Multidomain spectral methods based on spherical coordinates for numerical relativity

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ICOSAHOM 2004 (Brown University, Providence, USA, 21-25 June 2004)

Plan

1. General features of spectral methods developed in Meudon

2. Resolution of elliptic equations: the initial value problem of general relativity

3. Resolution of tensorial wave equations: spacetime dynamics

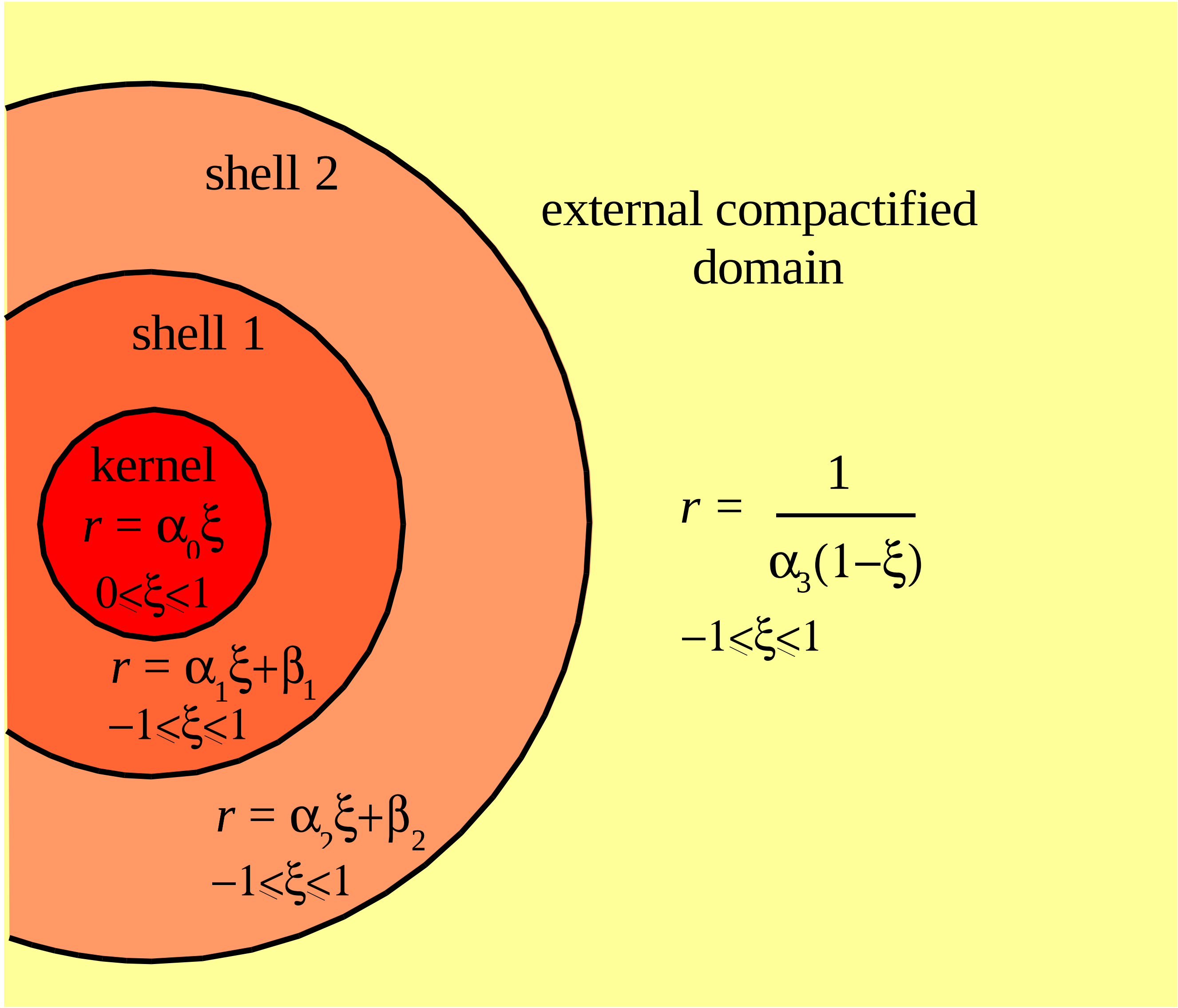
1

General features of spectral methods developed in Meudon

An overview

- Multidomain three-dimensional spectral method
- Spherical-type coordinates (r, θ, φ)
- Expansion functions: r : Chebyshev; θ : cosine/sine or associated Legendre functions; φ : Fourier
- Domains $=$ spherical shells $+$ 1 nucleus (contains $r = 0$)
- Entire space ($\mathbb{R}^3$) covered: compactification of the outermost shell
- Adaptative coordinates : domain decomposition with spherical topology
- Multidomain PDEs: patching method (strong formulation)
- Treatment of non-linear terms: pseudospectral method

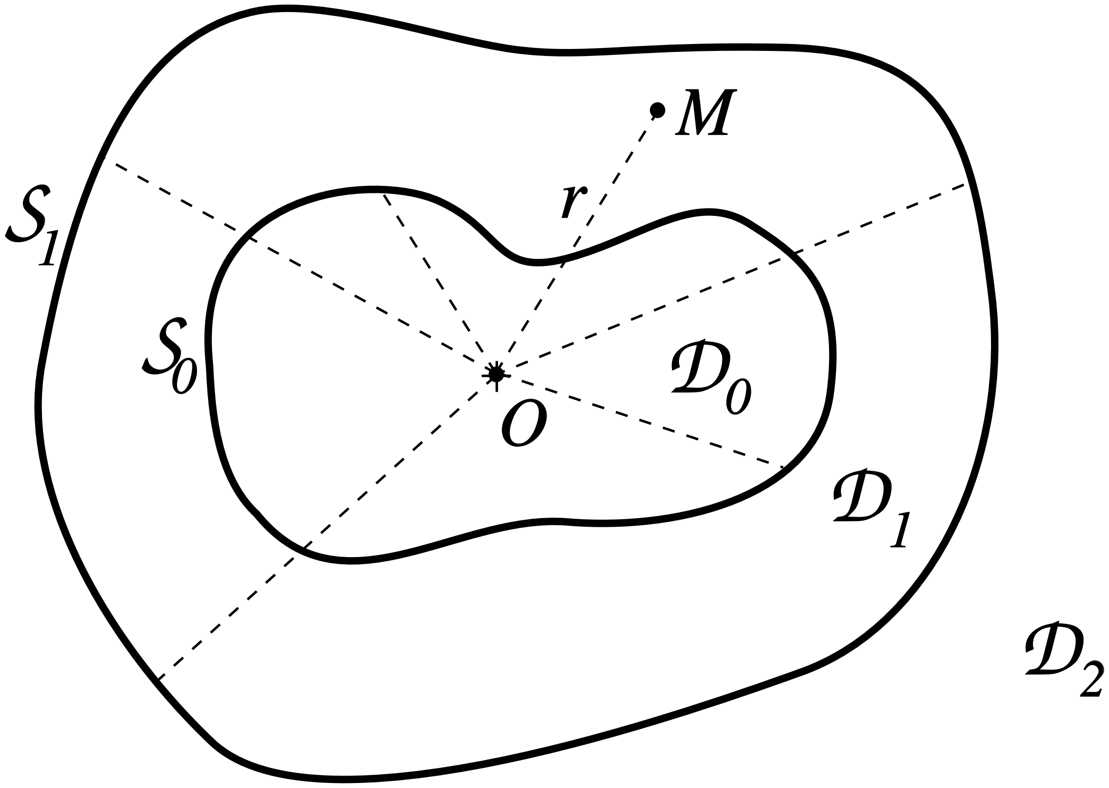
Domain decomposition



physical coordinates
(r, θ, φ)

comput. coordinates
(ξ, θ', φ')

Starlike domain decomposition



$\mathcal{N}$ nonoverlapping starlike domains:

- $\mathcal{D}_0$: *nucleus*
- $\mathcal{D}_q \quad (1 \leq q \leq \mathcal{N} - 2)$: *shell*
- $\mathcal{D}_{\mathcal{N}-1}$: *external domain*

$$\mathcal{D}_0 \cup \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_{\mathcal{N}-1} = \mathbb{R}^3$$

Mapping computational space → physical space

Mapping for domain $\mathcal{D}_q$:
$$\begin{array}{rcl} [-1+\delta_{0q},1]\times[0,\pi]\times[0,2\pi[& \longrightarrow & \mathcal{D}_q \\ (\xi,\theta',\varphi') & \longmapsto & (r,\theta,\varphi) \end{array}$$

Radial mapping : $\theta = \theta'$ and $\varphi = \varphi'$

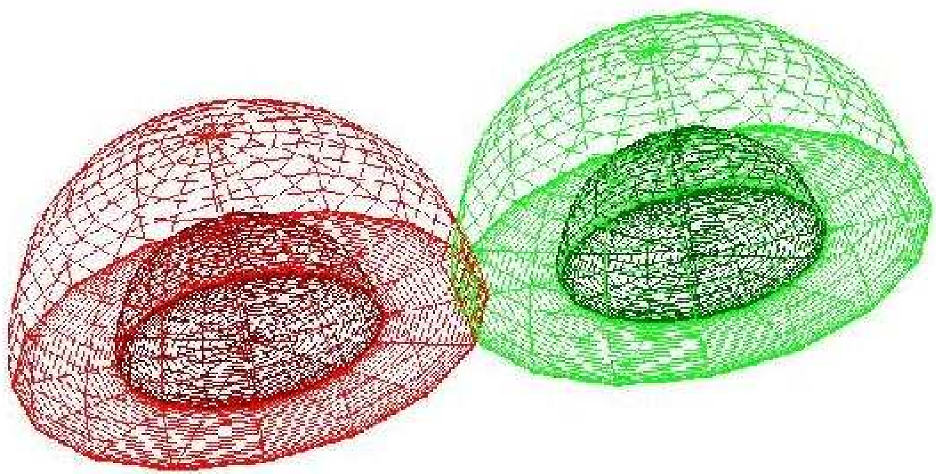
- in the nucleus: $\xi \in [0,1]$
$$r = \alpha_0 \left[\xi + \left(3\xi^4 - 2\xi^6\right) F_0(\theta,\varphi) + \frac{1}{2}\left(5\xi^3 - 3\xi^5\right) G_0(\theta,\varphi) \right]$$

- in the shells: $\xi \in [-1,1]$
$$r = \alpha_q \left[\xi + \frac{1}{4}\left(\xi^3 - 3\xi + 2\right) F_q(\theta,\varphi) + \frac{1}{4}\left(-\xi^3 + 3\xi + 2\right) G_q(\theta,\varphi) \right] + \beta_q$$

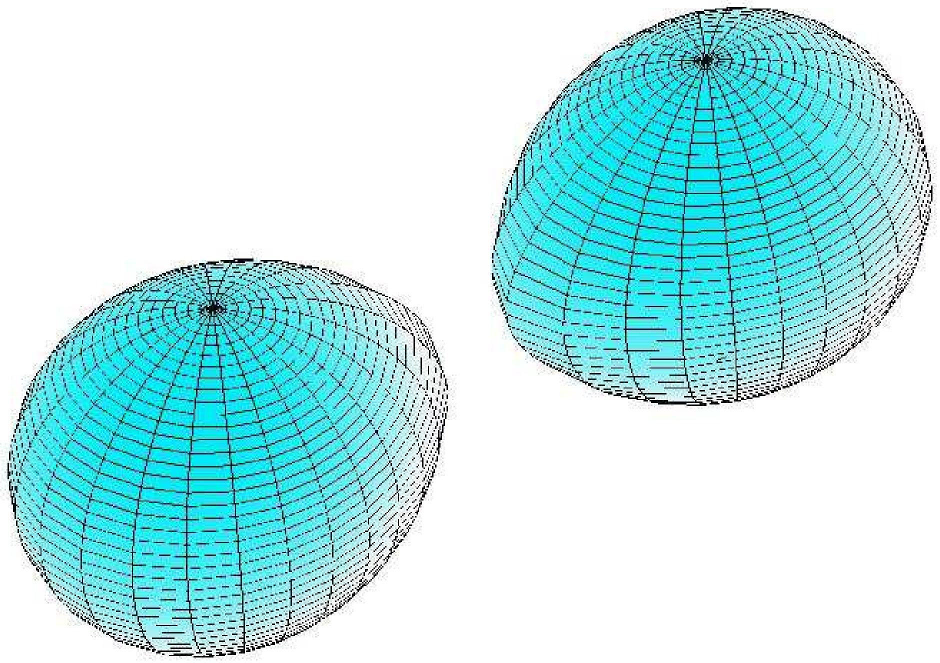
- in the external domain: $\xi \in [-1,1]$
$$\frac{1}{r} = \alpha_{\rm ext} \left[\xi + \frac{1}{4}\left(\xi^3 - 3\xi + 2\right) F_{\rm ext}(\theta,\varphi) - 1 \right]$$

[Bonazzola, Gourgoulhon & Marck, Phys. Rev. D **58**, 104020 (1998)]

Example: binary star with surface fitted coordinates



Double domain decomposition



[Taniguchi, Gourgoulhon & Bonazzola, Phys. Rev. D **64**, 064012 (2001)]

Surface fitted coordinates:

$F_0(\theta, \varphi)$ and $G_0(\theta, \varphi)$ chosen so that

$\xi = 1 \Leftrightarrow$ surface of the star

Basis functions

Polynomial interpolant of a field u in a given domain $\mathcal{D}_q$:

$$I_N\, u_q(\xi,\theta,\varphi) = \sum_{m=0}^{N_\varphi/2} \sum_{j=0}^{N_\theta-1} \sum_{i=0}^{N_r-1} \hat{u}_{qmji}\; X_i(\xi)\; \Theta_j(\theta)\; e^{im\varphi} \qquad \text{with } N := (N_r, N_\theta, N_\varphi)$$

Regularity at the origin and on the axis $\theta = 0$ + equatorial symmetry:

- φ expansion: **Fourier series**
- θ expansion: **Trigonometric polynomials** or **associated Legendre functions**
  - ⋆ for m even: $\Theta_j(\theta) = \cos(2j\theta)$ or $\Theta_j(\theta) = P_{2j}^m(\cos\theta)$
  - ⋆ for m odd: $\Theta_j(\theta) = \sin((2j+1)\theta)$ or $\Theta_j(\theta) = P_{2j+1}^m(\cos\theta)$
- ξ expansion: **Chebyshev polynomials**
  - ⋆ in the kernel: $X_i(\xi) = T_{2i}(\xi)$ for m even, $X_i(\xi) = T_{2i+1}(\xi)$ for m odd
  - ⋆ in the shells and the external compactified domain: $X_i(\xi) = T_i(\xi)$

Numerical implementation: LORENE

Langage **O**bjet pour la **RE**lativite **N**umeriqu**E**

A library of C++ classes devoted to multi-domain spectral methods, with adaptive spherical coordinates.

- 1997 : start of Lorene project (Jean-Alain Marck, EG)
- 1999 : Accurate models of rapidly rotating strange quark stars
- 1999 : Neutron star binaries on closed circular orbits
- 2001 : Public domain (GPL), Web page: `http://www.lorene.obspm.fr`
- 2001 : Black hole binaries on closed circular orbits
- 2002 : 3-D wave equation with non-reflecting boundary conditions
- 2002 : Maclaurin-Jacobi bifurcation point in general relativity
- 2004 : 3-D time evolution of Einsteins equations

2

Resolution of elliptic equations: the initial value problem of general relativity

Resolution of Poisson equation with noncompact source

Consider the three-dimensional Poisson equation on $\mathbb{R}^3$:

$$\Delta u(r,\theta,\varphi) = s(r,\theta,\varphi) \tag{1}$$

with the boundary condition

$$u(r,\theta,\varphi) \to 0 \quad \text{when } r \to +\infty \tag{2}$$

The source s has a non-compact support and obeys to the fall-off conditions

$$s(r,\theta,\varphi) \sim \sum_{q=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} \frac{Y_\ell^m(\theta,\varphi)}{r^{\ell+4}} \quad \text{when } r \to +\infty \tag{3}$$

Spherical harmonics expansions

Interpolant of the source in a domain $\mathcal{D}_q$ (notation:$s_q := s|_{\mathcal{D}_q}$) :

$$I_N s_q(\xi, \theta, \varphi) = \sum_{\ell=0}^{N_\theta - 1} \sum_{m=-\ell}^{\ell} \hat{s}_{q\ell m}(\xi) \, Y_\ell^m(\theta, \varphi)$$

Search for a numerical solution under the form

$$\bar{u}_q(\xi, \theta, \varphi) = \sum_{\ell=0}^{N_\theta - 1} \sum_{m=-\ell}^{\ell} \hat{u}_{q\ell m}(\xi) \, Y_\ell^m(\theta, \varphi)$$

Shorthand notation: $u_\bullet(\xi) := \hat{u}_{q\ell m}(\xi)$.

Eq. (1) becomes an ODE system:

- In the nucleus ($r = \alpha\xi$) :

$$\frac{d^2 u_\bullet}{d\xi^2} + \frac{2}{\xi}\left(\frac{du_\bullet}{d\xi} - \frac{du_\bullet}{d\xi}(0)\right) - \frac{\ell(\ell+1)}{\xi^2}\left(u_\bullet - u_\bullet(0) - \xi\frac{du_\bullet}{d\xi}(0)\right) = \alpha^2 \hat{s}_{0\ell m}(\xi)$$

- In the shells ($r = \alpha\xi + \beta$):

$$\left(\xi + \frac{\beta}{\alpha}\right)^2 \frac{d^2 u_\bullet}{d\xi^2} + 2\left(\xi + \frac{\beta}{\alpha}\right)\frac{du_\bullet}{d\xi} - \ell(\ell+1)u_\bullet = (\alpha\xi + \beta)^2 \hat{s}_{q\ell m}(\xi)$$

- In the external domain ($r^{-1} = \alpha(\xi - 1)$) :

$$\frac{d^2 u_\bullet}{d\xi^2} - \frac{\ell(\ell+1)}{(\xi-1)^2}\left(u_\bullet - u_\bullet(1) - (\xi-1)\frac{du_\bullet}{d\xi}(1)\right) = \frac{\hat{s}_{q\ell m}(\xi)}{\alpha^4(\xi-1)^4}$$

Resolution by means of a Chebyshev tau method

- In the nucleus : $$u_\bullet(\xi) = \sum_{i=0}^{N_r-1} \hat{u}_{q\ell mi}\, T_{2i}(\xi) \text{ for } \ell \text{ even}$$
$$u_\bullet(\xi) = \sum_{i=0}^{N_r-2} \hat{u}_{q\ell mi}\, T_{2i+1}(\xi) \text{ for } \ell \text{ odd}$$

- In the shells and external domain : $u_\bullet(\xi) = \sum_{i=0}^{N_r-1} \hat{u}_{q\ell mi} T_i(\xi)$

Linear combinations → banded matrices (5 bands)

Patching method

Number of solutions of the homogeneous equation:

- In the nucleus : 1 (r^ℓ)
- In the shells : 2 (r^ℓ and $r^{-(\ell+1)}$)
- In the external domain : 1 ($r^{-(\ell+1)}$)

Total : $1 + 2(\mathcal{N} - 2) + 1 = 2\mathcal{N} - 2$

Matching conditions: continuity of u and its first radial derivative accross the $\mathcal{N} - 1$ boundaries between the domains $\mathcal{D}_q \Longrightarrow 2\mathcal{N} - 2$ conditions

Behavior of the numerical error

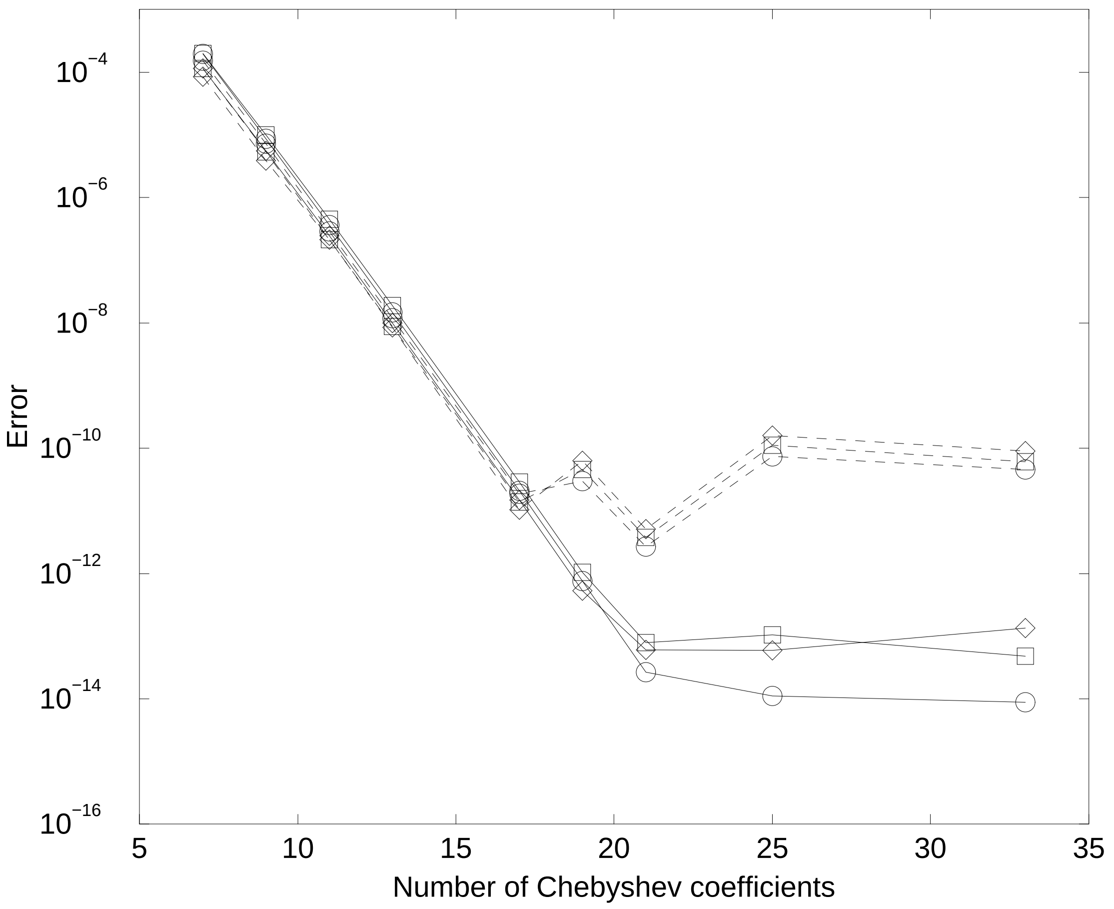
Source with a non-compact support, decaying as r^{-k}:

- evanescent error (error $\propto \exp(-N_r)$) if the source does not contain any spherical harmonics of index $\ell \geq k-3$
- error decreasing as $N^{-2(k-2)}$ otherwise

[Grandclément, Bonazzola, Gourgoulhon & Marck, J. Comp. Phys. **170**, 231 (2001)]

Extension to vector Poisson-type equations

Minimal distortion equation for the shift vector: $\Delta\vec{\beta} + \frac{1}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{\beta}) = \vec{S}$

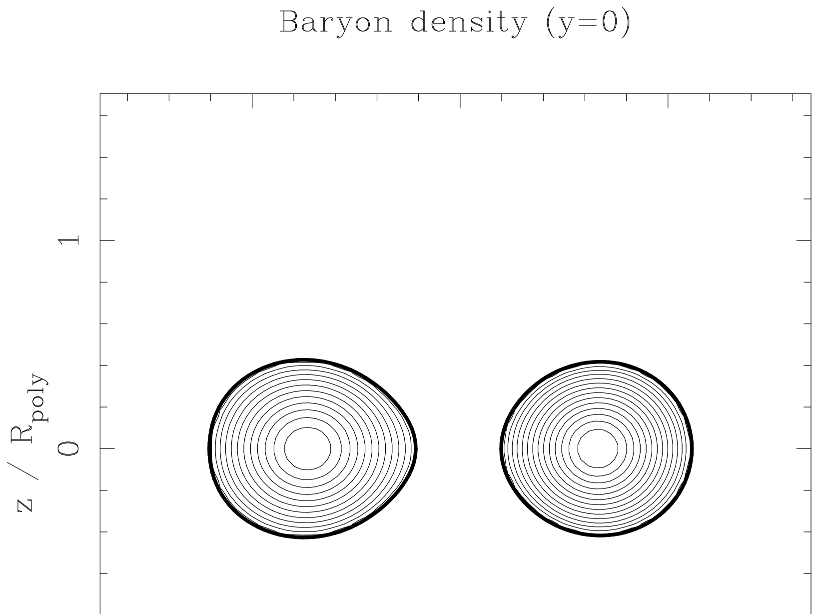


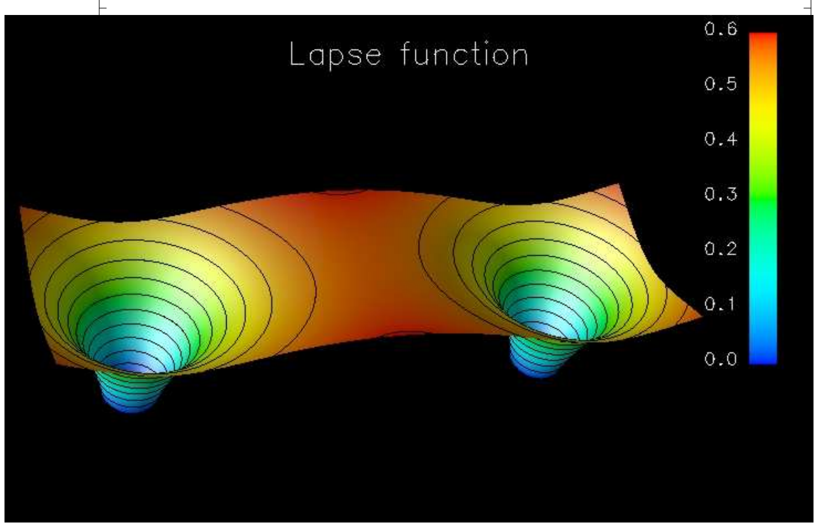
Error on the z component of the solution of the minimal distortion equation with a non-compact source

[Grandclément, Bonazzola, Gourgoulhon & Marck, J. Comp. Phys. **170**, 231 (2001)]

Application to the Cauchy data for 3+1 numerical relativity

Quasi-equilibrium sequences of orbiting binary black holes and neutrons stars





Initial data within the conformal thin sandwich framework: a set of two scalar and one vectorial elliptic equations (conformal factor Ψ, lapse function N and shift vector $\boldsymbol{\beta}$).

← **binary neutron star system** ($M/R = 0.16$ and $M/R = 0.18$, EOS $\gamma = 2.5$)

[Taniguchi & Gourgoulhon, PRD **68**, 124025 (2003)]

← **binary black hole system**

[Grandclément, Gourgoulhon, Bonazzola, PRD **65**, 044021 (2002)]

3

Resolution of tensorial wave equations: spacetime dynamics

Scalar wave equation with nonreflecting boundary conditions

Consider the wave equation

$$\Box\, u(t,r,\theta,\varphi) = s(t,r,\theta,\varphi) \tag{4}$$

with the radiating boundary condition

$$\lim_{r\to\infty} \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial t} \right) (r\, u) = 0. \tag{5}$$

Solve (4) in a finite ball $\mathcal{D}$ of radius R with some boundary conditions which approximate (5) when $R \to \infty$.

Decompose $\mathcal{D}$ in $\mathcal{N}$ spherical subdomains $\mathcal{D}_q$ with $\mathcal{D}_0 =$ nucleus and the other domains $=$ shells (no external compactified domain).

Finite-differencing in time: second-order implicit Crank-Nicolson scheme.
Space part: patching with Chebyshev tau

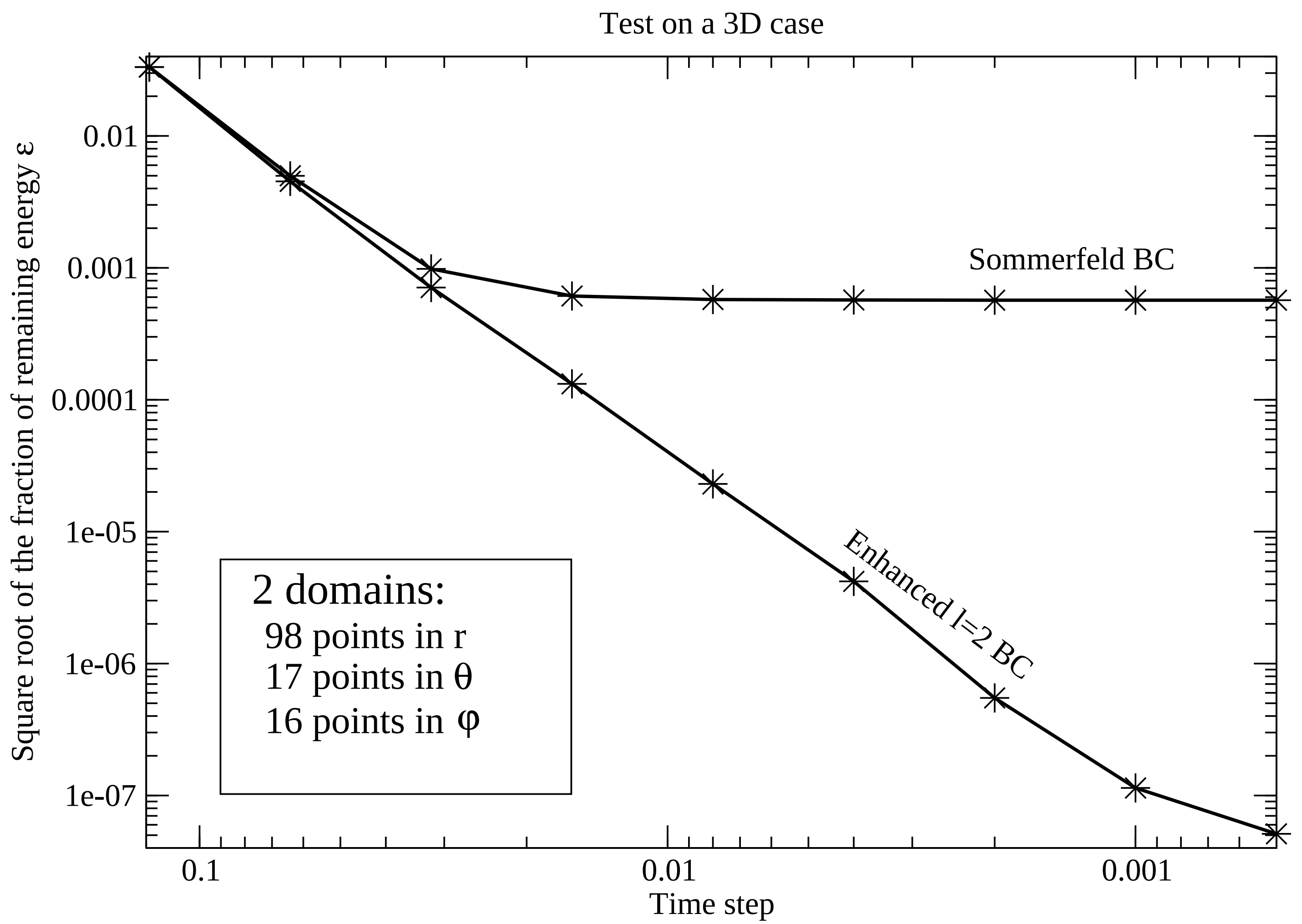
Non reflecting BC up to $\ell = 2$

Method of Bayliss & Turkel [Comm. Pure Appl. Math. **33**, 707 (1980)]:

$$
\begin{aligned}
B_1 u &:= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial r} + \frac{u}{r} \\
B_2 u &:= \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{3}{r} \right) B_1 u \\
B_3 u &:= \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{5}{r} \right) B_2 u
\end{aligned}
$$

Boundary condition : $B_3 u|_{r=R} = 0$.
$\Rightarrow$ ensures that spherical harmonics with $\ell = 0$, $\ell = 1$ and $\ell = 2$ are perfectly outgoing.
This is important for gravitational waves.

Comparison with Sommerfeld boundary condition



[Novak & Bonazzola, J. Comp. Phys. **197**, 186 (2004)]

Tensorial wave equation

Tensorial wave equations $\Box h^{\mu\nu} = \sigma^{\mu\nu}$ occurs in general relativity in various cases:

- in **harmonic coordinates** (4-dimensional tensor)
- in the **TT gauge** of linearized gravity (3-dimensional tensor)
- in the **Dirac gauge** within the 3+1 formalism (3-dimensional tensor) [Bonazzola, Gourgoulhon, Grandclément & Novak, gr-qc/0307082]

3+1 spacetime evolution in Dirac gauge

Conformal decomposition of the metric γ_{ij} of the spacelike hypersurfaces Σ_t of the 3+1 formalism of general relativity (cf. S.A. Teukolsky's talk):

$$\gamma^{ij} =: \Psi^4 (f^{ij} + h^{ij})$$

where f^{ij} is a flat metric on Σ_t, h^{ij} a symmetric tensor and Ψ a scalar field defined by $\Psi = \left(\frac{\det \gamma_{ij}}{\det f_{ij}} \right)^{1/12}$

The **Dirac gauge** is expressed as a divergence-free condition on h^{ij}: $\boxed{\mathcal{D}_j h^{ij} = 0}$ where $\mathcal{D}_j$ denotes the covariant derivative with respect to f_{ij}.

$\Longrightarrow$ Ricci tensor of space metric γ_{ij} becomes an elliptic operator for h^{ij}

$\Longrightarrow$ the dynamical Einstein equations become a **wave equation** for h^{ij}

Resolution of the tensor wave equation

Rewrite the evolution equation for h^{ij} as

$$\frac{\partial^2 h^{ij}}{\partial t^2} - \underline{\Delta} h^{ij} = \sigma^{ij}$$

Split h^{ij} into its trace $\boxed{h := f_{ij} h^{ij}}$ and its traceless-transverse (TT) part:

$\boxed{\bar{h}^{ij} := h^{ij} - \frac{1}{2}\left(h\, f^{ij} - \mathcal{D}^i \mathcal{D}^j \Phi\right)}$, with $\underline{\Delta}\Phi = h$.

The TT part of the wave equation is

$$\frac{\partial^2 \bar{h}^{ij}}{\partial t^2} - \underline{\Delta} \bar{h}^{ij} = \bar{\sigma}^{ij}$$

Taking advantage of spherical components

In spherical components, the TT tensor wave equation is reduced to two scalar wave equations:

$$\frac{\partial^2 \chi}{\partial t^2} - \underline{\Delta}\chi = \sigma_\chi$$

$$\frac{\partial^2 \mu}{\partial t^2} - \underline{\Delta}\mu = \sigma_\mu$$

Thanks to its TT character, all the components of $\bar{h}^{ij}$ can be deduced from χ and μ quasi-algebraically. For instance, in a spherical orthonormal basis,

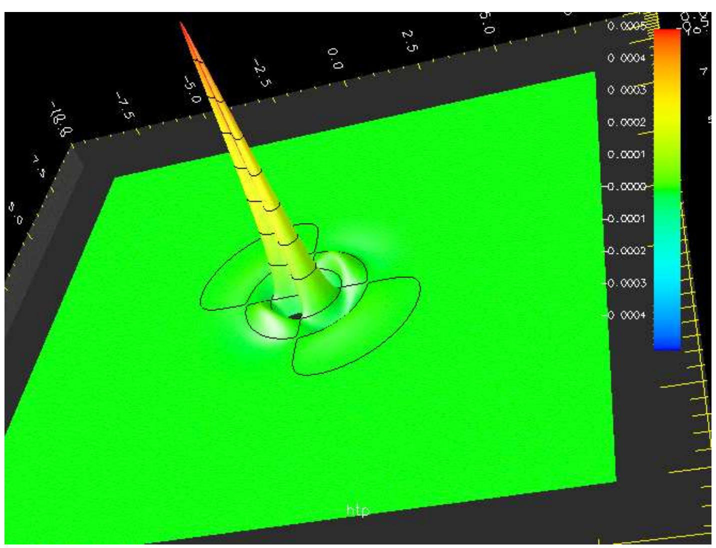
$$\begin{aligned} \bar{h}^{\hat{r}\hat{r}} &= \frac{\chi}{r^2} \\ \bar{h}^{\hat{r}\hat{\theta}} &= \frac{1}{r}\left(\frac{\partial \eta}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial \mu}{\partial \phi}\right) \qquad \text{with} \quad \Delta_{\theta\phi}\eta = -\frac{\partial \chi}{\partial r} - \frac{\chi}{r} \\ \bar{h}^{\hat{r}\hat{\phi}} &= \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial \eta}{\partial \phi} + \frac{\partial \mu}{\partial \theta}\right) \end{aligned}$$

Example: evolution of a vaccum spacetime

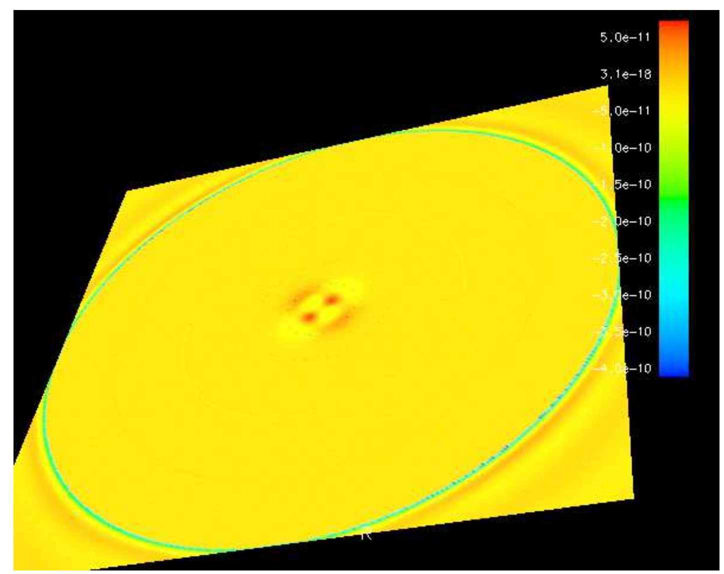
Pure gravitational wave spacetime

Initial data: same as [Baumgarte & Shapiro, PRD **59**, 024007 (1998)], namely a Teukolsky wave $\ell = 2$, $m = 2$: $\chi = 10^{-3}\, xy \exp(-r^2)$ and $\mu = 0$, momentarily static: $K_{ij} = 0$
Constraint equations solved within the conformal thin sandwich framework

Evolution: fully constrained scheme based on Dirac gauge and maximal slicing [Bonazzola, Gourgoulhon, Grandclément & Novak, gr-qc/0307082]

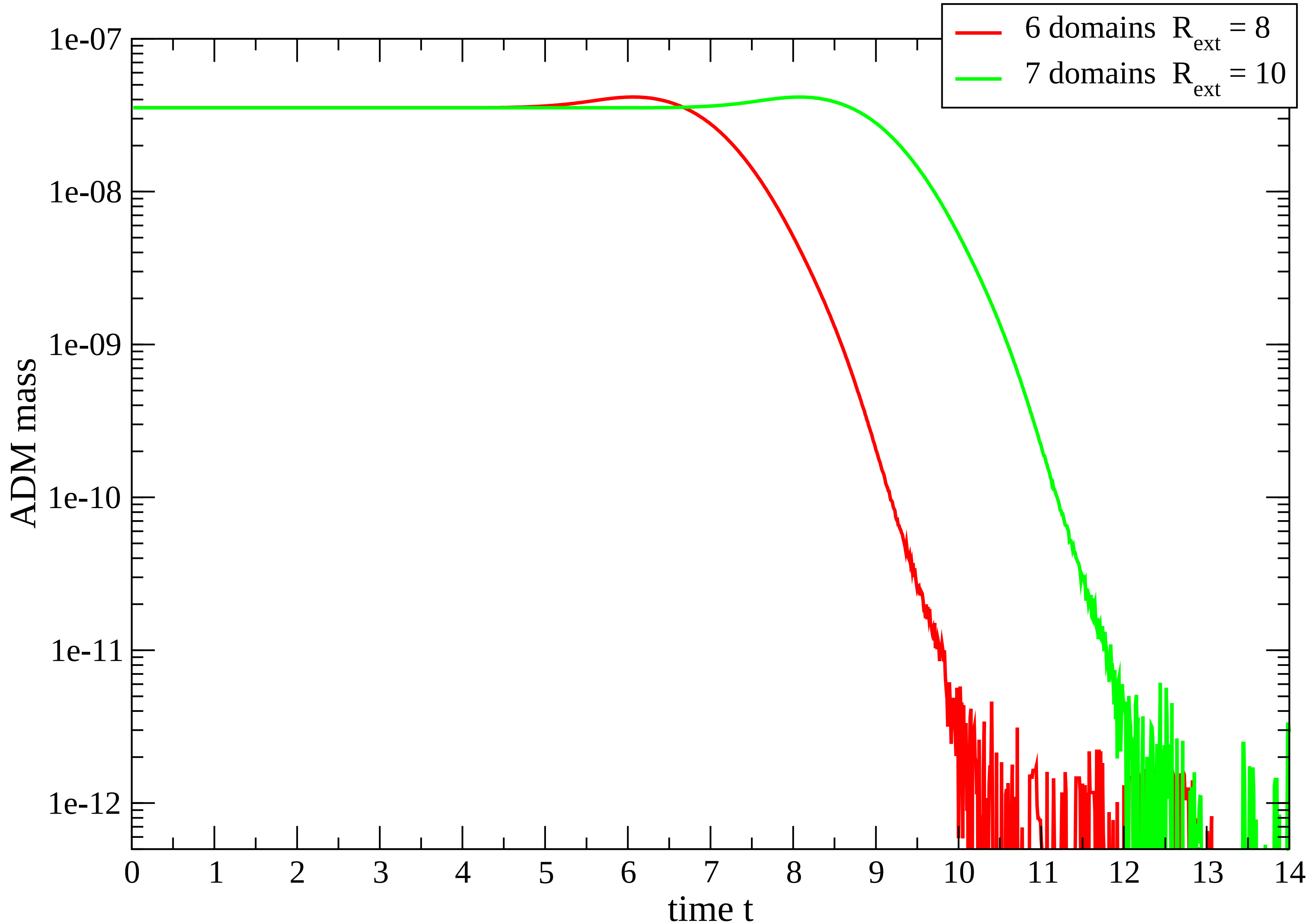


Evolution of $h^{\hat\phi\hat\phi}$ in the plane $z = 0$



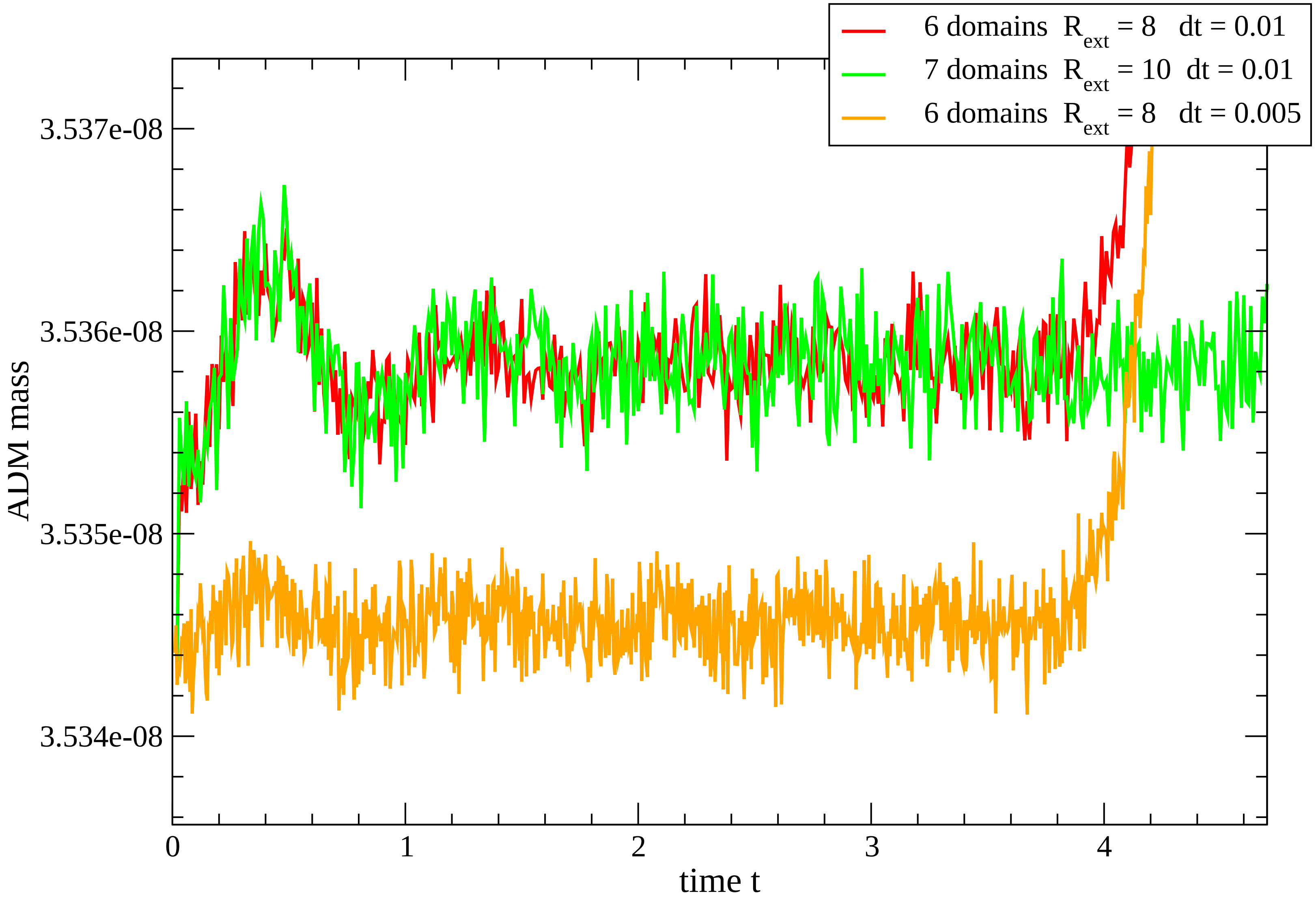
Evolution of the scalar curvature R of the hypersurface Σ_t in the plane $z = 0$

Test of the code: conservation of the ADM mass



Number of coefficients in each domain: $N_r = 17$, $N_\theta = 9$, $N_\varphi = 8$

Test of the code: conservation of the ADM mass (zoom)



For $dt = 5\,10^{-3}$, the ADM is conserved within a relative error lower than 10^{-4}.