometry (VLBI) in (sub)millimeter waves

The best result so far: VLBI observations at 1.3 mm have shown that the size of the emitting region in Sgr A* is only $37\ \mu$as [Doeleman et al., Nature **455**, 78 (2008)]

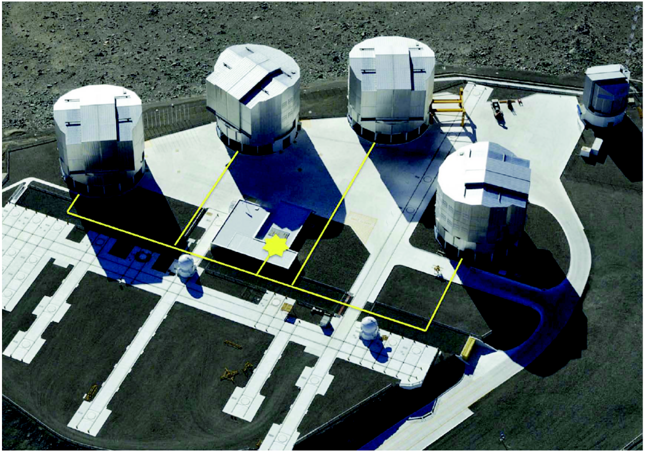
The near future: the Event Horizon Telescope

To go further:

- shorten the wavelength: $1.3\ \mathrm{mm} \to 0.8\ \mathrm{mm}$
- increase the number of stations; in particular add ALMA

Atacama Large Millimeter Array (ALMA)
part of the Event Horizon Telescope (EHT) to be completed by 2020

Near-infrared optical interferometry: GRAVITY



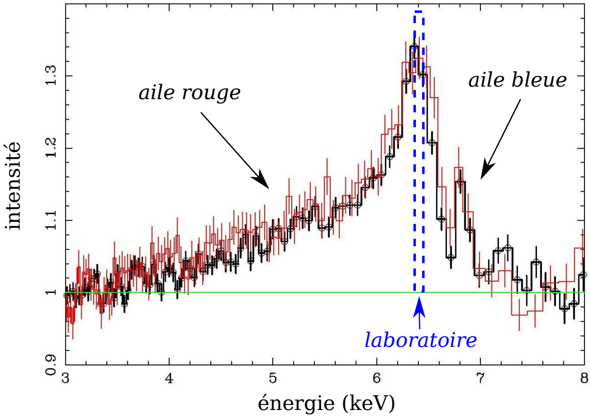
[Gillessen et al. 2010]

GRAVITY instrument at VLTI (late 2015)

Beam combiner (the four 8 m telescopes + four auxiliary telescopes) $\Longrightarrow$ astrometric precision of $10\ \mu\text{as}$

X-ray observations (Athena)

The accretion disk as a spacetime probe



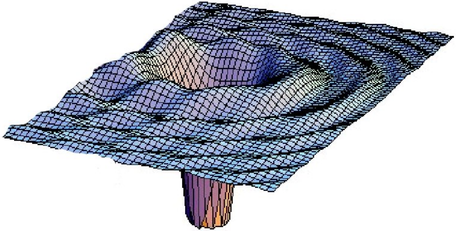
Kα line in the nucleus of the galaxy MCG-6-30-15 observed by XMM-Newton (red) and Suzaku (black) (adapted from [Miller (2007)])

Kα line: X fluorescence line of Fe atoms in the accretion disk (the Fe atoms are excited by the X-ray emitted from the plasma corona surrounding the disk)

Redshift $\Rightarrow$ time dilatation

Athena X-ray observatory selected in 2014 for ESA L2 mission $\Longrightarrow$ launch $\sim$ 2028

Another way to “see” BHs: gravitational waves



Link between black holes and gravitational waves:

Black holes and gravitational waves are both spacetime distortions:

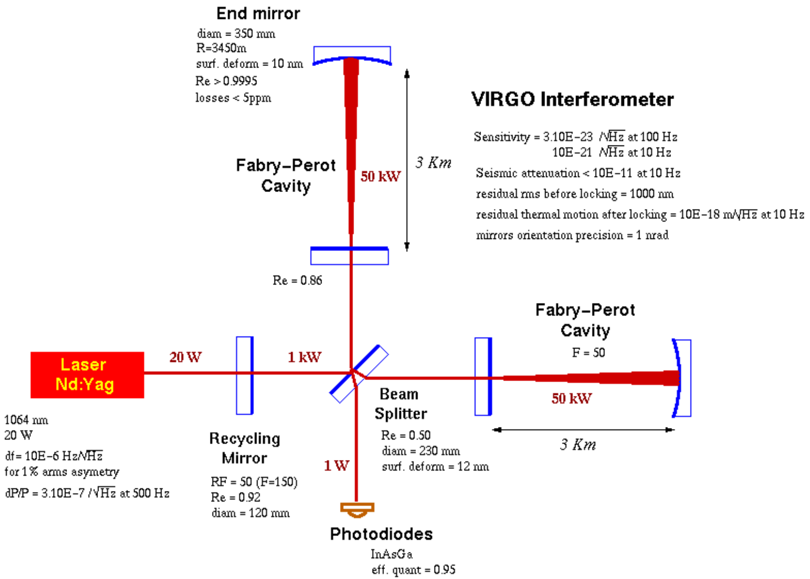
- extreme distortions (black holes)
- small distortions (gravitational waves)

In particular, black holes and gravitational waves are both vacuum solutions of general relativity equations (Einstein equations)

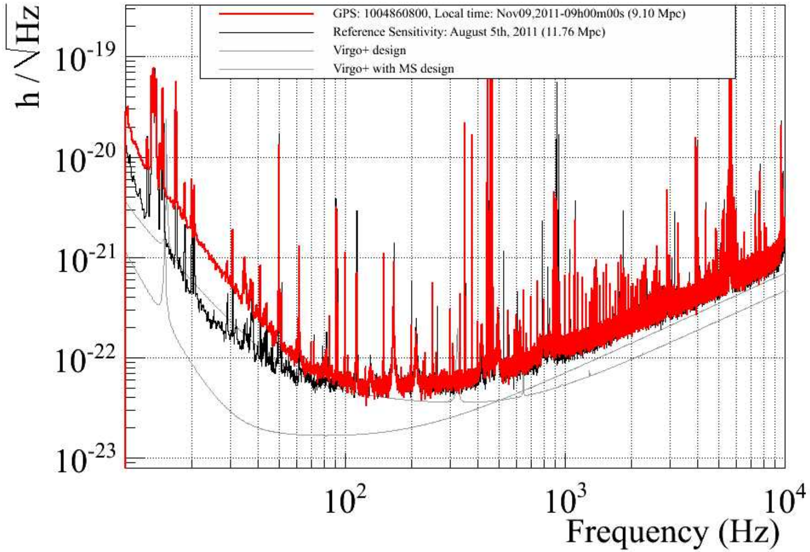
VIRGO: a giant Michelson interferometer...

Gravitational wave detector VIRGO in Cascina, near Pisa (Italy) [CNRS/INFN]

Optical scheme of the VIRGO interferometer

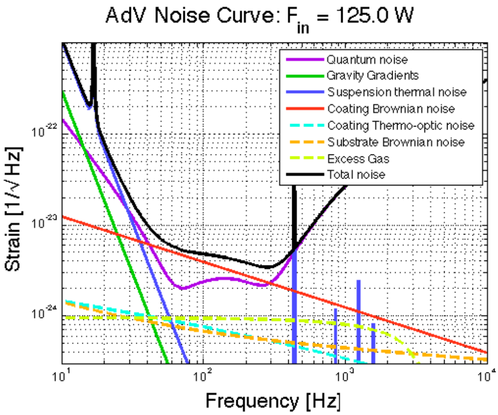


VIRGO sensitivity curve



Advanced VIRGO

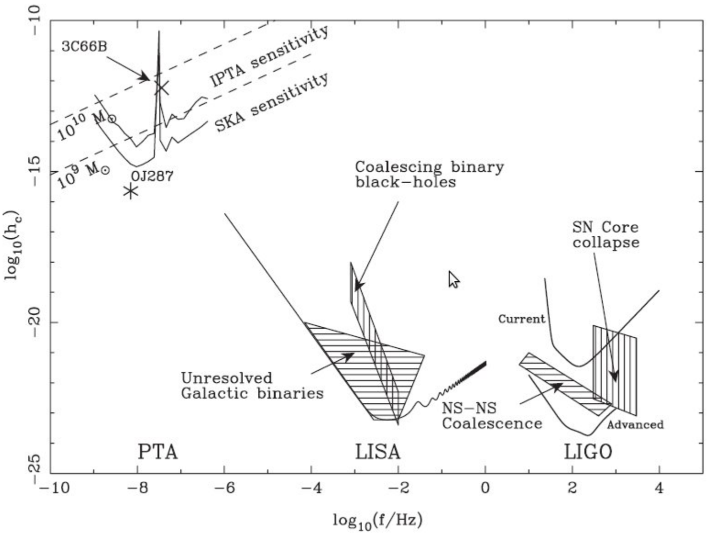
Advanced VIRGO: dual recycled (power + signal) interferometer with laser power $\sim$ 125 W



[CNRS/INFN/NIKHEF]

- VIRGO+ decommissioned in Nov. 2011
- Construction of Advanced VIRGO underway
- First lock in 2015
- Sensitivity $\sim 10 \times$ VIRGO
- $\Longrightarrow$ explored Universe volume 10^3 times larger !

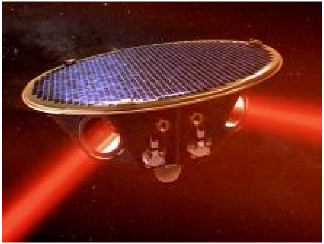
International Pulsar Timing Array (IPTA)

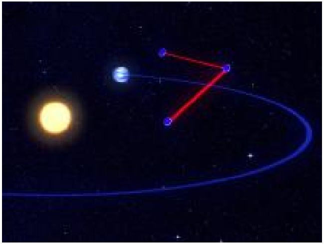


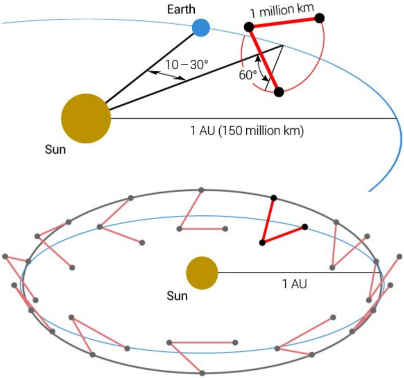
[Hobbs et al., CQG **27**, 084013 (2010)]

eLISA

Gravitational wave detector in space $\Longrightarrow$ low frequency range: $[10^{-3}, 10^{-1}]$ Hz







[http://www.elisascience.org/]

- eLISA scientific theme selected in Nov. 2013 for ESA L3 mission $\Longrightarrow$ launch in 2028
- LISA Pathfinder: launch in Sept.-Nov. 2015

Outline

Theoretical alternatives to the Kerr black hole

Within general relativity

The compact object is not a black hole but

- a boson star
- a gravastar
- a dark star
- ...

Beyond general relativity

The compact object is a black hole but in a theory that differs from GR:

- Einstein-Gauss-Bonnet with dilaton
- Chern-Simons gravity
- Hořava-Lifshitz gravity
- Einstein-Yang-Mills
- ...

How to test the alternatives to the Kerr black hole?

Search for

- stellar orbits deviating from Kerr timelike geodesics (GRAVITY)
- accretion disk spectra different from those arising in Kerr metric (X-ray observatories)
- images of the black hole shadow different from that of a Kerr black hole (EHT)

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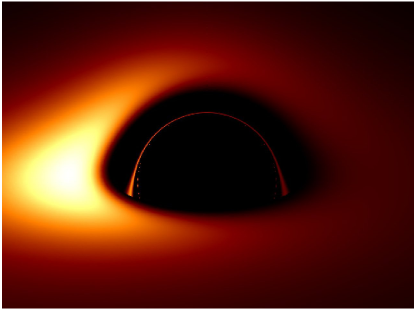
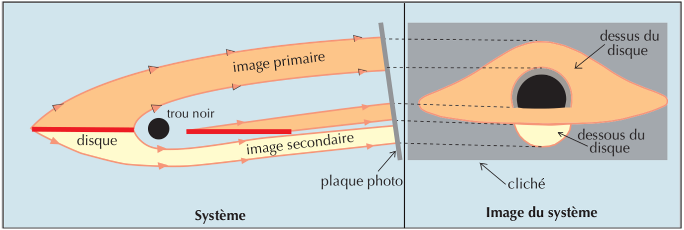
Need for a good and versatile geodesic integrator
to compute timelike geodesics (orbits) and null geodesics (ray-tracing) in any kind of metric

Black hole images from null geodesics



Simulated image of a thin accretion disk around a Schwarzschild black hole

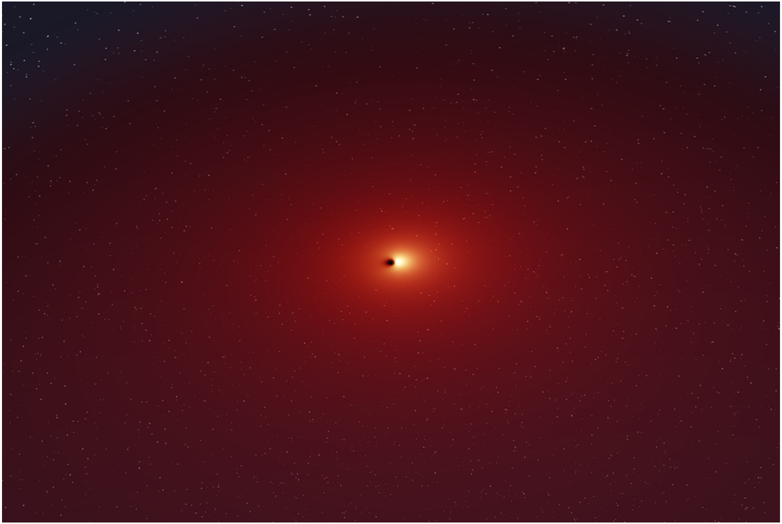
[Vincent, Paumard, Gourgoulhon & Perrin, CQG **28**, 225011 (2011)]



Light ray trajectories
http://luth.obspm.fr/~luminet/

Flight to a black hole

Images computed by J.-A. Marck [Marck, CQG **13**, 393 (1996)]



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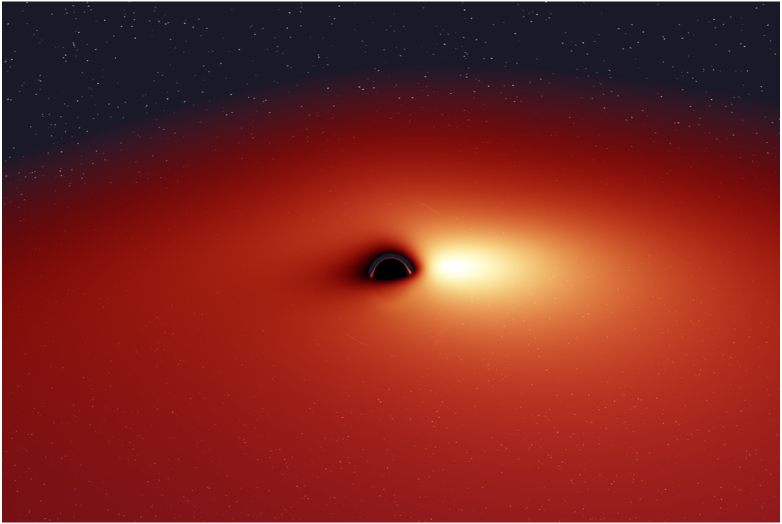

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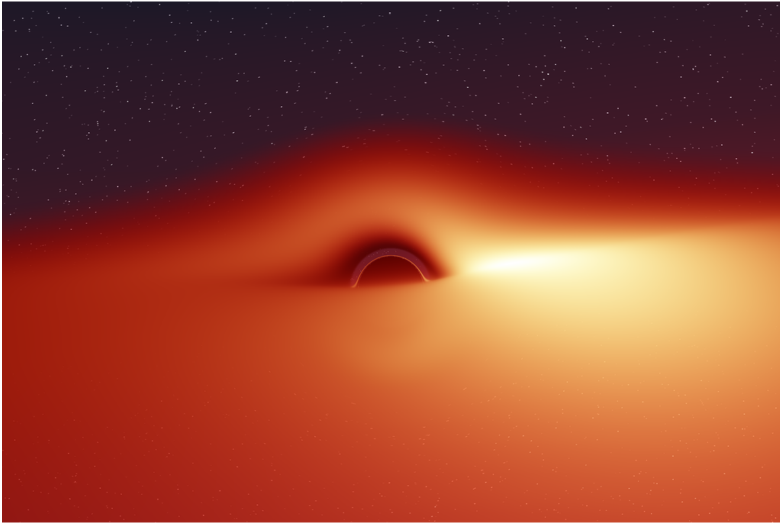

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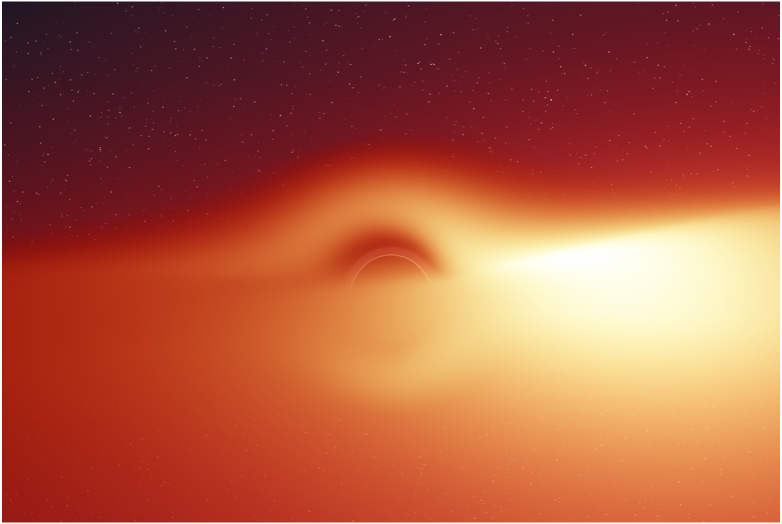

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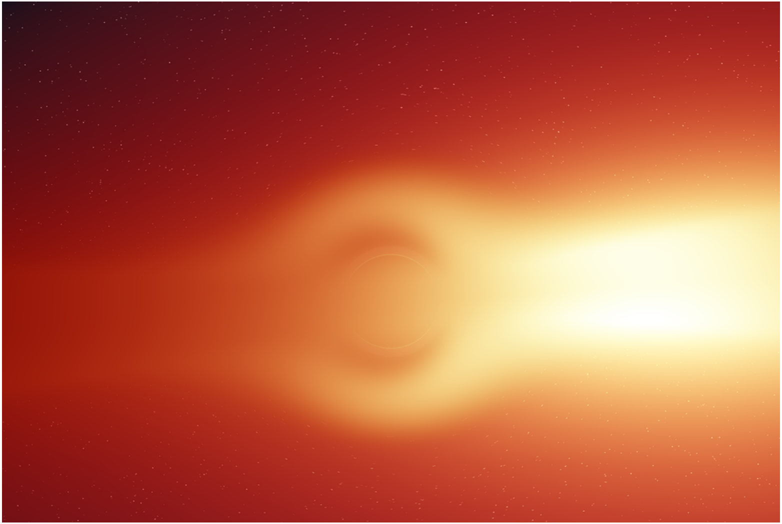

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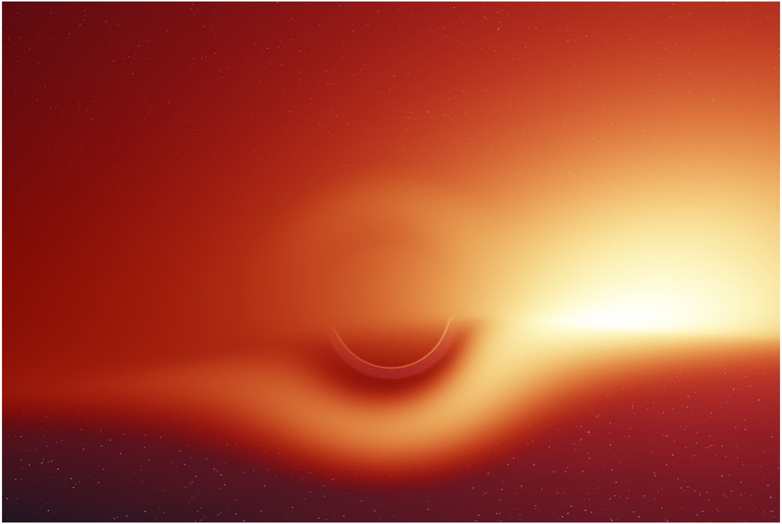

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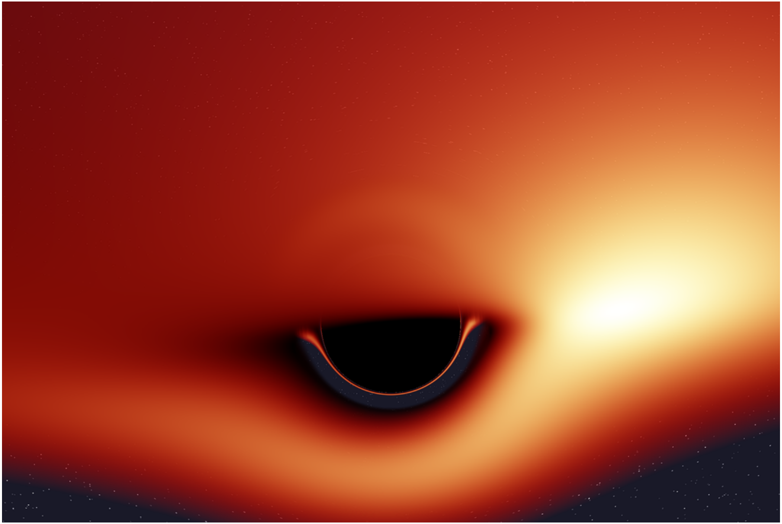

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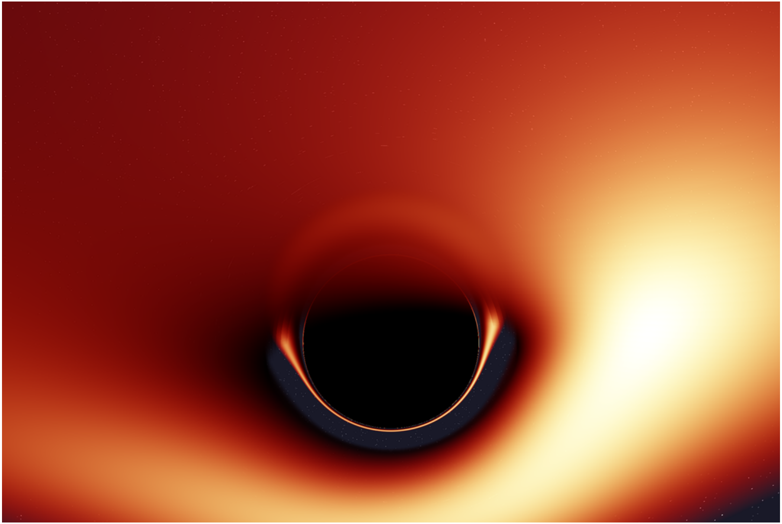

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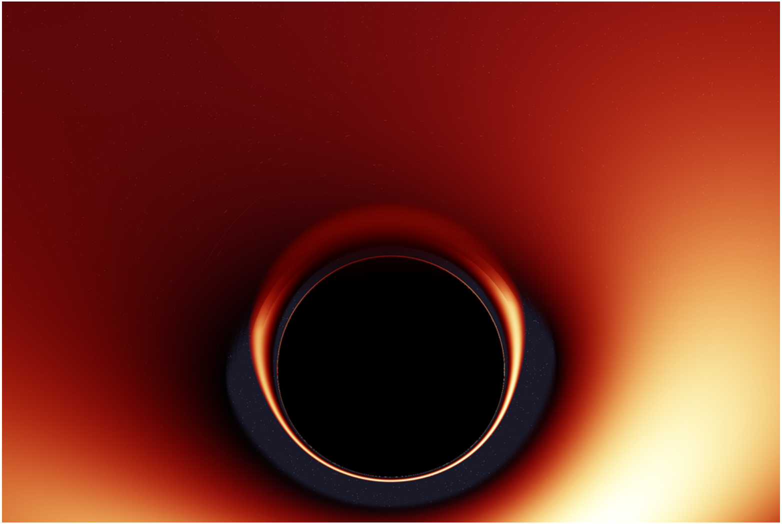

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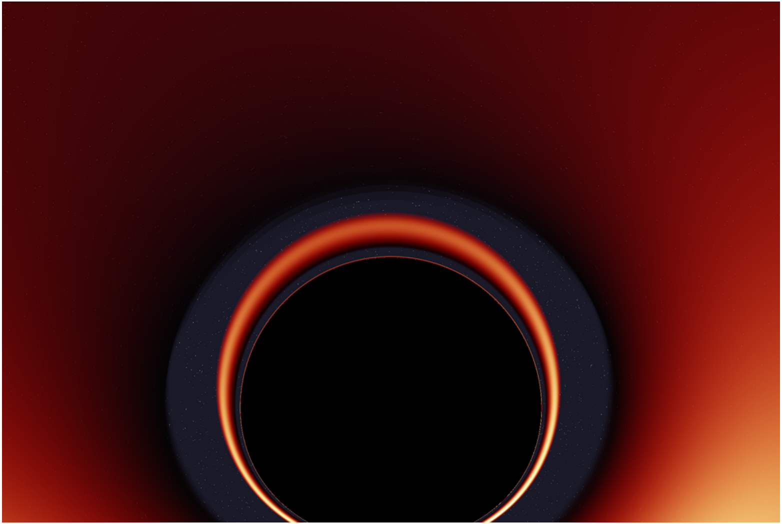

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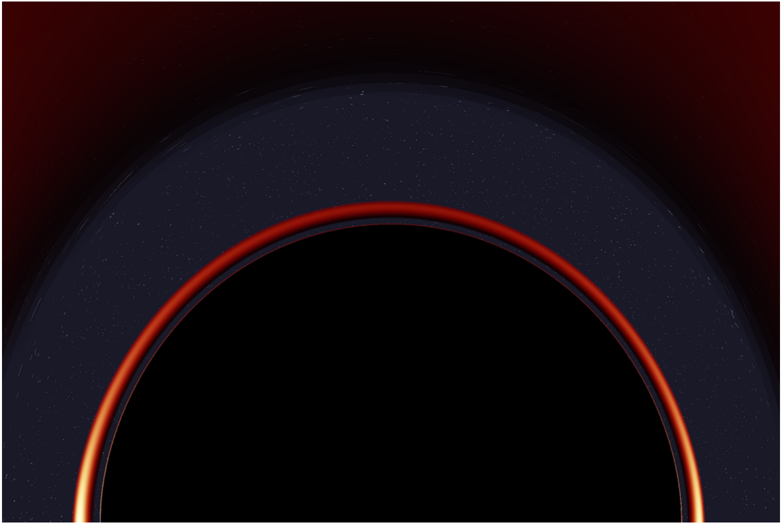
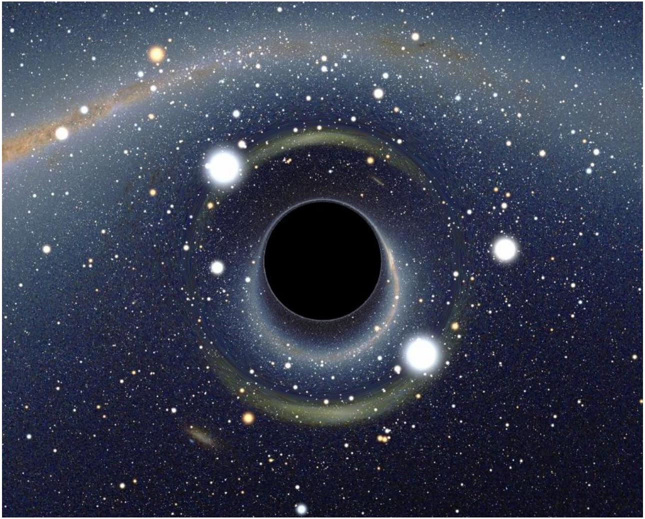


Isolated black hole in front of a stellar background

Image computed by A. Riazuello (IAP) [Riazuelo, 2007]



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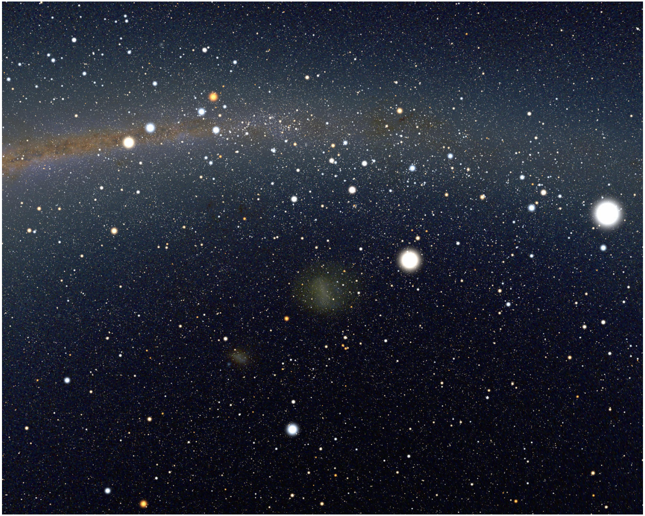

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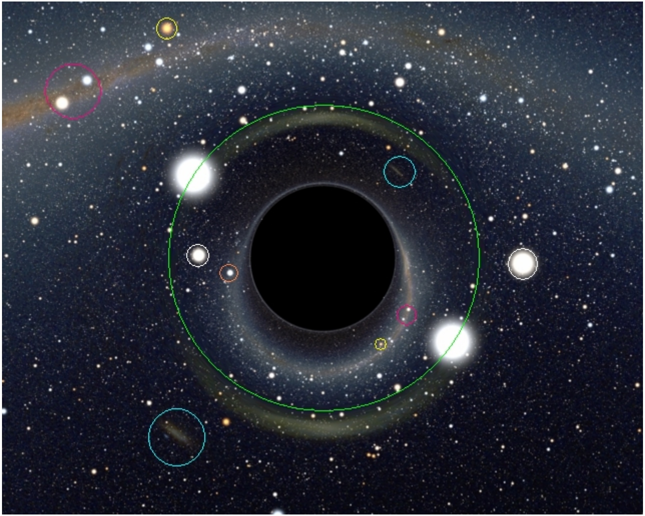
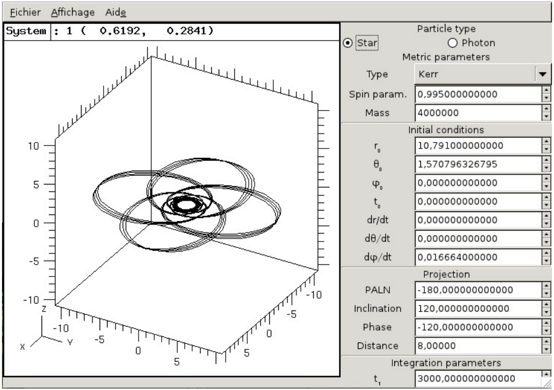


Gyoto code

Main developers: T. Paumard & F. Vincent

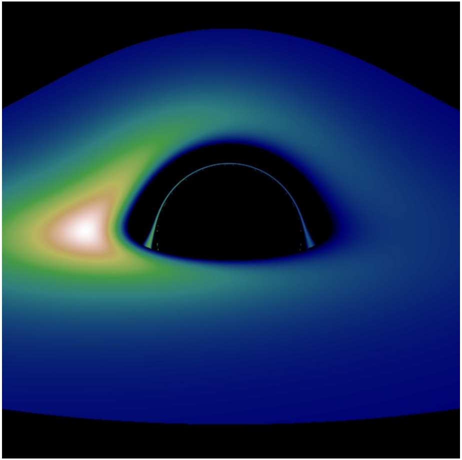


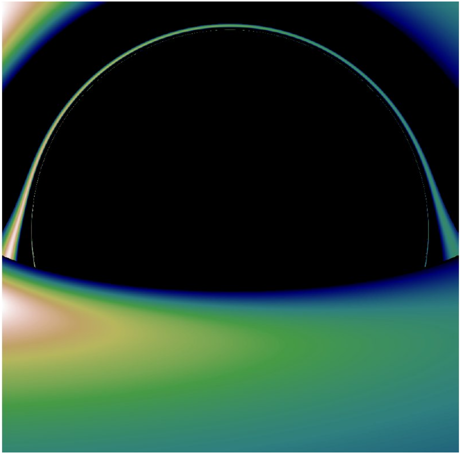
- Integration of geodesics in Kerr metric
- Integration of geodesics in any numerically computed 3+1 metric
- Radiative transfer included in optically thin media
- Very modular code (C++)
- Yorick interface
- Free software (GPL) : http://gyoto.obspm.fr/

[Vincent, Paumard, Gourgoulhon & Perrin, CQG **28**, 225011 (2011)]

[Vincent, Gourgoulhon & Novak, CQG **29**, 245005 (2012)]

Gyoto code



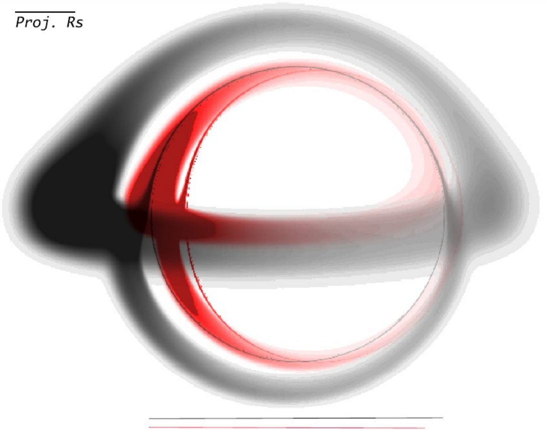


Computed images of a thin accretion disk around a Schwarzschild black hole

Measuring the spin from the black hole silhouette

Ray-tracing in the Kerr metric (spin parameter a)

Accretion structure around Sgr A* modelled as a ion torus, derived from the *polish doughnut* class [Abramowicz, Jaroszynski & Sikora (1978)]



Radiative processes included: thermal synchrotron, bremsstrahlung, inverse Compton

← Image of an ion torus computed with Gyoto for the inclination angle $i = 80°$:

- black: $a = 0.5M$
- red: $a = 0.9M$

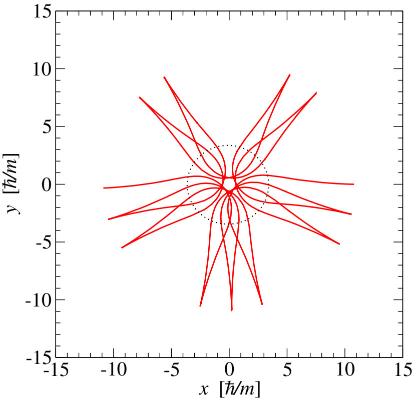
[Straub, Vincent, Abramowicz, Gourgoulhon & Paumard, A&A **543**, A83 (2012)]

Orbits around a rotating boson star

Boson star = localized configurations of a self-gravitating complex scalar field Φ ≡ *"Klein-Gordon geons"* [Bonazzola & Pacini (1966), Kaup (1968)]

Boson stars may behave as black-hole mimickers

- Solutions of the *Einstein-Klein-Gordon* system computed by means of Kadath [Grandclément, JCP **229**, 3334 (2010)]
- Timelike geodesics computed by means of Gyoto



Zero-angular-momentum orbit around a rotating boson star based on a free scalar field $\Phi = \phi(r, \theta) e^{i(\omega t + 2\varphi)}$ with $\omega = 0.75 \, m/\hbar$.

[Granclément, Somé & Gourgoulhon, PRD **90**, 024068 (2014)]

Another tool to explore black hole spacetimes: SageManifolds

Symbolic differential geometry based on the modern **free open-source** mathematics software system **Sage**

- Sage is based on the Python programming language
- it makes use of **many pre-existing open-sources packages**, among which
  - Maxima (symbolic calculations, since 1968!)
  - GAP (group theory)
  - PARI/GP (number theory)
  - Singular (polynomial computations)
  - matplotlib (high quality 2D figures)

  and provides a **uniform interface** to them
- William Stein (Univ. of Washington) created Sage in 2005; since then, ∼**100 developers** (mostly mathematicians) have joined the Sage team

The mission

Create a viable free open source alternative to Magma, Maple, Mathematica and Matlab.

The SageManifolds project

http://sagemanifolds.obspm.fr/

Aim

Implement the concept of real smooth manifolds of arbitrary dimension in Sage and tensor calculus on them, in a coordinate/frame-independent manner

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Aim

Implement the concept of real smooth manifolds of arbitrary dimension in Sage and tensor calculus on them, in a coordinate/frame-independent manner

In practice, this amounts to introducing new Python classes in Sage, basically one class per mathematical concept, for instance:

- `Manifold`: differentiable manifolds over $\mathbb{R}$, of arbitrary dimension
- `Chart`: coordinate charts
- `Point`: points on a manifold
- `DiffMapping`: differential mappings between manifolds
- `ScalarField`, `VectorField`, `TensorField`: tensor fields on a manifold
- `DiffForm`: p-forms
- `AffConnection`, `LeviCivitaConnection`: affine connections
- `Metric`: pseudo-Riemannian metrics

SM_Mars-Simon -- Sage - Mozilla Firefox

SageManifolds: examples | SM_Mars-Simon – Sage

```
M = Manifold(4, 'M', latex_name=r'\mathcal{M}')
print M
```

4-dimensional manifold 'M'

We introduce the standard **Boyer-Lindquist coordinates** as follows:

```
X.<t,r,th,ph> = M.chart(r't r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')
print X ; X
```

chart (M, (t, r, th, ph))

$(\mathcal{M}, (t, r, \theta, \phi))$

Metric tensor

The 2 parameters m and a of the Kerr spacetime are declared as symbolic variables:

```
var('m, a')
```

(m, a)

Let us introduce the spacetime metric g and set its components in the coordinate frame associated with Boyer-Lindquist coordinates, which is the current manifold's default frame:

```
g = M.lorentz_metric('g')
rho2 = r^2 + (a*cos(th))^2
Delta = r^2 -2*m*r + a^2
g[0,0] = -(1-2*m*r/rho2)
g[0,3] = -2*a*m*r*sin(th)^2/rho2
g[1,1], g[2,2] = rho2/Delta, rho2
g[3,3] = (r^2+a^2+2*m*r*(a*sin(th))^2/rho2)*sin(th)^2
g.view()
```

$$g = \left(-\frac{a^2 \cos(\theta)^2 - 2\,mr + r^2}{a^2 \cos(\theta)^2 + r^2} \right) \mathrm{d}t \otimes \mathrm{d}t + \left(-\frac{2\,amr \sin(\theta)^2}{a^2 \cos(\theta)^2 + r^2} \right) \mathrm{d}t \otimes \mathrm{d}\phi + \left(\frac{a^2 \cos(\theta)^2 + r^2}{a^2 - 2\,mr + r^2} \right) \mathrm{d}r \otimes \mathrm{d}r + \left(a^2 \cos(\theta)^2 + r^2 \right) \mathrm{d}\theta \otimes \mathrm{d}\theta + \left(-\frac{2\,amr \sin}{a^2 \cos(\theta}\right.$$

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SageManifolds: examples | SM_Mars-Simon – Sage

```
g[:]
```

$$\begin{pmatrix} -\frac{a^2\cos(\theta)^2 - 2mr + r^2}{a^2\cos(\theta)^2 + r^2} & 0 & 0 & -\frac{2amr\sin(\theta)^2}{a^2\cos(\theta)^2 + r^2} \\ 0 & \frac{a^2\cos(\theta)^2 + r^2}{a^2 - 2mr + r^2} & 0 & 0 \\ 0 & 0 & a^2\cos(\theta)^2 + r^2 & 0 \\ -\frac{2amr\sin(\theta)^2}{a^2\cos(\theta)^2 + r^2} & 0 & 0 & \frac{2a^2mr\sin(\theta)^4 + \left(a^2r^2 + r^4 + \left(a^4 + a^2r^2\right)\cos(\theta)^2\right)\sin(\theta)^2}{a^2\cos(\theta)^2 + r^2} \end{pmatrix}$$

The Levi-Civita connection ∇ associated with g:

```
nab = g.connection() ; print nab
```

```
Levi-Civita connection 'nabla_g' associated with the Lorentzian metric
'g' on the 4-dimensional manifold 'M'
```

As a check, we verify that the covariant derivative of g with respect to ∇ vanishes identically:

```
nab(g).view()
```

$$\nabla_g g = 0$$

Killing vector

The default vector frame on the spacetime manifold is the coordinate basis associated with Boyer-Lindquist coordinates:

```
M.default_frame() is X.frame()
```

True

```
X.frame()
```

$$\left(\mathcal{M}, \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right)\right)$$

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Let us consider the first vector field of this frame:

```
xi = X.frame()[0] ; xi
```

$$\frac{\partial}{\partial t}$$

```
print xi
```

```
vector field 'd/dt' on the 4-dimensional manifold 'M'
```

The 1-form associated to it by metric duality is

```
xi_form = xi.down(g)
xi_form.set_name('xi_form', r'\underline{\xi}')
print xi_form ; xi_form.view()
```

```
1-form 'xi_form' on the 4-dimensional manifold 'M'
```

$$\underline{\xi} = \left(-\frac{a^2 \cos(\theta)^2 - 2\, mr + r^2}{a^2 \cos(\theta)^2 + r^2} \right) \mathrm{d}t + \left(-\frac{2\, amr \sin(\theta)^2}{a^2 \cos(\theta)^2 + r^2} \right) \mathrm{d}\phi$$

Its covariant derivative is

```
nab_xi = nab(xi_form)
print nab_xi ; nab_xi.view()
```

```
tensor field 'nabla_g xi_form' of type (0,2) on the 4-dimensional
manifold 'M'
```

$$\nabla_g \underline{\xi} = \left(\frac{a^2 m \cos(\theta)^2 - mr^2}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4} \right) \mathrm{d}t \otimes \mathrm{d}r + \left(\frac{2\, a^2 mr \cos(\theta) \sin(\theta)}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4} \right) \mathrm{d}t \otimes \mathrm{d}\theta + \left(-\frac{a^2 m \cos(\theta)^2 - mr^2}{a^4 \cos(\theta)^4 + 2\, a^2 r^2 \cos(\theta)^2 + r^4} \right) \mathrm{d}r \otimes \mathrm{d}t + \left(\frac{(a^3 m \cos}{a^4 \cos(\theta)} \right.$$

Let us check that the Killing equation is satisfied:

```
nab_xi.symmetrize().view()
```

$$0$$

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Equivalently, we check that the Lie derivative of the metric along ξ vanishes:

```
g.lie_der(xi).view()
```

0

Thank to Killing equation, $\nabla_g \xi$ is antisymmetric. We may therefore define a 2-form by $F := -\nabla_g \xi$. Here we enforce the antisymmetry by calling the function `antisymmetrize()` on `nab_xi`:

```
F = - nab_xi.antisymmetrize()
F.set_name('F')
print F
F.view()
```

```
2-form 'F' on the 4-dimensional manifold 'M'
```

$$F = \left(-\frac{a^2 m \cos(\theta)^2 - m r^2}{a^4 \cos(\theta)^4 + 2 a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}t \wedge \mathrm{d}r + \left(-\frac{2 a^2 m r \cos(\theta) \sin(\theta)}{a^4 \cos(\theta)^4 + 2 a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}t \wedge \mathrm{d}\theta + \left(-\frac{\left(a^3 m \cos(\theta)^2 - a m r^2\right) \sin(\theta)^2}{a^4 \cos(\theta)^4 + 2 a^2 r^2 \cos(\theta)^2 + r^4}\right) \mathrm{d}r \wedge \mathrm{d}\phi + \left(-\frac{2 (a}{a^4 \mathrm{c}}\right.$$

We check that

```
F == - nab_xi
```

True

The squared norm of the Killing vector is:

```
lamb = - g(xi,xi)
lamb.set_name('lambda', r'\lambda')
print lamb
lamb.view()
```

```
scalar field 'lambda' on the 4-dimensional manifold 'M'
```

$$\begin{array}{llcl} \lambda: & \mathcal{M} & \longrightarrow & \mathbf{R} \\ & (t, r, \theta, \phi) & \longmapsto & \frac{a^2 \cos(\theta)^2 - 2 m r + r^2}{a^2 \cos(\theta)^2 + r^2} \end{array}$$

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SageManifolds: examples | SM_Mars-Simon – Sage

Instead of invoking $g(\xi, \xi)$, we could have evaluated λ by means of the 1-form $\underline{\xi}$ acting on the vector field ξ:

```
lamb == - xi_form(xi)
```

True

or, in index notation,

```
lamb == - ( xi_form['_a']*xi['^a'] )
```

True

Curvature

The Riemann curvature tensor associated with g is:

```
Riem = g.riemann()
print Riem
```

```
tensor field 'Riem(g)' of type (1,3) on the 4-dimensional manifold 'M'
```

The component ${R^0}_{123}$ is

```
Riem[0,1,2,3]
```

$$-\frac{(a^7m-2\,a^5m^2r+a^5mr^2)\cos(\theta)\sin(\theta)^5+(a^7m+2\,a^5m^2r+6\,a^5mr^2-6\,a^3m^2r^3+5\,a^3mr^4)\cos(\theta)\sin(\theta)^3-2\,(a^7m-a^5mr^2-5\,a^3mr^4-3\,amr^6)\cos(\theta)\sin(\theta)}{a^2r^6-2\,mr^7+r^8+(a^8-2\,a^6mr+a^6r^2)\cos(\theta)^6+3\,(a^6r^2-2\,a^4mr^3+a^4r^4)\cos(\theta)^4+3\,(a^4r^4-2\,a^2mr^5+a^2r^6)\cos(\theta)^2}$$

The Ricci tensor:

```
Ric = g.ricci()
print Ric
```

```
field of symmetric bilinear forms 'Ric(g)' on the 4-dimensional manifold
'M'
```