Special relativity from an accelerated observer perspective

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Plan

Outline

A brief history of special relativity

- 1898 : H. Poincaré : *simultaneity* must result from some *convention*
- 1900 : H. Poincaré : synchronization of clocks by exchange of light signals
- 1905 : A. Einstein : funding article based on 2 axioms, both related to *inertial observers:* (i) the relavitity principle, (ii) the constancy of the velocity of light
- 1905 : H. Poincaré : mathematical use of time as a fourth dimension
- 1907 : A. Einstein : first mention of an *accelerated observer* (uniform acceleration)
- 1908 : H. Minkowsky : 4-dimensional spacetime, generic accelerated observer
- 1909 : M. Born : detailed study of uniformly accelerated motion
- 1909 : P. Ehrenfest : paradox on the circumference of a disk set to rotation
- 1911 : A. Einstein, P. Langevin : round-trip motion and differential aging ($\Longrightarrow$ *twin paradox*)
- 1911 : M. Laue : prediction of the *Sagnac effect* within special relativity
- 1956 : J. L. Synge : fully geometrical exposure of special relativity

Standard exposition of special relativity

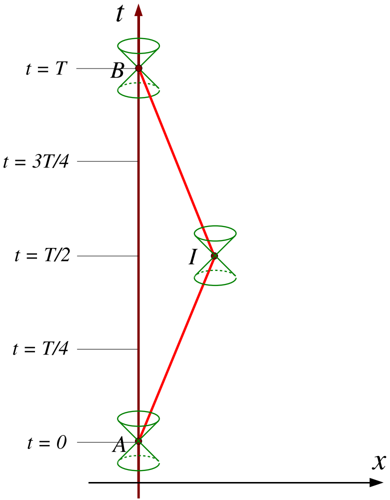
Standard textbook presentations of special relativity are based on inertial observers.

For these privileged observers, there exists a global 3+1 decomposition of spacetime, i.e. a split between some *time* and some *3-dimensional Euclidean space* This could make people comfortable to think in a "Newtonian way".

Special relativity differs then from Newtonian physics only in the manner one moves from one inertial observer to another one :

Lorentz transformations ↔ Galilean transformations

Some drawback of this approach: the twin paradox



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$$T' = T\sqrt{1 - \frac{V^2}{c^2}} \leq T$$

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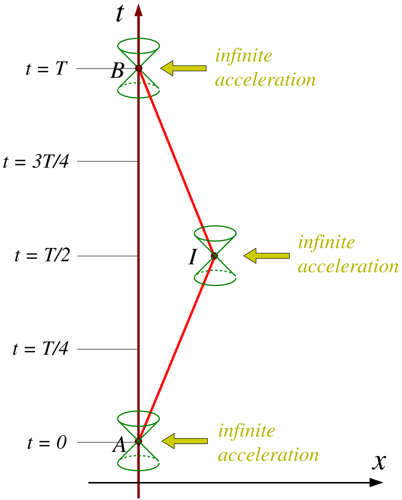

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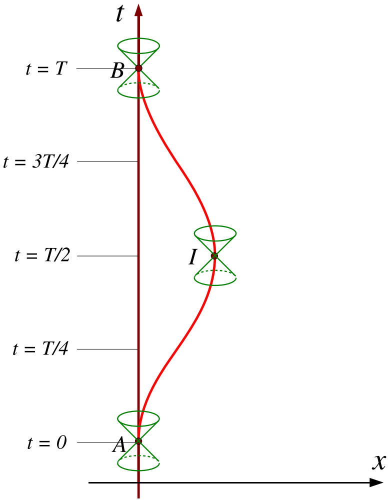

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A more satisfactory presentation would require an accelerated observer.

Other arguments for considering accelerated observers

- The real world is made of accelerated / rotating observers.

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- The real world is made of accelerated / rotating observers.
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- Explaining the above effects by relying only on inertial observers is tricky; it seems *logically more appropriate* to introduce *generic (accelerated) observers* first, considering inertial observers as a special subcase.
- Often students learning *general* relativity discover notions like *Fermi-Walker transport* or *Rindler horizon* which have nothing to do with spacetime curvature and actually pertain to the realm of *special* relativity.

Outline

The good framework: Minkowsky spacetime

When limiting the discussion to inertial observers, one can stick to a 3+1 point of view and avoid to refer to Minkowsky spacetime

On the contrary, the appropriate framework for introducing accelerated observers is Minkowsky spacetime, that is the quadruplet $(\mathscr{E}, \boldsymbol{g}, \mathcal{I}^+, \boldsymbol{\epsilon})$ where

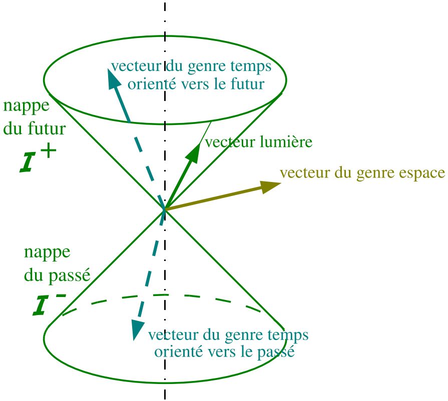
- $\mathscr{E}$ is a 4-dimensional affine space on $\mathbb{R}$ (associate vector space : E)
- $\boldsymbol{g}$ is the **metric tensor**, i.e. a bilinear form on E that is symmetric, non-degenerate and has signature $(-,+,+,+)$
- $\mathcal{I}^+$ is one of the two sheets of $\boldsymbol{g}$'s null cone, definiting the **time orientation** of spacetime
- $\boldsymbol{\epsilon}$ is the **Levi-Civita alternating tensor**, i.e. a quadrilinear form on E that is antisymmetric and results in ± 1 when applied to any vector basis which is orthonormal with respect to $\boldsymbol{g}$

The null cone and vector gender

E: space of vectors on spacetime *(4-vectors)*

Metric tensor:

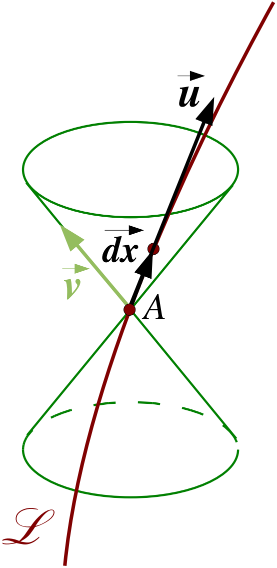
$$\begin{array}{rccl} \boldsymbol{g}: & E \times E & \longrightarrow & \mathbb{R} \\ & (\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) & \longmapsto & \boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) =: \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} \end{array}$$



A vector $\vec{\boldsymbol{v}} \in E$ is

- **spacelike** iff $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} > 0$
- **timelike** iff $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} < 0$
- **null** iff $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} = 0$

Worldlines and the metric tensor



Physical interpretation of the metric tensor 1:

Proper time along a (massive) particle worldline = length given by the metric tensor:

$$d\tau = \frac{1}{c}\sqrt{-\boldsymbol{g}(d\vec{\boldsymbol{x}}, d\vec{\boldsymbol{x}})}$$

4-velocity $\vec{\boldsymbol{u}}$ = unit timelike future-directed tangent to the worldline :

$$\vec{\boldsymbol{u}} := \frac{1}{c}\frac{d\vec{\boldsymbol{x}}}{d\tau}, \qquad \boldsymbol{g}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{u}}) = -1$$

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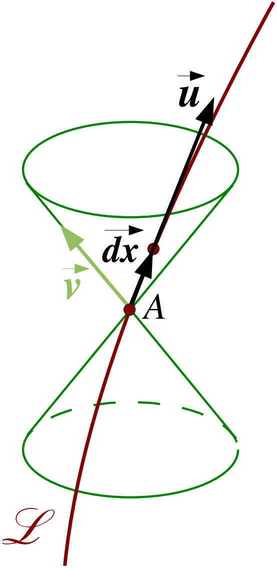

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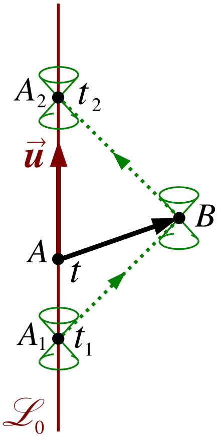
Physical interpretation of the metric tensor 2:

The worldline of massless particles (e.g. photons) are null lines of $\boldsymbol{g}$ (i.e. straight lines with a null tangent vector)

Einstein-Poincaré simultaneity

Observer $\mathcal{O}$ of worldline $\mathscr{L}_0$

A event on $\mathscr{L}_0$, B distant event



Using only proper times measured by $\mathcal{O}$ and a round-trip light signal:

Einstein-Poincaré definition of simultaneity

B is simultaneous with $A \iff \boxed{t = \frac{1}{2}(t_1 + t_2)}$

t: proper time of A

t_1 (resp. t_2): proper time of signal emission (resp. reception)

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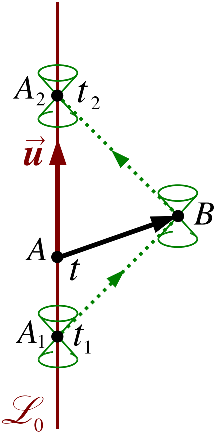

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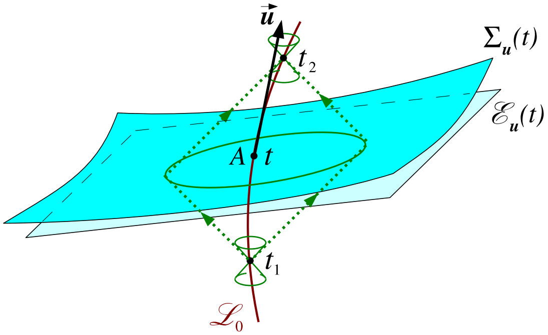
Geometrical characterization

If B is "closed" to $\mathcal{O}$'s worldline,

B is simultaneous with A $\iff$ $\boxed{\vec{u}(A) \cdot \overrightarrow{AB} = 0}$

Local rest space of an observer

Observer $\mathcal{O}$: worldline $\mathscr{L}_0$, 4-velocity $\vec{\boldsymbol{u}}$, proper time t



Given an event $A \in \mathscr{L}_0$ of proper time t,

- **hypersurface of simultaneity of A for $\mathcal{O}$:** set $\Sigma_{\boldsymbol{u}}(t)$ of all events simultaneous to A according to $\mathcal{O}$
- **local rest space of $\mathcal{O}$:** hyperplane $\mathscr{E}_{\boldsymbol{u}}(t)$ tangent to $\Sigma_{\boldsymbol{u}}(t)$ at A

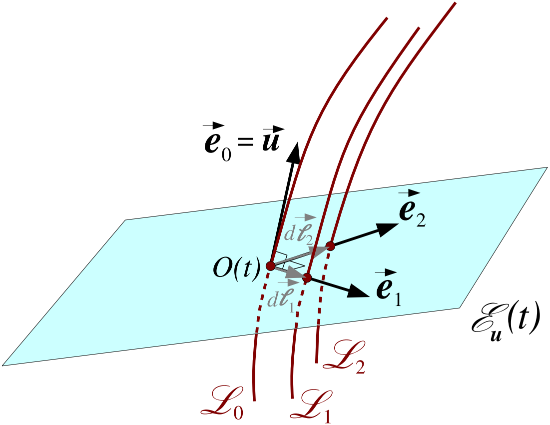
According to the geometrical characterization of Einstein-Poincaré simultaneity:

$\mathscr{E}_{\boldsymbol{u}}(t)$ is the spacelike hyperplane orthogonal to $\vec{\boldsymbol{u}}(t)$

Notation: $E_{\boldsymbol{u}}(t)$ = 3-dimensional vector space associated with the affine space $\mathscr{E}_{\boldsymbol{u}}(t)$; $E_{\boldsymbol{u}}(t)$ is a subspace of E

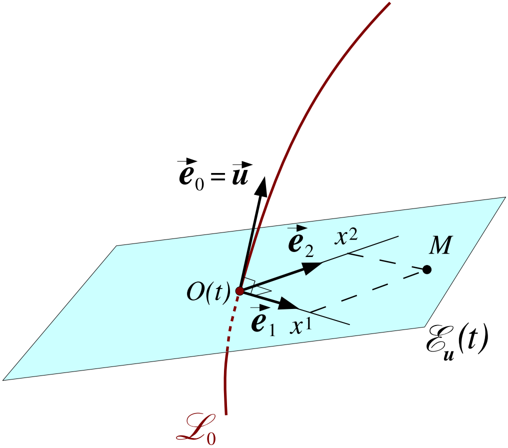
Local frame of an observer

An observer is defined not only by its wordline, but also by an orthonormal basis $(\vec{e}_1(t), \vec{e}_2(t), \vec{e}_3(t))$ of its local rest space $E_{\boldsymbol{u}}(t)$ at each instant t



$(\vec{e}_\alpha(t)) = (\vec{u}(t), \vec{e}_1(t), \vec{e}_2(t), \vec{e}_3(t))$ is then an orthonormal basis of E: it is $\mathcal{O}$'s **local frame**.

Coordinates associated with an observer



Observer $\mathcal{O}$:

- proper time t
- local frame $(\vec{e}_\alpha(t))$

$M \in \mathscr{E}$ "close" to $\mathcal{O}$'s worldline $\mathscr{L}_0$

Coordinates (t, x^1, x^2, x^3) of M with respect to $\mathcal{O}$:

- t defined by

$$M \in \mathscr{E}_{\boldsymbol{u}}(t)$$

- (x^1, x^2, x^3) defined by

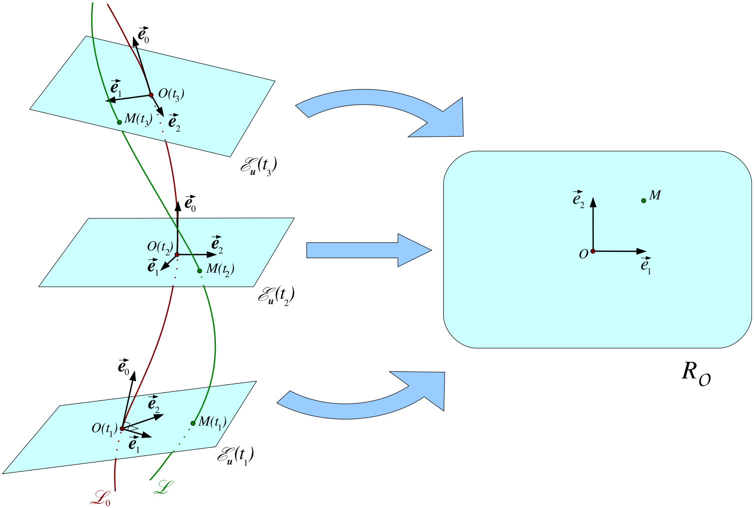
$$\overrightarrow{O(t)M} = x^i \, \vec{e}_i(t)$$

Misner, Thorne & Wheeler's generalization (1973) of coordinates introduced by Synge (1956) (called by him *Fermi coordinates*)

Reference space of observer $\mathcal{O}$

3-dim. Euclidean space $R_{\mathcal{O}}$ with mapping

$$\varphi : \begin{array}{ccc} \mathscr{E} & \longrightarrow & R_{\mathcal{O}} \\ M(t, x^i) & \longmapsto & \vec{x} = x^i \vec{e}_i \end{array}$$



Variation of the local frame (1/2)

Expand $d\vec{\boldsymbol{e}}_\alpha/dt$ on the basis $(\vec{\boldsymbol{e}}_\alpha)$: $\dfrac{d\vec{\boldsymbol{e}}_\alpha}{dt} = \Omega^\beta{}_\alpha\, \vec{\boldsymbol{e}}_\beta$

Introduce $\boldsymbol{\Omega}$ endomorphism of E whose matrix in the $(\vec{\boldsymbol{e}}_\alpha)$ basis is $(\Omega^\alpha{}_\beta)$. Then

$$\frac{d\vec{\boldsymbol{e}}_\alpha}{dt} = \boldsymbol{\Omega}(\vec{\boldsymbol{e}}_\alpha)$$

From the property $\vec{\boldsymbol{e}}_\alpha \cdot \vec{\boldsymbol{e}}_\beta = \eta_{\alpha\beta}$ and $d\eta_{\alpha\beta}/dt = 0$ one gets immediately

$$\boldsymbol{\Omega}(\vec{\boldsymbol{e}}_\alpha) \cdot \vec{\boldsymbol{e}}_\beta = -\vec{\boldsymbol{e}}_\alpha \cdot \boldsymbol{\Omega}(\vec{\boldsymbol{e}}_\beta)$$

$\Longrightarrow$ the bilinear form $\underline{\boldsymbol{\Omega}}$ defined by $\forall(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) \in E^2, \quad \underline{\boldsymbol{\Omega}}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}) := \vec{\boldsymbol{v}} \cdot \boldsymbol{\Omega}(\vec{\boldsymbol{w}})$ is antisymmetric, i.e. $\underline{\boldsymbol{\Omega}}$ is a 2-form.

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$\Longrightarrow$ $\exists$ a unique 1-form $\boldsymbol{a}$ and a unique vector $\vec{\boldsymbol{\omega}}$ such that

$$\boxed{\underline{\boldsymbol{\Omega}} = c\, \underline{\boldsymbol{u}} \otimes \boldsymbol{a} - c\, \boldsymbol{a} \otimes \underline{\boldsymbol{u}} - \boldsymbol{\epsilon}(\vec{u}, \vec{\boldsymbol{\omega}}, ., .)}, \qquad \boldsymbol{a} \cdot \vec{u} = 0, \qquad \vec{\boldsymbol{\omega}} \cdot \vec{u} = 0$$

This is similar to the electric / magnetic decomposition of the electromagnetic field tensor $\boldsymbol{F}$ with respect to an observer:

$$\boldsymbol{F} = \underline{\boldsymbol{u}} \otimes \boldsymbol{E} - \boldsymbol{E} \otimes \underline{\boldsymbol{u}} + \boldsymbol{\epsilon}(\vec{u}, c\vec{\boldsymbol{B}}, ., .), \qquad \boldsymbol{E} \cdot \vec{u} = 0, \qquad \vec{\boldsymbol{B}} \cdot \vec{u} = 0$$

Variation of the local frame (2/2)

Accordingly

$$\frac{d\vec{\boldsymbol{e}}_\alpha}{dt} = \underbrace{c(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{e}}_\alpha)\,\vec{\boldsymbol{u}} - c(\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{e}}_\alpha)\,\vec{\boldsymbol{a}}}_{\text{Fermi-Walker}} \;+\; \underbrace{\vec{\boldsymbol{\omega}}\times_{\boldsymbol{u}}\vec{\boldsymbol{e}}_\alpha}_{\text{spatial rotation}} \qquad (1)$$

with $\vec{\boldsymbol{v}}\times_{\boldsymbol{u}}\vec{\boldsymbol{w}} := \vec{\boldsymbol{\epsilon}}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}},\vec{\boldsymbol{w}},.\,)$

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$$\frac{d\vec{u}}{dt} = c\,\vec{a}$$

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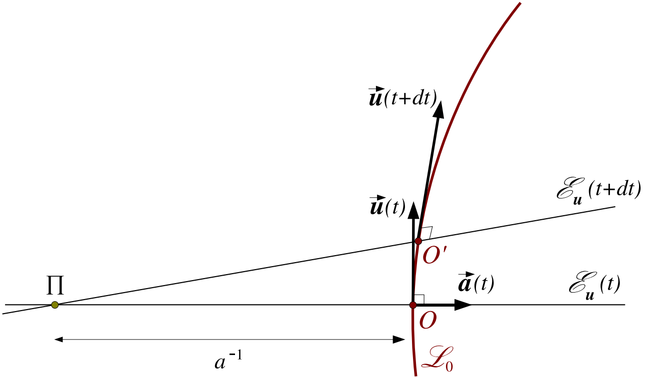
As for the 4-velocity, the 4-acceleration and the 4-rotation are *absolute quantities*

$\mathcal{O}$ **inertial observer** $\iff$ $\vec{a} = 0$ and $\vec{\omega} = 0$ $\iff$ $\frac{d\vec{e}_\alpha}{dt} = 0$

Non-globlality of the local frame



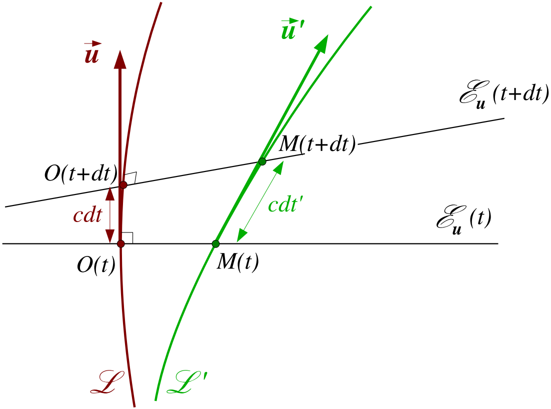
The local frame of observer $\mathcal{O}$ is valid within a range
$r \ll a^{-1} = \|\vec{a}\|_g^{-1} = (\vec{a}\cdot\vec{a})^{-1/2}$

$a = \gamma/c^2$ with γ acceleration of $\mathcal{O}$ relative to a tangent inertial observer

$\gamma = 10\ \mathrm{m\,s^{-2}} \Longrightarrow c^2/\gamma \simeq 9\times 10^{15}\ \mathrm{m} \simeq 1$ light-year !

Outline

Lorentz factor



Observer $\mathcal{O}$:
- worldline $\mathscr{L}$
- 4-velocity $\vec{u}$
- 4-acceleration $\vec{a}$
- 4-rotation $\vec{\omega}$
- proper time t
- local rest space $\mathscr{E}_{u}(t)$

Massive particle $\mathscr{P}$:
- worldline $\mathscr{L}'$
- 4-velocity $\vec{u}'$
- proper time t'

Lorentz factor of $\mathscr{P}$ with respect to $\mathcal{O}$: $\boxed{\Gamma := \frac{dt}{dt'}}$

One can show

$$\boxed{\Gamma = -\frac{\vec{u}\cdot\vec{u}'}{1+\vec{a}\cdot\overrightarrow{OM}}}$$

Relative velocity

Monitoring the motion of particle $\mathscr{P}$ within $\mathcal{O}$'s local coordinates (t, x^i):

$$\overrightarrow{O(t)M(t)} = x^i(t)\,\vec{\boldsymbol{e}}_i(t)$$

The **velocity of $\mathscr{P}$ relative to $\mathcal{O}$** is

$$\boxed{\vec{\boldsymbol{V}}(t) := \frac{dx^i}{dt}\,\vec{\boldsymbol{e}}_i(t)}$$

By construction $\vec{\boldsymbol{V}}(t) \in E_{\boldsymbol{u}}(t)$: $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{V}} = 0$

The 4-velocity of $\mathscr{P}$ is expressible in terms of Γ and $\vec{\boldsymbol{V}}$ as

$$\boxed{\vec{\boldsymbol{u}}' = \Gamma\left[(1 + \vec{\boldsymbol{a}} \cdot \overrightarrow{OM})\,\vec{\boldsymbol{u}} + \frac{1}{c}\left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}} \overrightarrow{OM}\right)\right]} \quad (2)$$

The normalization relation $\vec{\boldsymbol{u}}' \cdot \vec{\boldsymbol{u}}' = -1$ is then equivalent to

$$\boxed{\Gamma = \left[(1 + \vec{\boldsymbol{a}} \cdot \overrightarrow{OM})^2 - \frac{1}{c^2}\left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}} \overrightarrow{OM}\right) \cdot \left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}} \overrightarrow{OM}\right)\right]^{-1/2}} \quad (3)$$

Relative acceleration

The **acceleration of $\mathscr{P}$ relative to $\mathcal{O}$** is

$$\vec{\gamma}(t) := \frac{d^2 x^i}{dt^2}\, \vec{\boldsymbol{e}}_i(t)$$

By construction $\vec{\gamma}(t) \in E_{\boldsymbol{u}}(t)$: $\vec{\boldsymbol{u}} \cdot \vec{\gamma} = 0$

The 4-acceleration of $\mathscr{P}$ reads

$$\begin{aligned}
\vec{\boldsymbol{a}}' \quad = \quad & \frac{\Gamma^2}{c^2}\Bigg\{ \vec{\gamma} + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}} \left(\vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}} \overrightarrow{OM} \right) + 2\vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}} \vec{\boldsymbol{V}} + \frac{d\vec{\boldsymbol{\omega}}}{dt} \times_{\boldsymbol{u}} \overrightarrow{OM} \\
& + c^2 (1 + \vec{\boldsymbol{a}} \cdot \overrightarrow{OM})\, \vec{\boldsymbol{a}} + \frac{1}{\Gamma}\frac{d\Gamma}{dt} \left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}} \overrightarrow{OM} \right) \\
& + c \left[2\vec{\boldsymbol{a}} \cdot \left(\vec{\boldsymbol{V}} + \vec{\boldsymbol{\omega}} \times_{\boldsymbol{u}} \overrightarrow{OM} \right) + \frac{d\vec{\boldsymbol{a}}}{dt} \cdot \overrightarrow{OM} + \frac{1}{\Gamma}\frac{d\Gamma}{dt} (1 + \vec{\boldsymbol{a}} \cdot \overrightarrow{OM}) \right] \vec{\boldsymbol{u}} \Bigg\}.
\end{aligned}$$

Special case of an inertial observer

If $\mathcal{O}$ is inertial, $\vec{\boldsymbol{a}} = 0$, $\vec{\boldsymbol{\omega}} = 0$, and we recover well known formulæ :

$$\vec{\boldsymbol{u}}' = \Gamma \left(\vec{\boldsymbol{u}} + \frac{1}{c} \vec{\boldsymbol{V}} \right)$$

$$\Gamma = \left(1 - \frac{1}{c^2} \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{V}} \right)^{-1/2}$$

$$\vec{\boldsymbol{a}}' = \frac{\Gamma^2}{c^2} \left[\vec{\boldsymbol{\gamma}} + \frac{\Gamma^2}{c^2} (\vec{\boldsymbol{\gamma}} \cdot \vec{\boldsymbol{V}}) \left(\vec{\boldsymbol{V}} + c \vec{\boldsymbol{u}} \right) \right]$$

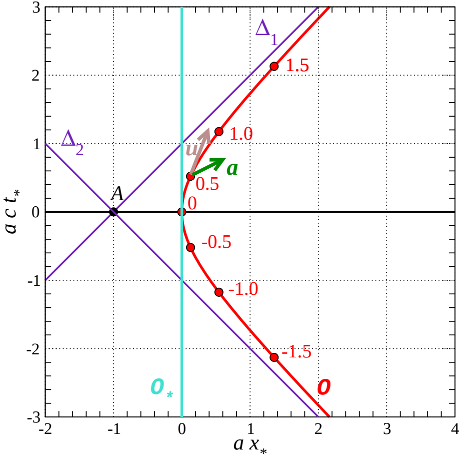
$$\vec{\boldsymbol{a}}' = \frac{1}{c^2} \vec{\boldsymbol{\gamma}} \qquad (\vec{\boldsymbol{V}} = 0)$$

Outline

Uniformly accelerated observer

Definition: the observer $\mathcal{O}$ is **uniformly accelerated** iff

- its worldline stays in a plane $\Pi \subset \mathscr{E}$
- the *norm* of its 4-acceleration is constant $a := \|\vec{\boldsymbol{a}}\|_{\boldsymbol{g}} = \sqrt{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{a}}} = \mathrm{const}$
- its 4-rotation vanishes : $\vec{\boldsymbol{\omega}} = 0$



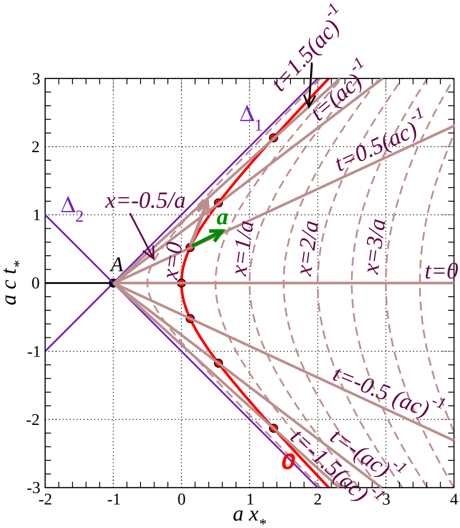
Worldline in terms of the coordinates (ct_*, x_*, y_*, z_*) associated with an inertial observer $\mathcal{O}_*$:

$$\begin{cases} ct_* = a^{-1} \sinh(act) \\ x_* = a^{-1} [\cosh(act) - 1] \\ y_* = 0 \\ z_* = 0. \end{cases}$$

$$(ax_* + 1)^2 - (act_*)^2 = 1$$

$$\begin{aligned} \vec{\boldsymbol{u}}(t) &= \cosh(act)\, \vec{\boldsymbol{e}}_0^* + \sinh(act)\, \vec{\boldsymbol{e}}_1^* \\ \vec{\boldsymbol{a}}(t) &= a \left[\sinh(act)\, \vec{\boldsymbol{e}}_0^* + \cosh(act)\, \vec{\boldsymbol{e}}_1^*\right] \end{aligned}$$

Coordinates associated with the accelerated observer



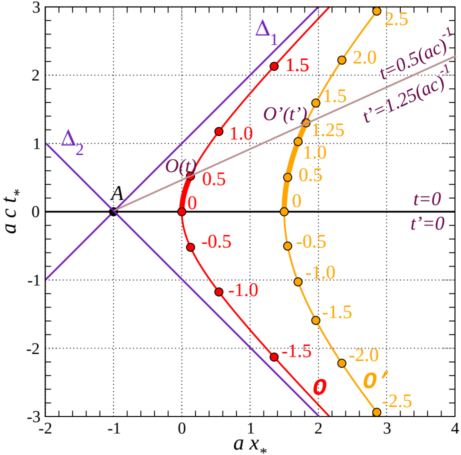
Relation between the coordinates (t, x, y, z) associated with $\mathcal{O}$ and the inertial coordinates (t_*, x_*, y_*, z_*):

$$\begin{cases} ct_* &= (x + a^{-1}) \sinh(act) \\ x_* &= (x + a^{-1}) \cosh(act) - a^{-1} \\ y_* &= y \\ z_* &= z. \end{cases}$$

with $x > -a^{-1}$

The coordinates (t, x, y, z) are called **Rindler coordinates**

Time dilation at rest



Observer $\mathcal{O}'$ at rest with respect to $\mathcal{O}$, located at coord. $(x, y, z) = (x_0, 0, 0)$

$\Longrightarrow \vec{\boldsymbol{V}} = 0$

(3) $\Longrightarrow \Gamma = \left[1 + \vec{\boldsymbol{a}}(t) \cdot \overrightarrow{O(t)O'(t')}\right]^{-1}$

(2) $\Longrightarrow \vec{\boldsymbol{u}}'(t') = \vec{\boldsymbol{u}}(t)$

$\Longrightarrow$ the local rest spaces of $\mathcal{O}$ and $\mathcal{O}'$ coincide: $\boxed{\mathscr{E}_{\boldsymbol{u}'}(t') = \mathscr{E}_{\boldsymbol{u}}(t)}$

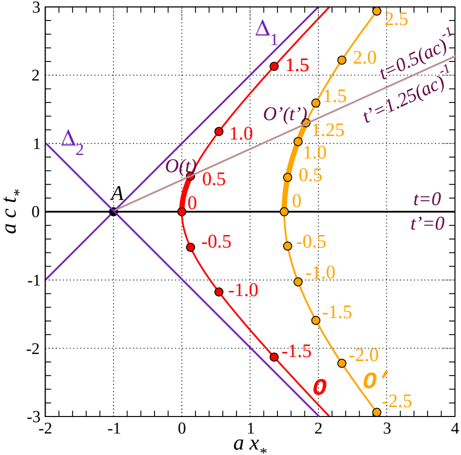
$\vec{\boldsymbol{a}}(t) = a\, \vec{\boldsymbol{e}}_1(t)$ and $\overrightarrow{O(t)O'(t')} = x_0 \vec{\boldsymbol{e}}_1(t)$

$\Rightarrow \Gamma = (1 + a x_0)^{-1}$ & $dt' = (1 + a x_0)\, dt$

Since $x_0 = \mathrm{const}$, this relation can be integrated:

$$\boxed{t' = (1 + a x_0)\, t}$$

Time dilation at rest



Observer $\mathcal{O}'$ at rest with respect to $\mathcal{O}$, located at coord. $(x, y, z) = (x_0, 0, 0)$

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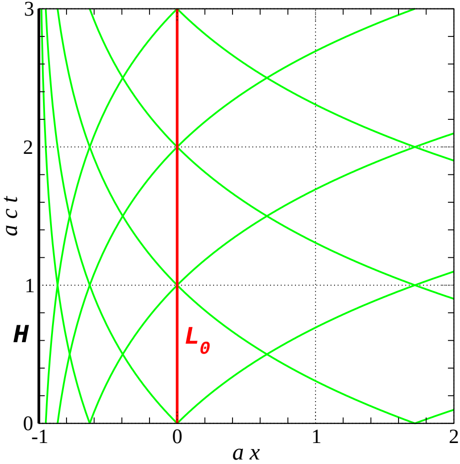
Analogous to *Einstein effect* in general relativity

Photon trajectories



Null geodesics in terms of inertial coordinates:

$$ct_* = \pm(x_* - b), \quad b \in \mathbb{R}$$

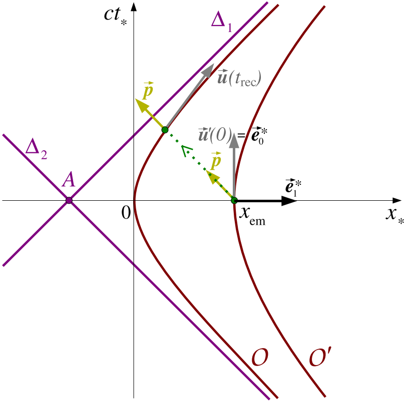
in terms of $\mathcal{O}$'s coordinates:

$$ct = \pm a^{-1} \ln\left(\frac{1+ax}{1+ab}\right)$$

$x = -a^{-1}$: **Rindler horizon**

Redshift

Reception by $\mathcal{O}$ of a photon emitted by $\mathcal{O}'$ at $t' = 0$



If $\vec{p}$ is the photon 4-momentum, the energy measured by $\mathcal{O}$ is

$$E_{rec} = -c\,\vec{p} \cdot \vec{u}(t_{rec})$$

with

$$\vec{p} = \frac{E_{em}}{c}\left(\vec{u}'(0) + \vec{n}'\right) = \frac{E_{em}}{c}\left(\vec{e}_0^* - \vec{e}_1^*\right)$$

$$\vec{u}(t_{rec}) = \cosh(act_{rec})\,\vec{e}_0^* + \sinh(act_{rec})\,\vec{e}_1^*$$

$$ct_{rec} = a^{-1}\ln(1 + ax_{em})$$

$$\Longrightarrow \boxed{E_{rec} = E_{em}(1 + ax_{em})}$$

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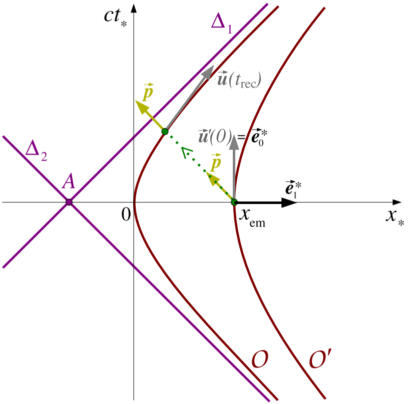

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$$\vec{u}(t_{rec}) = \cosh(act_{rec})\,\vec{e}_0^* + \sinh(act_{rec})\,\vec{e}_1^*$$

$$ct_{rec} = a^{-1}\ln(1 + ax_{em})$$

$$\Longrightarrow \boxed{E_{rec} = E_{em}(1 + ax_{em})}$$

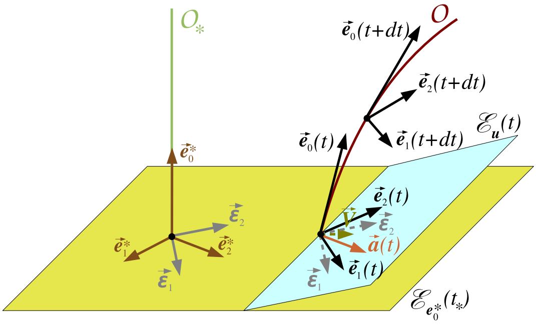
$\Longrightarrow$ spectral shift $\boxed{z = \dfrac{1}{1 + ax_{em}} - 1}$

$$\begin{cases} z > 0 \text{ for } x_{em} < 0 \\ z < 0 \text{ for } x_{em} > 0 \end{cases}$$

Thomas precession

$\mathcal{O}_*$ = inertial observer ; proper time t_*; (local) frame $(\vec{e}^{\,*}_\alpha)$

$\mathcal{O}$ = accelerated observer *without rotation*; proper time t; local frame $(\vec{e}_\alpha(t))$



$\boldsymbol{S}_t$: the boost from $\vec{e}^{\,*}_0$ to $\vec{e}_0(t)$:

$$\vec{e}_0(t) = \boldsymbol{S}_t(\vec{e}^{\,*}_0)$$

Let

$$\vec{\varepsilon}_\alpha(t_*) := \boldsymbol{S}_t^{-1}(\vec{e}_\alpha(t))$$

$$\iff \boxed{\vec{e}_\alpha(t) = \boldsymbol{S}_t(\vec{\varepsilon}_\alpha(t_*))}$$

$\vec{\varepsilon}_0 = \vec{e}^{\,*}_0$

$(\vec{\varepsilon}_i)$ = triad in $\mathcal{O}_*$'s rest space which is "quasi-parallel" to the triad $(\vec{e}_i)$ of $\mathcal{O}$'s local rest frame.

Thomas precession

Evolution of $\mathcal{O}$'s local rest frame:

$$\vec{\boldsymbol{e}}_\alpha(t+dt) = \boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_\alpha)$$

According to (1) with $\vec{\boldsymbol{\omega}} = 0$, $\boldsymbol{\Lambda}(\vec{\boldsymbol{e}}_\alpha) = \vec{\boldsymbol{e}}_\alpha + c\,dt[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{e}}_\alpha)\,\vec{\boldsymbol{u}} - (\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{e}}_\alpha)\,\vec{\boldsymbol{a}}]$

$\boldsymbol{\Lambda}$ is an infinitesimal boost

Hence

$$\vec{\boldsymbol{e}}_\alpha(t+dt) = \boldsymbol{\Lambda}\circ\boldsymbol{S}_t(\vec{\boldsymbol{\varepsilon}}_\alpha(t_*))$$

Now in general, the composition of the boosts $\boldsymbol{\Lambda}$ and $\boldsymbol{S}_t$ is a boost times a rotation — **Thomas rotation**:

$$\boldsymbol{\Lambda}\circ\boldsymbol{S}_t = \boldsymbol{S}'\circ\boldsymbol{R}$$

In the present case, $\boldsymbol{R}(\vec{\boldsymbol{e}}_0^*) = \vec{\boldsymbol{e}}_0^*$, so that necessarily $\boldsymbol{S}' = \boldsymbol{S}_{t+dt}$. Hence

$$\vec{\boldsymbol{e}}_\alpha(t+dt) = \boldsymbol{S}_{t+dt}\circ\boldsymbol{R}(\vec{\boldsymbol{\varepsilon}}_\alpha(t_*))$$

$$\Longrightarrow \boxed{\vec{\boldsymbol{\varepsilon}}_\alpha(t_*+dt_*) = \boldsymbol{R}(\vec{\boldsymbol{\varepsilon}}_\alpha(t_*))}$$

Thomas precession

Thus

$$\frac{d\vec{\varepsilon}_i}{dt_*} = \vec{\omega}_{\mathrm{T}} \times_{e_0^*} \vec{\varepsilon}_i$$

The following expression can be established for the rotation vector:

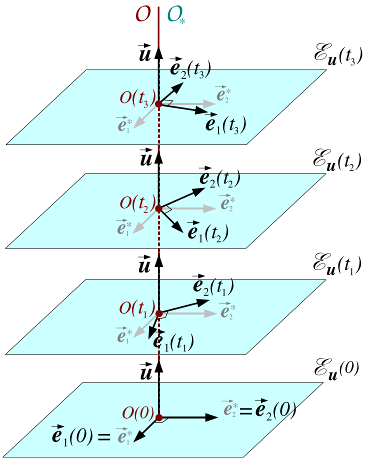
$$\vec{\omega}_{\mathrm{T}} = \frac{\Gamma^2}{c^2(1+\Gamma)} \, \vec{\gamma} \times_{e_0^*} \vec{V}$$

with
$\vec{V}$ = velocity of $\mathcal{O}$ with respect to $\mathcal{O}_*$
$\vec{\gamma}$ = acceleration of $\mathcal{O}$ with respect to $\mathcal{O}_*$
Γ = Lorentz factor of $\mathcal{O}$ with respect to $\mathcal{O}_*$

Remark: if $\mathcal{O}$ is a uniformly accelerated observer, $\vec{V}$ and $\vec{\gamma}$ are parallel, so that $\vec{\omega}_{\mathrm{T}} = 0$

Outline

Uniformly rotating observer



Observer $\mathcal{O}$ in **uniform rotation**:

$\vec{\boldsymbol{a}} = 0$ and $\vec{\boldsymbol{\omega}} = \mathrm{const}$

Local frame of $\mathcal{O}$:

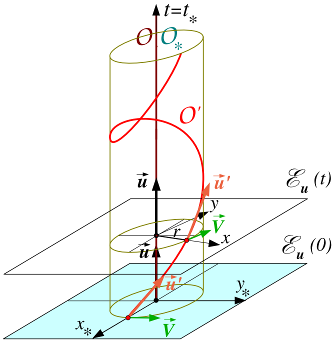
$$
\begin{aligned}
\vec{\boldsymbol{e}}_0(t) &= \vec{\boldsymbol{e}}^*_0 \\
\vec{\boldsymbol{e}}_1(t) &= \cos\omega t\, \vec{\boldsymbol{e}}^*_1 + \sin\omega t\, \vec{\boldsymbol{e}}^*_2 \\
\vec{\boldsymbol{e}}_2(t) &= -\sin\omega t\, \vec{\boldsymbol{e}}^*_1 + \cos\omega t\, \vec{\boldsymbol{e}}^*_2 \\
\vec{\boldsymbol{e}}_3(t) &= \vec{\boldsymbol{e}}^*_3 = \omega^{-1}\, \vec{\boldsymbol{\omega}}
\end{aligned}
$$

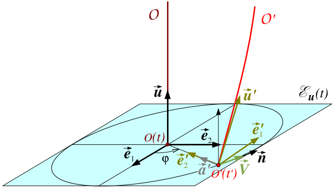
with $(\vec{\boldsymbol{e}}^*_\alpha)$ reference frame of inertial observer $\mathcal{O}_*$

Coordinate system of $\mathcal{O}$: (t, x, y, z) such that

$$
\begin{cases}
x_* = x\cos\omega t - y\sin\omega t \\
y_* = x\sin\omega t + y\cos\omega t \\
z_* = z
\end{cases}
$$

Corotating observer





Observer $\mathcal{O}'$ at rest with respect to $\mathcal{O}$, i.e. at fixed values of $x = r\cos\varphi$ and $y = r\cos\varphi$ $(z = 0)$

Worldline in term of inertial coordinates:

$$\left\{ \begin{array}{lcl} x_*(t) & = & r\cos(\omega t + \varphi) \\ y_*(t) & = & r\sin(\omega t + \varphi) \\ z_*(t) & = & 0. \end{array} \right.$$

Velocity of $\mathcal{O}'$ w.r.t. $\mathcal{O}_*$:

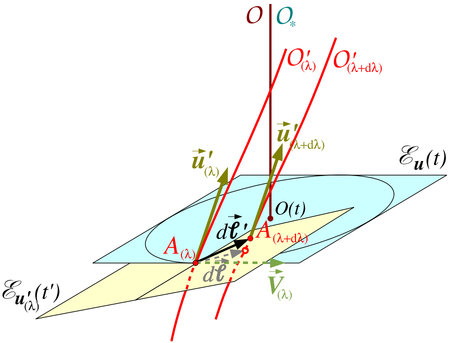
$\vec{\boldsymbol{V}} = r\omega\,\vec{\boldsymbol{n}}$, with $\vec{\boldsymbol{n}} := -\sin\varphi\,\vec{\boldsymbol{e}}_1 + \cos\varphi\,\vec{\boldsymbol{e}}_2$

4-acceleration of $\mathcal{O}'$:

$\vec{\boldsymbol{a}}' = \frac{\Gamma^2}{c^2} r\omega^2\,\vec{\boldsymbol{e}}'_2$, $\vec{\boldsymbol{e}}'_2 = -\cos\varphi\,\vec{\boldsymbol{e}}_1 - \sin\varphi\,\vec{\boldsymbol{e}}_2$

The problem of clock synchronization

1-parameter family of corotating observers $\mathcal{O}'_{(\lambda)}$



Moving from $\mathcal{O}'_{(\lambda)}$ to $\mathcal{O}'_{(\lambda+d\lambda)}$

$A_{(\lambda)}$: event on $\mathcal{O}'_{(\lambda)}$'s worldline

$A_{(\lambda+d\lambda)}$: event on $\mathcal{O}'_{(\lambda+d\lambda)}$'s worldline simultaneous to $A_{(\lambda)}$ for $\mathcal{O}'_{(\lambda)}$:

$$\vec{\boldsymbol{u}}'_{(\lambda)} \cdot \overrightarrow{A_{(\lambda)}A_{(\lambda+d\lambda)}} = 0 \qquad (4)$$

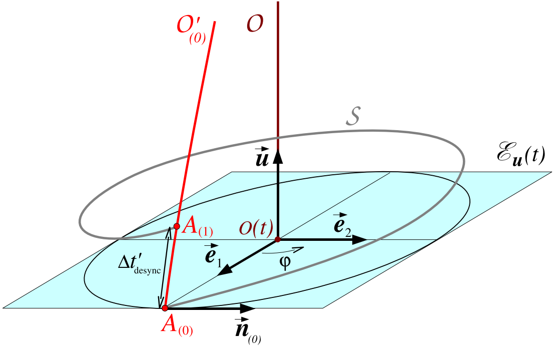
with $\overrightarrow{A_{(\lambda)}A_{(\lambda+d\lambda)}} = c\,dt\,\vec{\boldsymbol{u}} + d\vec{\boldsymbol{\ell}} + dt\,\vec{\boldsymbol{V}}$

$d\vec{\boldsymbol{\ell}} := dx^i\,\vec{\boldsymbol{e}}_i(t)$, separation between $\mathcal{O}'_{(\lambda)}$ and $\mathcal{O}'_{(\lambda+d\lambda)}$ from the point of view of $\mathcal{O}$

Expanding (4) yields $\boxed{dt = \Gamma^2 \dfrac{\vec{\boldsymbol{V}} \cdot d\vec{\boldsymbol{\ell}}}{c^2}}$

The problem of clock synchronization

Integrating on a closed contour

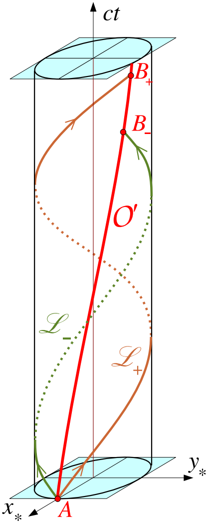


Desynchronization lapse:

$$\Delta t'_{\rm desync} = \frac{1}{c^2 \Gamma_{(0)}} \oint_{\mathcal{C}} \Gamma^2 \, \vec{\boldsymbol{V}} \cdot d\vec{\boldsymbol{\ell}}$$

Synchronization helix

Sagnac effect



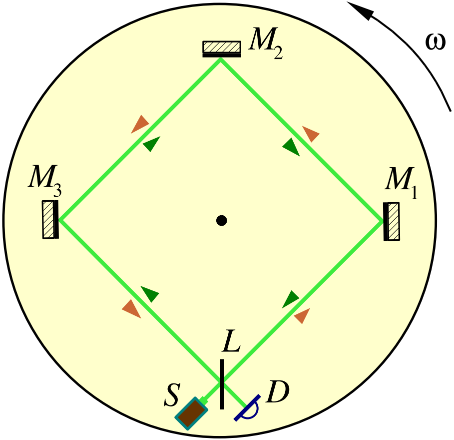
Two signals of *same velocity* w.r.t. $\mathcal{O}$

After a round trip, discrepancy between the two arrival times (t': proper time of emitter $\mathcal{O}'$):

$$\Delta t' := t'_+ - t'_- = 2\Delta t'_{\rm desync}$$

$$\Longrightarrow \boxed{\Delta t' = \frac{2}{c^2\Gamma_{(0)}} \oint_{\mathcal{C}} \Gamma^2\, \vec{V}\cdot d\vec{\ell}}$$ **Sagnac delay**

Sagnac experiment



Phase shift:

$$\Delta\phi = \frac{4\pi f}{c^2\Gamma_{(0)}} \oint_{\mathcal{C}} \Gamma^2 \, \vec{\boldsymbol{V}} \cdot d\vec{\boldsymbol{\ell}}$$

Slow rotation limit ($r\omega \ll c$):

$$\Delta\phi = \frac{8\pi f}{c^2} \, \vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{A}}$$

Application: gyrometers

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