Trapping Horizons as Inner Boundary Conditions for Black Hole Spacetimes

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Plan

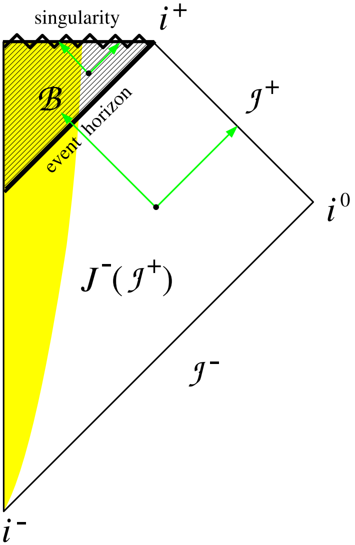
1. New horizons
2. Application to 3+1 numerical relativity

Outline

1. New horizons

2. Application to 3+1 numerical relativity

Classical definition of a black hole



black hole:

$$\mathcal{B} := \mathscr{M} - J^-(\mathscr{I}^+)$$

i.e. the region of spacetime where light rays cannot escape to infinity

- $\mathscr{M}$ = asymptotically flat manifold
- $\mathscr{I}^+$ = future null infinity
- $J^-(\mathscr{I}^+)$ = causal past of $\mathscr{I}^+$

event horizon: $\mathcal{H} := \dot{J}^-(\mathscr{I}^+)$
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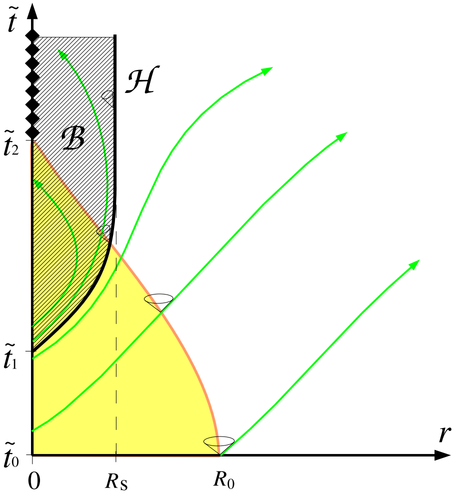

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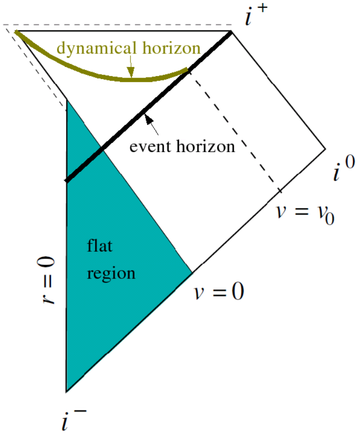
The determination of the boundary of $J^{-}(\mathscr{I}^{+})$ requires the knowledge of the entire future null infinity. Moreover this is not locally linked with the notion of strong gravitational field:



Example of event horizon in a **flat** region of spacetime:

Vaidya metric, describing incoming radiation from infinity:

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right) dv^2 + 2dv\, dr + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

with
$m(v) = 0$ for $v < 0$
$dm/dv > 0$ for $0 \leq v \leq v_0$
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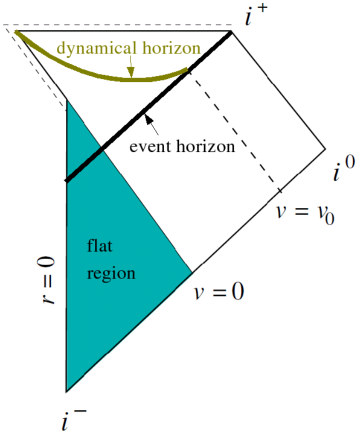

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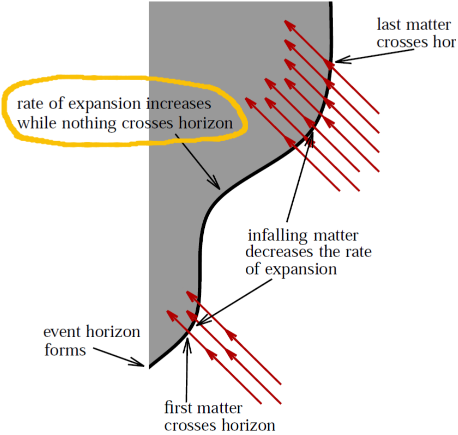
$\Rightarrow$ no local physical experiment whatsoever can locate the event horizon

Another non-local feature: teleological nature of event horizons



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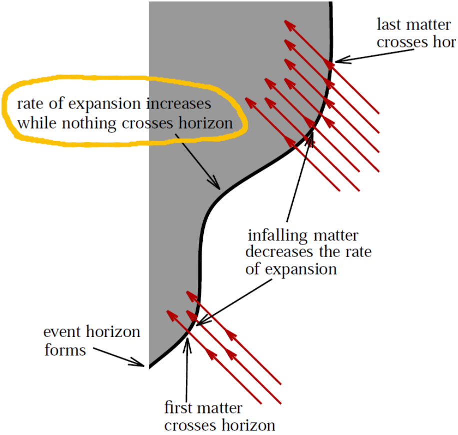

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To deal with black holes as physical objects, a local definition would be desirable

Local characterizations of black holes

Recently a **new paradigm** appeared in the theoretical approach of black holes: instead of *event horizons*, black holes are described by

- trapping horizons (Hayward 1994)
- isolated horizons (Ashtekar et al. 1999)
- dynamical horizons (Ashtekar and Krishnan 2002)

All these concepts are **local** and are based on the notion of trapped surfaces

Motivations: quantum gravity, numerical relativity

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1/ Expansion of a surface along a normal vector field

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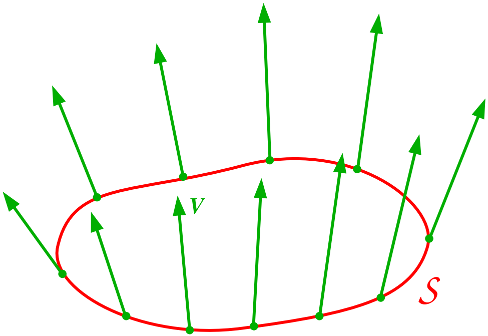

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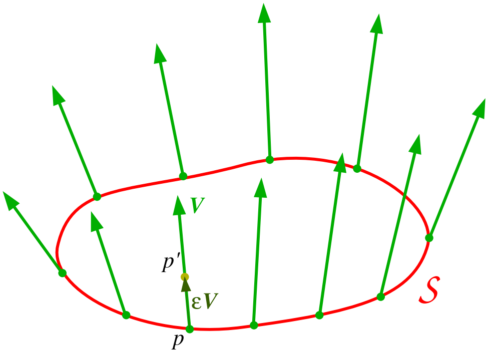

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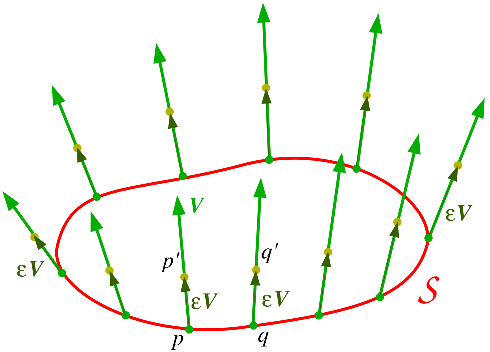

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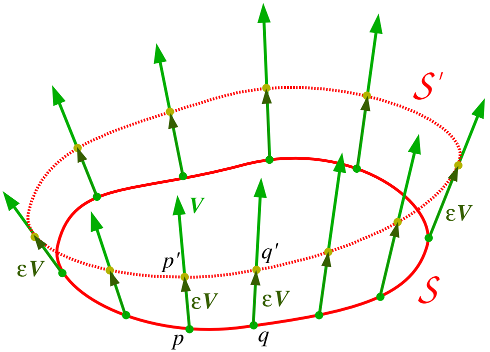

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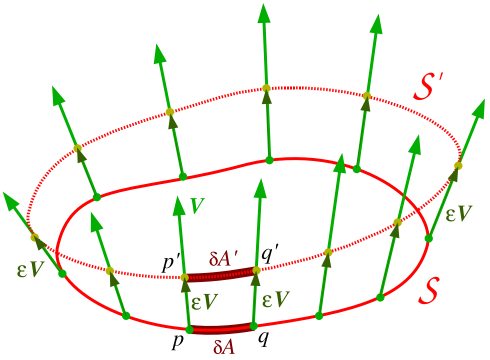

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At each point, the **expansion of $\mathcal{S}$ along $\boldsymbol{v}$** is defined from the relative change in the area element δA:

$$\theta^{(\boldsymbol{v})} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \frac{\delta A' - \delta A}{\delta A} = \mathcal{L}_{\boldsymbol{v}} \ln \sqrt{q} = q^{\mu\nu} \nabla_\mu v_\nu$$

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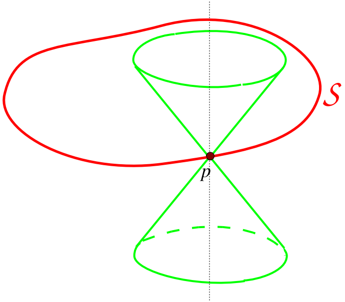

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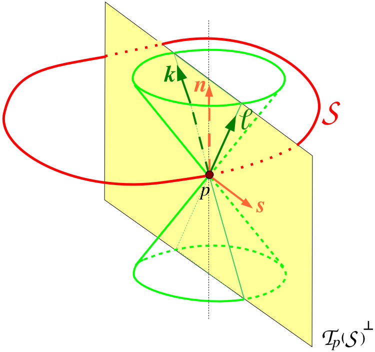

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$\exists$ two future-directed null directions orthogonal to $\mathcal{S}$:

$\boldsymbol{\ell}$ = outgoing, expansion $\theta^{(\boldsymbol{\ell})}$

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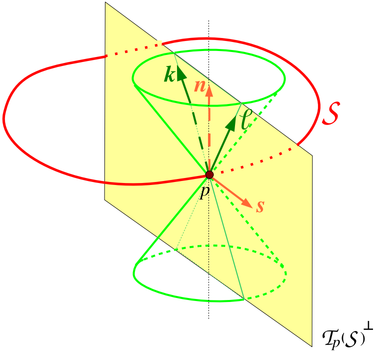

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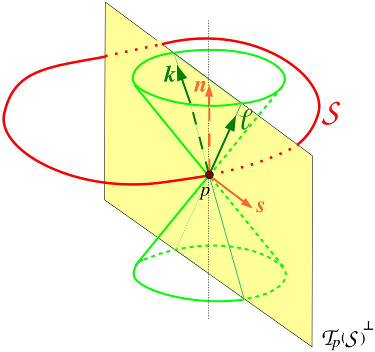

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trapped surface = **local** concept characterizing very strong gravitational fields

Link with apparent horizons

A closed spacelike 2-surface $\mathcal{S}$ is said to be outer trapped (resp. marginally outer trapped (MOTS)) iff [Hawking & Ellis 1973]

- the notions of *interior* and *exterior* of $\mathcal{S}$ can be defined (for instance spacetime asymptotically flat) $\Rightarrow$ $\boldsymbol{\ell}$ is chosen to be the *outgoing* null normal and $\boldsymbol{k}$ to be the *ingoing* one
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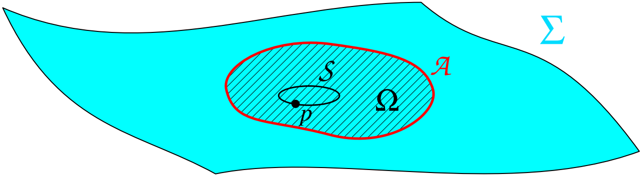


Σ: spacelike hypersurface extending to spatial infinity (Cauchy surface)

outer trapped region of Σ : Ω = set of points $p \in \Sigma$ through which there is a outer trapped surface $\mathcal{S}$ lying in Σ

apparent horizon in Σ: $\mathcal{A}$ = connected component of the boundary of Ω

Proposition [Hawking & Ellis 1973]: $\mathcal{A}$ smooth $\Longrightarrow$ $\mathcal{A}$ is a MOTS

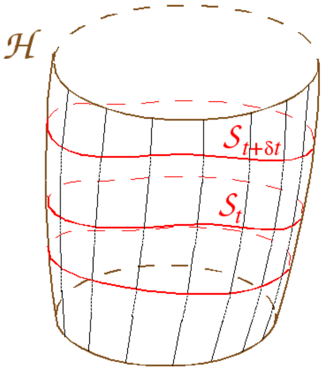
Connection with singularities and black holes

Proposition [Penrose (1965)]:
provided that the weak energy condition holds,
$\exists$ a trapped surface $\mathcal{S} \Longrightarrow \exists$ a singularity in $(\mathscr{M}, \boldsymbol{g})$ (in the form of a future inextendible null geodesic)

Proposition [Hawking & Ellis (1973)]:
provided that the cosmic censorship conjecture holds,
$\exists$ a trapped surface $\mathcal{S} \Longrightarrow \exists$ a black hole $\mathcal{B}$ and $\mathcal{S} \subset \mathcal{B}$

Local definitions of "black holes"

A hypersurface $\mathcal{H}$ of $(\mathscr{M}, \boldsymbol{g})$ is said to be



- a **future outer trapping horizon (FOTH)** iff
  (i) $\mathcal{H}$ foliated by marginally trapped 2-surfaces ($\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} = 0$)
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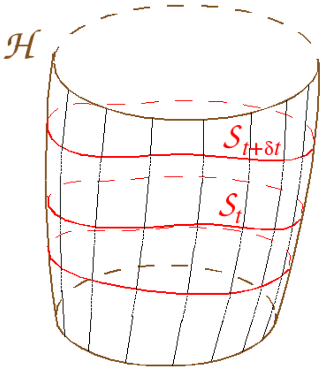

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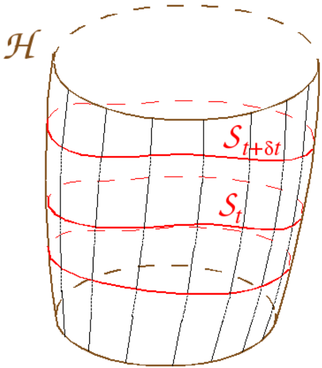

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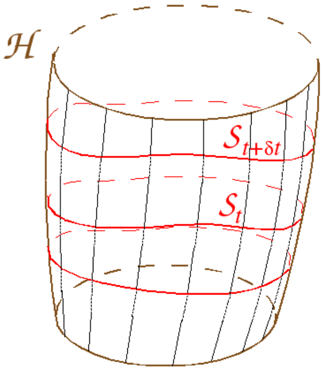

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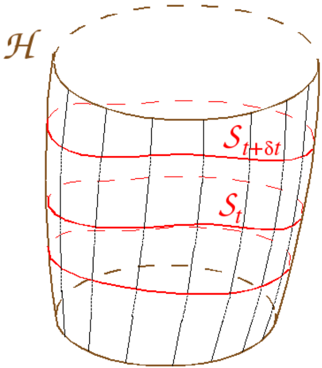

BH in equilibrium (e.g. Kerr) = IH
BH out of equilibrium = DH
generic BH = FOTH

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Dynamics of these new horizons

The *trapping horizons* and *dynamical horizons* have their **own dynamics**, ruled by Einstein equations.
In particular, one can establish for them

- existence and (partial) uniqueness theorems
  [Andersson, Mars & Simon, PRL **95**, 111102 (2005)],
  [Ashtekar & Galloway, Adv. Theor. Math. Phys. **9**, 1 (2005)]
- first and second laws of black hole mechanics
  [Ashtekar & Krishnan, PRD **68**, 104030 (2003)], [Hayward, PRD **70**, 104027 (2004)]
- a viscous fluid bubble analogy ("membrane paradigm" as for the event horizon), leading to a Navier-Stokes-like equation and a **positive** bulk viscosity *(event horizon = negative bulk viscosity)*
  [Gourgoulhon, PRD **72**, 104007 (2005)], [Gourgoulhon & Jaramillo, gr-qc/0607050]

Reviews: [Ashtekar & Krishnan, Liv. Rev. Relat. **7**, 10 (2004)], [Booth, Can. J. Phys. **83**, 1073 (2005)], [Gourgoulhon & Jaramillo, Phys. Rep. **423**, 159 (2006)]

Outline

The basic idea

Use the concepts of trapping/dynamical horizons in the very construction of a 3+1 black hole spacetime

... and not as *a posteriori* analysis tools as in e.g. [Dreyer, Krishnan, Shoemaker & Schnetter, PRD **67**, 024018 (2003)], [Schnetter, Krishnan & Beyer, gr-qc/0604015]

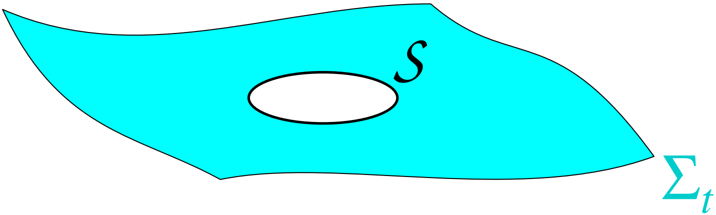
Related previous proposals *(prior to the introduction of trapping/dynamical horizons)* : use of a MOTS (apparent horizon) as inner boundary conditions for excision [Thornburg, CQG **4**, 1119 (1987)], [Eardley, PRD **57**, 2299 (1998)]

Already used for initial data (IH) *(cf. M. Ansorg's and H. Pfeiffer's talks)*

Excision

Framework: 3+1 formalism: spacetime slicing by a family $(\Sigma_t)_{t\in\mathbb{R}}$ of spacelile hypersurfaces

Excision method to deal with black holes: excise from the numerical domain a region containing the singularity



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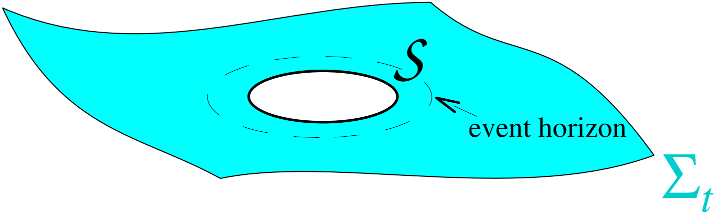


Provided that the excised region is located within the even horizon, the choice of it does not affect the exterior spacetime

Need for boundary conditions at the excision surface

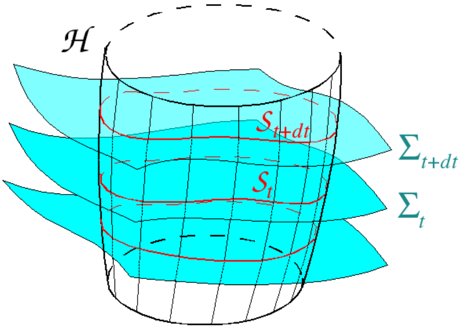
In the constrained scheme based on Dirac gauge + maximal slicing [Bonazzola, Gourgoulhon, Grandclément & Novak, PRD **70**, 104007 (2004)] *(cf. J. Novak's talk)*, boundary conditions are required for the **elliptic equations** governing

- the conformal factor Ψ
- the lapse function N
- the shift vector $\boldsymbol{\beta}$

NB: no need of boundary conditions for the metric potentials $h^{ij} := \tilde{\gamma}^{ij} - f^{ij}$ [I. Cordero Carrión (2006)]

Trapping horizon inner boundary

Choose the excision boundary $\mathcal{S}_t$ to be a **marginally trapped surface** for each time t



The tube $\mathcal{H} = \bigcup_{t\in\mathbb{R}} \mathcal{S}_t$

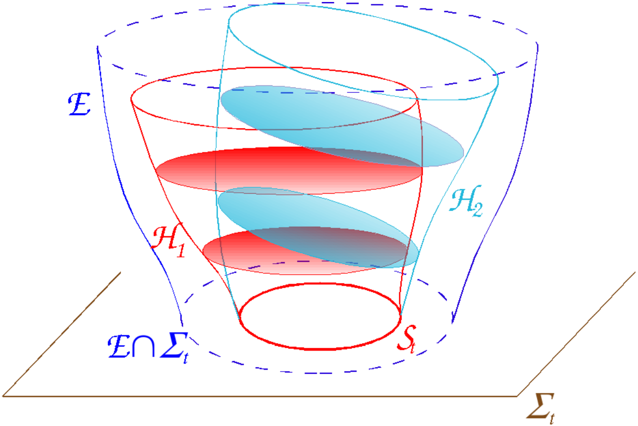
is then generically a smooth **trapping horizon**

[Andersson, Mars & Simon, PRL **95**, 111102 (2005)]

- geometrically well defined excision boundary
- ensures $\mathcal{S}_t$ is located inside the event horizon reminder
- easy to implement with spherical coordinates and spectral methods

Non-uniqueness of trapping horizons

Different 3+1 slicings may lead to different trapping horizons



NB: uniqueness in spherical symmetry

Geometrical setup

Hypersurface Σ_t :

- induced metric $\boldsymbol{\gamma}$ (positive definite); associated connection $\boldsymbol{D}$
- future directed timelike unit normal $\boldsymbol{n}$
- extrinsic curvature $\boldsymbol{K}$: $K_{\alpha\beta} = -\nabla_\mu n_\alpha \, \gamma^\mu{}_\beta$
- lapse function N : $\underline{\boldsymbol{n}} = -N\mathbf{d}t$

Geometrical setup

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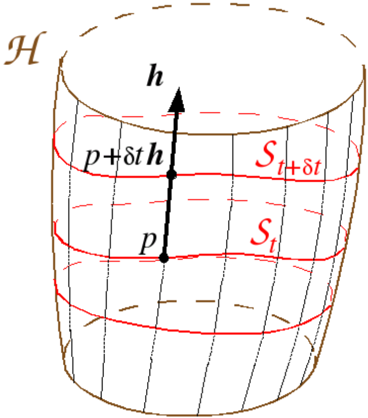
2-surface $\mathcal{S}_t$:

- induced metric $\boldsymbol{q}$ (positive definite); associated connection $\boldsymbol{\mathcal{D}}$
- normal vector pairs (basis of $\mathcal{T}_p(\mathcal{S}_t)^\perp$): see figure
  - orthonormal basis $(\boldsymbol{n}, \boldsymbol{s})$, where $\boldsymbol{s}$ is the outgoing spacelike unit normal to $\mathcal{S}_t$ in Σ_t
  - null basis $(\boldsymbol{\ell}, \boldsymbol{k})$ (not unique: $\boldsymbol{\ell} \mapsto \boldsymbol{\ell}' = \lambda \boldsymbol{\ell}$, $\boldsymbol{k} \mapsto \boldsymbol{k}' = \mu \boldsymbol{k}$)
- extrinsic curvature, as a hypersurface of Σ_t, $\boldsymbol{H}$: $H_{\alpha\beta} = D_\mu s_\alpha \, q^\mu{}_\beta$

Privileged evolution vector on $\mathcal{H}$



Vector field $\boldsymbol{h}$ on $\mathcal{H}$ defined by

- (i) $\boldsymbol{h}$ is tangent to $\mathcal{H}$
- (ii) $\boldsymbol{h}$ is orthogonal to $\mathcal{S}_t$
- (iii) $\mathcal{L}_{\boldsymbol{h}}\, t = h^\mu \partial_\mu t = \langle \mathbf{d}t, \boldsymbol{h}\rangle = 1$

NB: (iii) $\Longrightarrow$ the 2-surfaces $\mathcal{S}_t$ are Lie-dragged by $\boldsymbol{h}$

$\boldsymbol{h} \in \mathcal{T}_p(\mathcal{S}_t)^\perp = \mathrm{Vect}(\boldsymbol{n}, \boldsymbol{s})$ and can be decomposed as $\boxed{\boldsymbol{h} = N\boldsymbol{n} + b\boldsymbol{s}}$

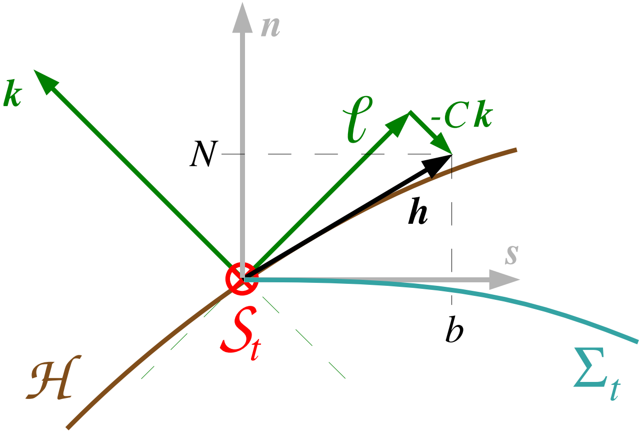
Norm of $\boldsymbol{h}$ and type of $\mathcal{H}$

Definition: $C := \frac{1}{2}\boldsymbol{h}\cdot\boldsymbol{h} = \frac{1}{2}\left(b^2 - N^2\right)$

$$
\begin{array}{lllllll}
\mathcal{H} \text{ is spacelike (DH)} & \Longleftrightarrow & \boldsymbol{h} \text{ is spacelike} & \Longleftrightarrow & C > 0 & \Longleftrightarrow & b > N \\
\mathcal{H} \text{ is null (IH)} & \Longleftrightarrow & \boldsymbol{h} \text{ is null} & \Longleftrightarrow & C = 0 & \Longleftrightarrow & b = N \\
\mathcal{H} \text{ is timelike} & \Longleftrightarrow & \boldsymbol{h} \text{ is timelike} & \Longleftrightarrow & C < 0 & \Longleftrightarrow & b < N.
\end{array}
$$

Null basis associated with $\boldsymbol{h}$

The vectors $\boldsymbol{\ell} := \frac{1}{2}(b+N)\,(\boldsymbol{n}+\boldsymbol{s})$ and $\boldsymbol{k} := \frac{1}{b+N}\,(\boldsymbol{n}-\boldsymbol{s})$ are the unique pair of null vectors normal to $\mathcal{S}_t$ such that $\boldsymbol{\ell}\cdot\boldsymbol{k} = -1$ and $\boxed{\boldsymbol{h} = \boldsymbol{\ell} - C\boldsymbol{k}}$



Spatial coordinates

Coordinates $(x^i)_{i\in\{1,2,3\}}$ on Σ_t $\Rightarrow$ defines the shift vector $\boldsymbol{\beta}$: $\boxed{\boldsymbol{\partial}_t = N\boldsymbol{n} + \boldsymbol{\beta}}$

2+1 orthogonal decomposition of the shift with respect to $\mathcal{S}_t$:

$\boldsymbol{\beta} = \beta^\perp \boldsymbol{s} - \boldsymbol{V}$ with $\boldsymbol{s}\cdot\boldsymbol{V} = 0$.

The coordinates (t, x^i) are comoving w.r.t. $\mathcal{H}$ iff there exists a function f not depending on t and such that

$\forall p = (t, x^1, x^2, x^3) \in \mathscr{M},\ p \in \mathcal{H} \iff f(x^1, x^2, x^3) = 0$

Special case: adapted coordinates: $f = f(x^1)$

Coordinates (t, x^i) comoving w.r.t. $\mathcal{H}$ $\iff$ $\boldsymbol{\partial}_t$ tangent to $\mathcal{H}$

$\iff$ $\boxed{\beta^\perp = b}$

$\iff$ $\boxed{\boldsymbol{h} = \boldsymbol{\partial}_t + \boldsymbol{V}}$

Condition $\theta^{(\ell)} = 0$ on $\mathcal{S}_t$

Preliminary: 2+1 orthogonal decomposition of the extrincic curvature of Σ_t:

$$\boldsymbol{K} = \underbrace{-\boldsymbol{\sigma}^{(\boldsymbol{n})} - \frac{1}{2}\theta^{(\boldsymbol{n})}\boldsymbol{q}}_{\text{part tangent to } \mathcal{S}_t} + \underbrace{\underline{\boldsymbol{s}} \otimes \boldsymbol{L} + \boldsymbol{L} \otimes \underline{\boldsymbol{s}}}_{\text{mixed part}} + \underbrace{\boldsymbol{K}(\boldsymbol{s},\boldsymbol{s})\, \underline{\boldsymbol{s}} \otimes \underline{\boldsymbol{s}}}_{\text{normal part}}$$

with $\mathrm{tr}\, \boldsymbol{\sigma}^{(\boldsymbol{n})} = 0$ (shear of $\mathcal{S}_t$ along $\boldsymbol{n}$) and $\boldsymbol{L} := \boldsymbol{K}(\boldsymbol{s}, \vec{\boldsymbol{q}})$

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One has $\theta^{(\ell)} = \frac{1}{2}(b+N)\left[\theta^{(\boldsymbol{n})} + \theta^{(\boldsymbol{s})}\right]$.

Since $\theta^{(\boldsymbol{n})} = \boldsymbol{K}(\boldsymbol{s},\boldsymbol{s}) - K$ (see above) and $\theta^{(\boldsymbol{s})} = H = \boldsymbol{D}\cdot\boldsymbol{s}$,

we get $\theta^{(\ell)} = \frac{1}{2}(b+N)\left[\boldsymbol{D}\cdot\boldsymbol{s} + \boldsymbol{K}(\boldsymbol{s},\boldsymbol{s}) - K\right]$.

Hence the well known marginally trapped surface condition:

$$\theta^{(\ell)} = 0 \iff \boxed{\boldsymbol{D}\cdot\boldsymbol{s} + \boldsymbol{K}(\boldsymbol{s},\boldsymbol{s}) - K = 0}$$

which yields, in a conformal decomposition ($\boldsymbol{\gamma} = \Psi^4 \tilde{\boldsymbol{\gamma}}$),

$$\boxed{4\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{D}}\Psi + \boldsymbol{K}(\tilde{\boldsymbol{s}},\tilde{\boldsymbol{s}})\Psi^{-2} - K\Psi^2 + \tilde{\boldsymbol{D}}\cdot\tilde{\boldsymbol{s}} = 0} \qquad (1)$$

Condition $\mathcal{L}_h \theta^{(\ell)} = 0$ on $\mathcal{S}_t$

i.e. not only $\mathcal{S}_t$ is a marginally trapped surface at time t, but remains marginally trapped at time $t + \delta t$:
Thanks to Einstein equation, the condition $\mathcal{L}_{\boldsymbol{h}} \theta^{(\boldsymbol{\ell})} = 0$, along with $\theta^{(\boldsymbol{\ell})} = 0$, is equivalent to [Eardley, PRD **57**, 2299 (1998)]

$$\boxed{-\mathcal{D}_a \mathcal{D}^a (b - N) - 2 L^a \mathcal{D}_a (b - N) + A(b - N) = B(b + N)} \qquad (2)$$

with
$$L_a := K_{ij} s^i q^j{}_a$$
$$A := \frac{1}{2}\mathcal{R} - \mathcal{D}_a L^a - L_a L^a - 4\pi T_{\mu\nu}(n^\mu + s^\mu)(n^\nu - s^\nu)$$
$\mathcal{R}$: Ricci scalar of the metric $\boldsymbol{q}$ on $\mathcal{S}_t$
$$B := \frac{1}{2}\hat{\sigma}_{ab}\hat{\sigma}^{ab} + 4\pi T_{\mu\nu}(n^\mu + s^\mu)(n^\nu + s^\nu)$$
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Remark: for an isolated horizon, $B = 0$ and the solution to Eq. (2) is $b - N = 0$, which, in comoving coordinates w.r.t. $\mathcal{H}$, translates to $\beta^\perp = N$ *(cf. H. Pfeiffer's talk)*

BC for the tangential part of the shift vector

Recall: $\boldsymbol{\beta} = \beta^{\perp}\boldsymbol{s} - \boldsymbol{V}$ and in comoving coord. w.r.t. $\mathcal{H}$, $\beta^{\perp} = b$ & $\boldsymbol{h} = \boldsymbol{\partial}_t + \boldsymbol{V}$

shear tensor $\boldsymbol{\sigma}^{(\boldsymbol{h})}$ of the surface $\mathcal{S}_t$ along its evolution = traceless part of the deformation tensor of $\mathcal{S}_t$: $\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\,\boldsymbol{q} =: \theta^{(\boldsymbol{h})}\boldsymbol{q} + 2\boldsymbol{\sigma}^{(\boldsymbol{h})}$

In comoving coord. $2\sigma^{(\boldsymbol{h})}_{ab} = \dfrac{\partial q_{ab}}{\partial t} - \dfrac{\partial}{\partial t}\ln\sqrt{q}\; q_{ab} + \mathcal{D}_a V_b + \mathcal{D}_b V_a - \mathcal{D}_c V^c\, q_{ab}$

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Demand: the components of the metric on $\mathcal{S}_t$ vary as less as possible, i.e. vary only to reflect the expansion of $\mathcal{S}_t$:

$$\frac{\partial q_{ab}}{\partial t} - \frac{\partial}{\partial t} \ln\sqrt{q}\, q_{ab} = 0 \iff \boxed{\mathcal{D}_a V_b + \mathcal{D}_b V_a - \mathcal{D}_c V^c\, q_{ab} = 2\sigma^{(\boldsymbol{h})}_{ab}} \qquad (3)$$

$\sigma^{(\boldsymbol{h})}_{ab}$ being determined via the evolution equation

$$\boldsymbol{\mathcal{L}_h\, \sigma}^{(\boldsymbol{h})} = -\vec{\boldsymbol{q}}^{*}\mathbf{Weyl}(\underline{\boldsymbol{\ell}}, ., \boldsymbol{\ell}, .) - C^2\vec{\boldsymbol{q}}^{*}\mathbf{Weyl}(\underline{\boldsymbol{k}}, ., \boldsymbol{k}, .) - 8\pi C\left[\vec{\boldsymbol{q}}^{*}\boldsymbol{T} - \frac{1}{2}(\boldsymbol{q} : \boldsymbol{T})\boldsymbol{q}\right] + \cdots$$

Remark: for an isolated horizon, $\boldsymbol{\sigma}^{(\boldsymbol{h})} = 0$ and Eq. (3) says that $\boldsymbol{V}$ must be a conformal Killing vector of $(\mathcal{S}_t, \boldsymbol{q})$ *(cf. H. Pfeiffer's talk)*

Choice of the 3+1 slicing

NB1: The trapping horizon condition by itself does specify the value of the lapse N, but only of the combination of $b - N$ and $b + N$ which appears in Eq. (2). Given an initial marginally trapped surface $\mathcal{S}_0 \subset \Sigma_0$, the choice of b and N on $\mathcal{S}_0$ determines a unique trapping horizon among all those which intersects Σ_0 in $\mathcal{S}_0$.

NB2: The 3+1 slicing $(\Sigma_t)_{t\in\mathbb{R}}$ is determined by
(i) a condition "in the bulk" (e.g. maximal slicing)
(ii) the value of the lapse on $\mathcal{S}_t$
In other words, (i) is not sufficient to specify uniquely the 3+1 slicing.

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In other words, (i) is not sufficient to specify uniquely the 3+1 slicing.

Having chosen (i), one can use (ii) to select a trapping horizon $\mathcal{H}$ with "good" properties.
For instance, we can demand that the area $A(t)$ of each section $\mathcal{S}_t$ is maximal [Gourgoulhon & Jaramillo, gr-qc/0607050]. This translate into

$$\boxed{b - N = \alpha\, \boldsymbol{D}\cdot\boldsymbol{s}}, \quad \alpha = \text{const.} \tag{4}$$

Other choices, based on the convexity of $A(t)$, are possible [gr-qc/0607050]

Summary

The trapping horizon conditions + some coordinate choice lead to **5** equations to set the values of the **5** fields $\Psi, N, \beta^1, \beta^2, \beta^3$ at the excision surface $\mathcal{S}_t$:

- *trapping horizon conditions:*
  - $\theta^{(\ell)} = 0 \Longrightarrow \boxed{\Psi}$ [Eq. (1)]
  - $\mathcal{L}_{\boldsymbol{h}}\, \theta^{(\ell)} = 0 \Longrightarrow \boxed{f_1(b - N,\ b + N)}$ [Eq. (2)]
- *coordinate choice:*
  - comoving coordinates w.r.t. $\mathcal{H}$: $\boxed{\beta^{\perp} = b}$
  - traceless part of $\dfrac{\partial q_{ab}}{\partial t} = 0 \Longrightarrow \boxed{\boldsymbol{V}}$ [Eq. (3)]
  - choice of slicing/lapse $\Longrightarrow \boxed{f_2(b - N,\ b + N)}$ [Eq. (4)]

General Relativity Trimester

Gravitational Waves, Relativistic Astrophysics and Cosmology

Centre Emile Borel

INSTITUT HENRI POINCARE

Paris

18 September - 15 December 2006

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Workshop

From Geometry to Numerics
IHP, Paris
20-24 November 2006

Ultra-preliminary list of speakers:

M. Ansorg
Y. Choquet-Bruhat
P. Chrusciel
J. Frauendiener
S. Hayward (tbc)
J. Isenberg
S. Klainerman
B. Krishnan
V. Moncrief
N. O' Murchadha
D. Pollack
J. York (tbc)