Spectral Methods in Numerical Relativity

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Plan

1. Spectral methods developed in Meudon
2. Applications to general relativity
3. Spectral methods in numerical relativity around the World

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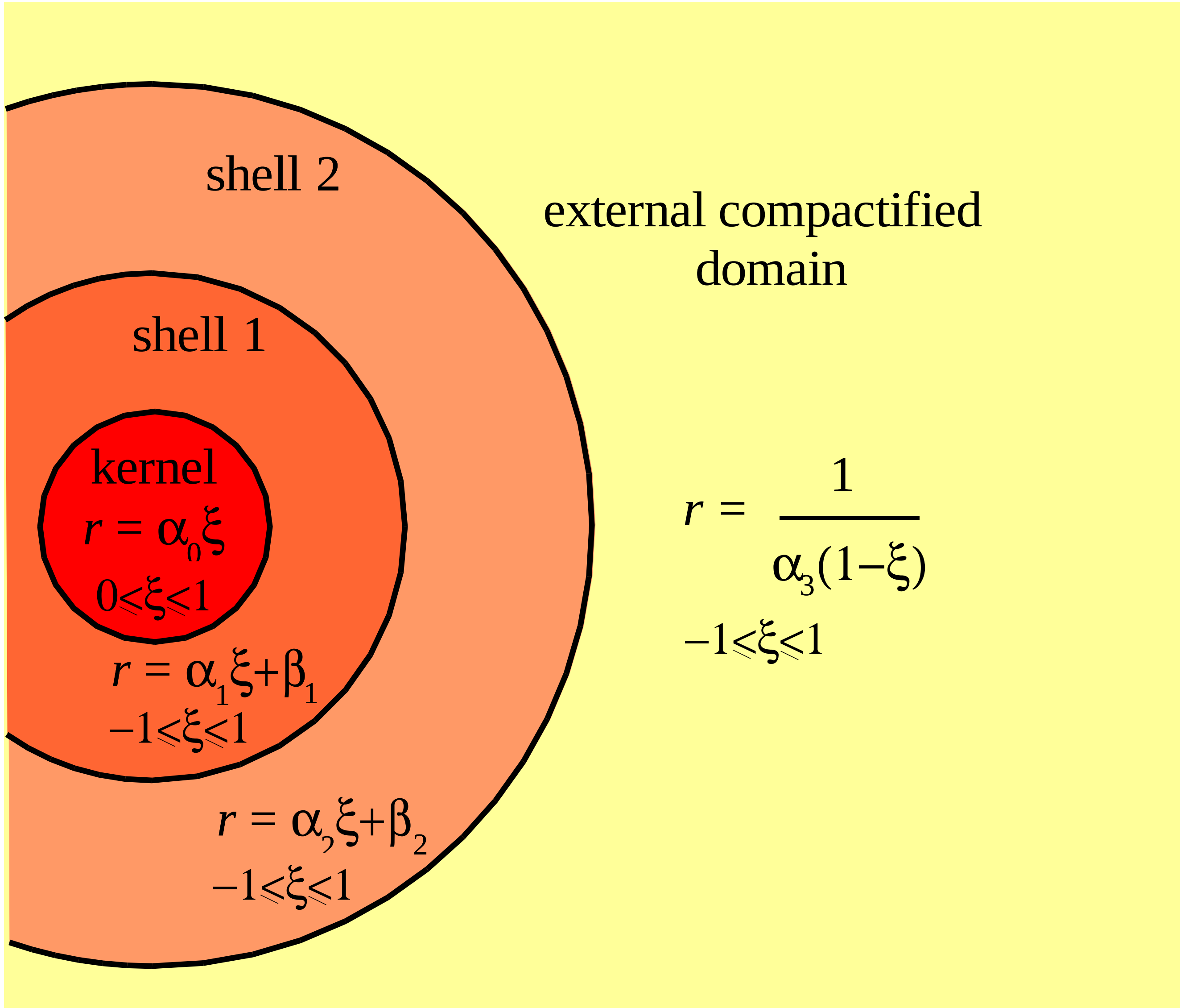
Spectral methods developed in Meudon

Basic features

- Multidomain three-dimensional spectral method
- Spherical-type coordinates (r, θ, φ)
- Expansion functions: r : Chebyshev; θ : cosine/sine or associated Legendre functions; φ : Fourier
- Domains = spherical shells + 1 nucleus (contains $r = 0$)
- Entire space ($\mathbb{R}^3$) covered: compactification of the outermost shell
- Adaptative coordinates : domain decomposition with spherical topology
- Multidomain PDEs: patching method (strong formulation)

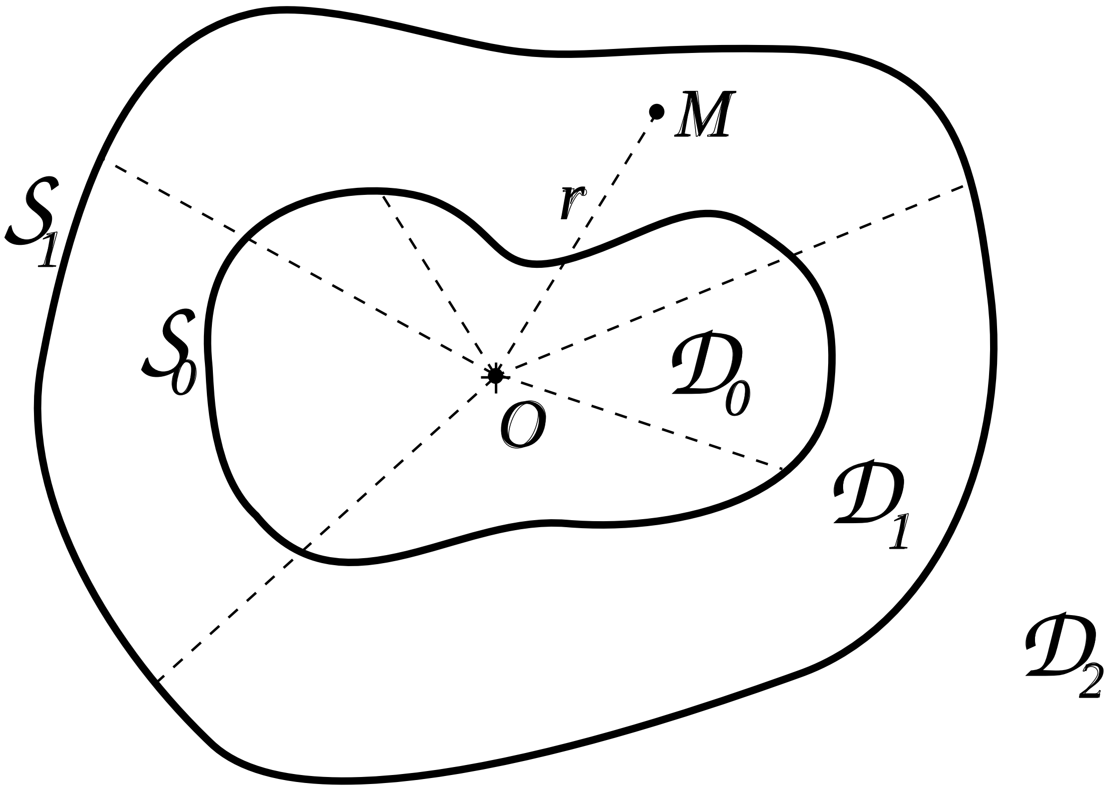
Domain decomposition



physical coordinates
(r, θ, φ)

comput. coordinates
(ξ, θ', φ')

Starlike domain decomposition



$\mathcal{N}$ nonoverlapping starlike domains:

- $\mathcal{D}_0$: *nucleus*
- $\mathcal{D}_q \quad (1 \leq q \leq \mathcal{N}-2)$: *shell*
- $\mathcal{D}_{\mathcal{N}-1}$: *external domain*

$$\mathcal{D}_0 \cup \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_{\mathcal{N}-1} = \mathbb{R}^3$$

Mapping computational space → physical space

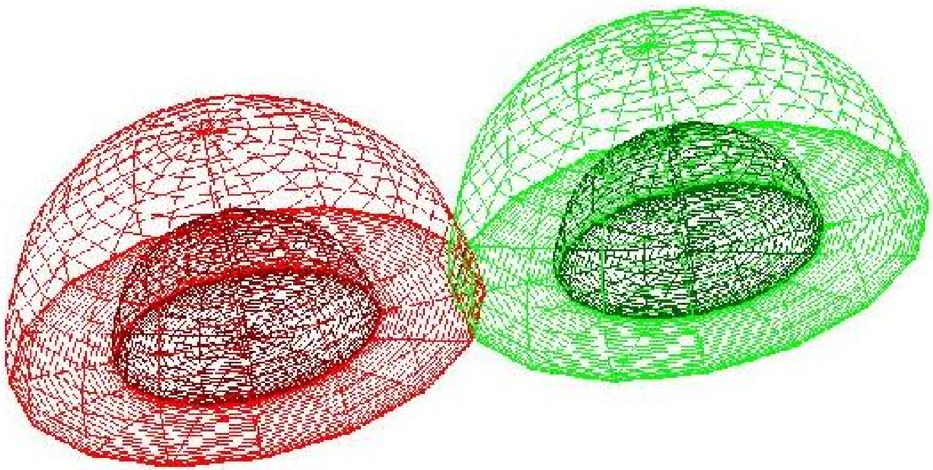
Mapping for domain $\mathcal{D}_q$:
$$\begin{array}{rcl} [-1+\delta_{0q}, 1] \times [0,\pi] \times [0, 2\pi[& \longrightarrow & \mathcal{D}_q \\ (\xi, \theta', \varphi') & \longmapsto & (r, \theta, \varphi) \end{array}$$

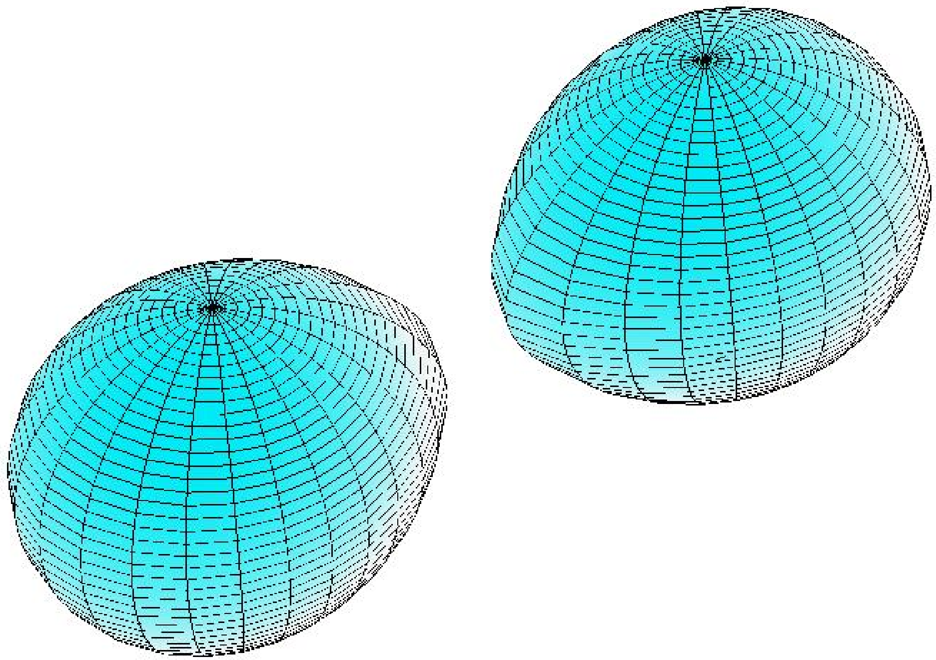
Radial mapping : $\theta = \theta'$ and $\varphi = \varphi'$

- in the nucleus: $\xi \in [0, 1]$ $$r = \alpha_0 \left[\xi + \left(3\xi^4 - 2\xi^6\right) F_0(\theta,\varphi) + \frac{1}{2}\left(5\xi^3 - 3\xi^5\right) G_0(\theta,\varphi) \right]$$
- in the shells: $\xi \in [-1, 1]$ $$r = \alpha_q \left[\xi + \frac{1}{4}\left(\xi^3 - 3\xi + 2\right) F_q(\theta,\varphi) + \frac{1}{4}\left(-\xi^3 + 3\xi + 2\right) G_q(\theta,\varphi) \right] + \beta_q$$
- in the external domain: $\xi \in [-1, 1]$ $$\frac{1}{r} = \alpha_{\rm ext} \left[\xi + \frac{1}{4}\left(\xi^3 - 3\xi + 2\right) F_{\rm ext}(\theta,\varphi) - 1 \right]$$

[Bonazzola, Gourgoulhon & Marck, Phys. Rev. D **58**, 104020 (1998)]

Example: binary star with surface fitted coordinates





[Taniguchi, Gourgoulhon & Bonazzola, Phys. Rev. D **64**, 064012 (2001)]

Double domain decomposition

Surface fitted coordinates:

$F_0(\theta, \varphi)$ and $G_0(\theta, \varphi)$ chosen so that

$\xi = 1 \Leftrightarrow$ surface of the star

Basis functions

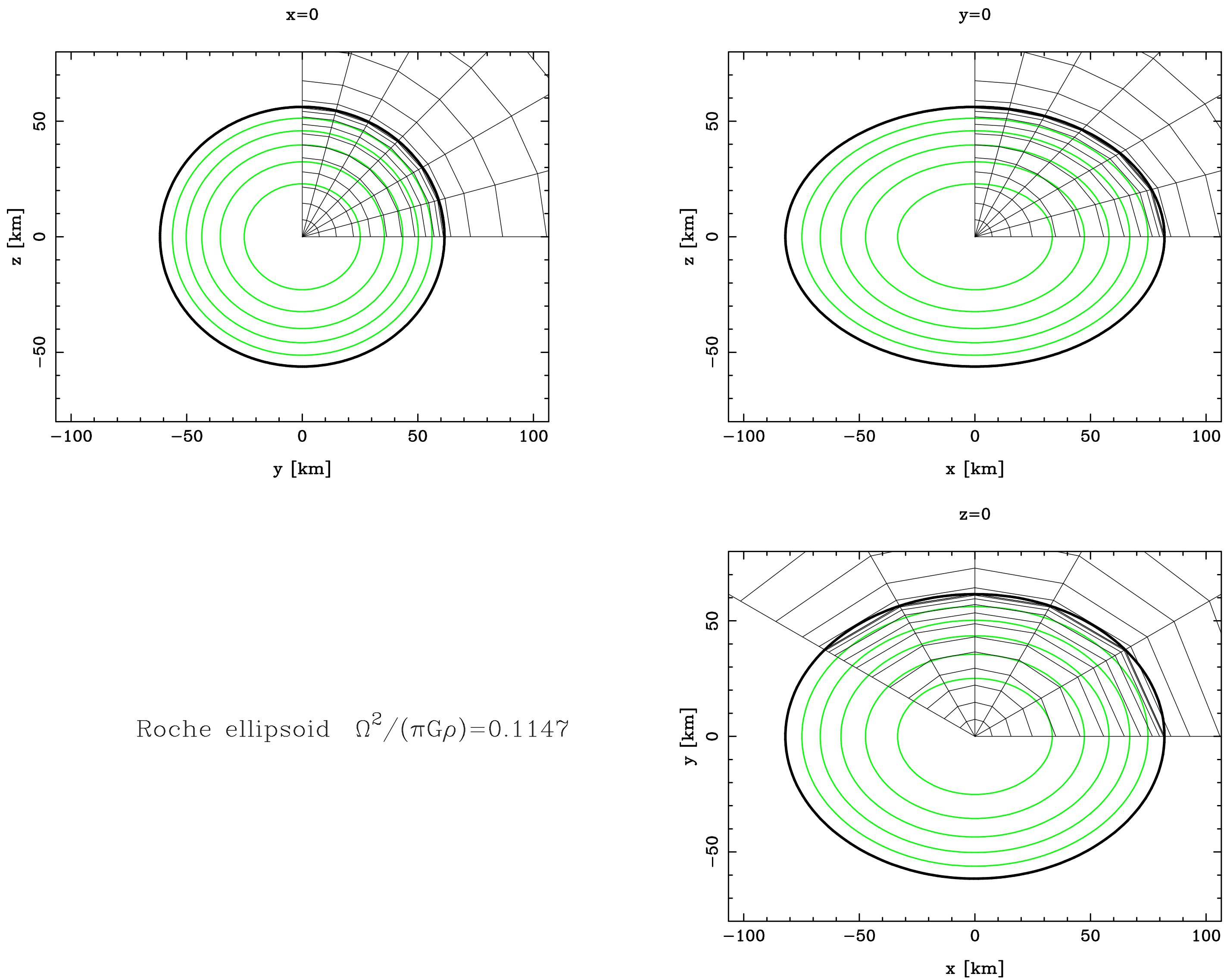
Polynomial interpolant of a field u in a given domain $\mathcal{D}_q$:

$$I_N u_q(\xi,\theta,\varphi) = \sum_{m=0}^{N_\varphi/2} \sum_{j=0}^{N_\theta-1} \sum_{i=0}^{N_r-1} \hat{u}_{qmji}\, X_i(\xi)\, \Theta_j(\theta)\, e^{im\varphi} \qquad \text{with } N := (N_r, N_\theta, N_\varphi)$$

Regularity at the origin and on the axis $\theta = 0$ + equatorial symmetry:

- φ expansion: **Fourier series**
- θ expansion: **Trigonometric polynomials** or **associated Legendre functions**
  - ⋆ for m even: $\Theta_j(\theta) = \cos(2j\theta)$ or $\Theta_j(\theta) = P_{2j}^m(\cos\theta)$
  - ⋆ for m odd: $\Theta_j(\theta) = \sin((2j+1)\theta)$ or $\Theta_j(\theta) = P_{2j+1}^m(\cos\theta)$
- ξ expansion: **Chebyshev polynomials**
  - ⋆ in the kernel: $X_i(\xi) = T_{2i}(\xi)$ for m even, $X_i(\xi) = T_{2i+1}(\xi)$ for m odd
  - ⋆ in the shells and the external compactified domain: $X_i(\xi) = T_i(\xi)$

Corresponding collocation points: an example



Roche ellipsoid $\Omega^2/(\pi G\rho)=0.1147$

[from Bonazzola, Gourgoulhon & Marck, Phys. Rev. D **58**, 104020 (1998)]

Resolution of Poisson equation with noncompact source

Consider the three-dimensional Poisson equation on $\mathbb{R}^3$:

$$\Delta u(r,\theta,\varphi) = s(r,\theta,\varphi) \qquad (1)$$

with the boundary condition

$$u(r,\theta,\varphi) \to 0 \quad \text{when } r \to +\infty \qquad (2)$$

The source s has a non-compact support and obeys to the fall-off conditions

$$s(r,\theta,\varphi) \sim \sum_{q=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} \frac{Y_\ell^m(\theta,\varphi)}{r^{\ell+4}} \quad \text{when } r \to +\infty \qquad (3)$$

Spherical harmonics expansions

Interpolant of the source in a domain $\mathcal{D}_q$ (notation:$s_q := s|_{\mathcal{D}_q}$) :

$$I_N s_q(\xi, \theta, \varphi) = \sum_{\ell=0}^{N_\theta - 1} \sum_{m=-\ell}^{\ell} \hat{s}_{q\ell m}(\xi)\, Y_\ell^m(\theta, \varphi)$$

Search for a numerical solution under the form

$$\bar{u}_q(\xi, \theta, \varphi) = \sum_{\ell=0}^{N_\theta - 1} \sum_{m=-\ell}^{\ell} \hat{u}_{q\ell m}(\xi)\, Y_\ell^m(\theta, \varphi)$$

Shorthand notation: $u_\bullet(\xi) := \hat{u}_{q\ell m}(\xi)$.

Eq. (1) becomes an ODE system:

- In the nucleus ($r = \alpha\xi$) :

$$\frac{d^2 u_\bullet}{d\xi^2} + \frac{2}{\xi}\left(\frac{du_\bullet}{d\xi} - \frac{du_\bullet}{d\xi}(0)\right) - \frac{\ell(\ell+1)}{\xi^2}\left(u_\bullet - u_\bullet(0) - \xi\frac{du_\bullet}{d\xi}(0)\right) = \alpha^2 \hat{s}_{0\ell m}(\xi)$$

- In the shells ($r = \alpha\xi + \beta$):

$$\left(\xi + \frac{\beta}{\alpha}\right)^2 \frac{d^2 u_\bullet}{d\xi^2} + 2\left(\xi + \frac{\beta}{\alpha}\right)\frac{du_\bullet}{d\xi} - \ell(\ell+1)u_\bullet = (\alpha\xi + \beta)^2 \hat{s}_{q\ell m}(\xi)$$

- In the external domain ($r^{-1} = \alpha(\xi - 1)$) :

$$\frac{d^2 u_\bullet}{d\xi^2} - \frac{\ell(\ell+1)}{(\xi-1)^2}\left(u_\bullet - u_\bullet(1) - (\xi - 1)\frac{du_\bullet}{d\xi}(1)\right) = \frac{\hat{s}_{q\ell m}(\xi)}{\alpha^4(\xi-1)^4}$$

Resolution by means of a Chebyshev tau method

- In the nucleus : $u_\bullet(\xi) = \sum_{i=0}^{N_r-1} \hat{u}_{q\ell m i}\, T_{2i}(\xi)$ for ℓ even

$$u_\bullet(\xi) = \sum_{i=0}^{N_r-2} \hat{u}_{q\ell m i}\, T_{2i+1}(\xi) \text{ for } \ell \text{ odd}$$

- In the shells and external domain : $u_\bullet(\xi) = \sum_{i=0}^{N_r-1} \hat{u}_{q\ell m i} T_i(\xi)$

Linear combinations $\longrightarrow$ banded matrices (5 bands)

Patching method

Number of solutions of the homogeneous equation:

- In the nucleus : 1 (r^ℓ)
- In the shells : 2 (r^ℓ and $r^{-(\ell+1)}$)
- In the external domain : 1 ($r^{-(\ell+1)}$)

Total : $1 + 2(\mathcal{N} - 2) + 1 = 2\mathcal{N} - 2$

Matching conditions: continuity of u and its first radial derivative accross the $\mathcal{N} - 1$ boundaries between the domains $\mathcal{D}_q \Longrightarrow 2\mathcal{N} - 2$ conditions

Behavior of the numerical error

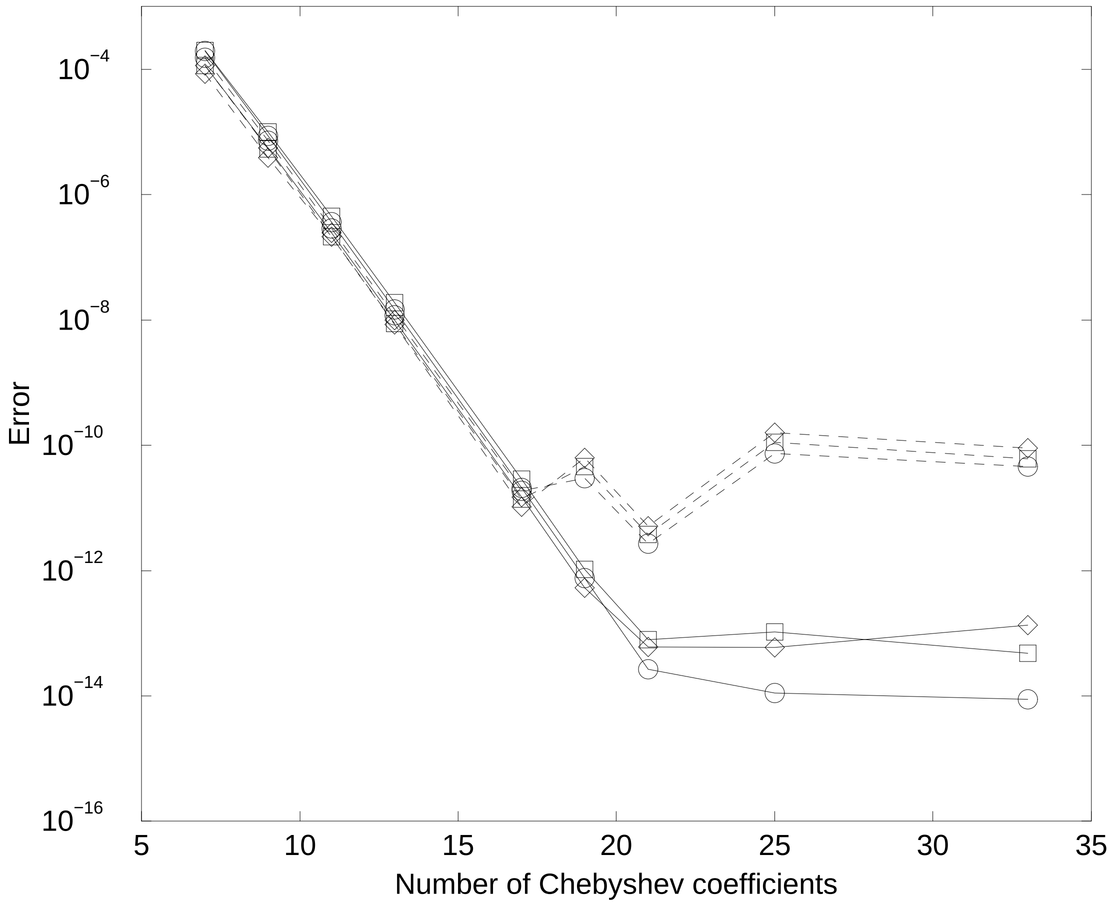
Source with a non-compact support, decaying as r^{-k}:

- evanescent error (error $\propto \exp(-N_r)$) if the source does not contain any spherical harmonics of index $\ell \geq k-3$
- error decreasing as $N^{-2(k-2)}$ otherwise

[Grandclément, Bonazzola, Gourgoulhon & Marck, J. Comp. Phys. **170**, 231 (2001)]

Extension to vector Poisson-type equations

Minimal distortion equation for the shift vector: $\Delta\vec{\beta} + \frac{1}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{\beta}) = \vec{S}$



Error on the z component of the solution of the minimal distortion equation with a non-compact source

[Grandclément, Bonazzola, Gourgoulhon & Marck, J. Comp. Phys. **170**, 231 (2001)]

Wave equation with nonreflecting boundary conditions

Consider the wave equation

$$\Box\, u(t, r, \theta, \varphi) = s(t, r, \theta, \varphi) \tag{4}$$

with the radiating boundary condition

$$\lim_{r\to\infty} \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial t}\right) (r\, u) = 0. \tag{5}$$

Solve (4) in a finite ball $\mathcal{D}$ of radius R with some boundary conditions which approximate (5) when $R \to \infty$.

Decompose $\mathcal{D}$ in $\mathcal{N}$ spherical subdomains $\mathcal{D}_q$ with $\mathcal{D}_0 =$ nucleus and the other domains $=$ shells (no external compactified domain).

Finite-differencing in time: second-order implicit Crank-Nicolson scheme.
Space part: patching with Chebyshev tau

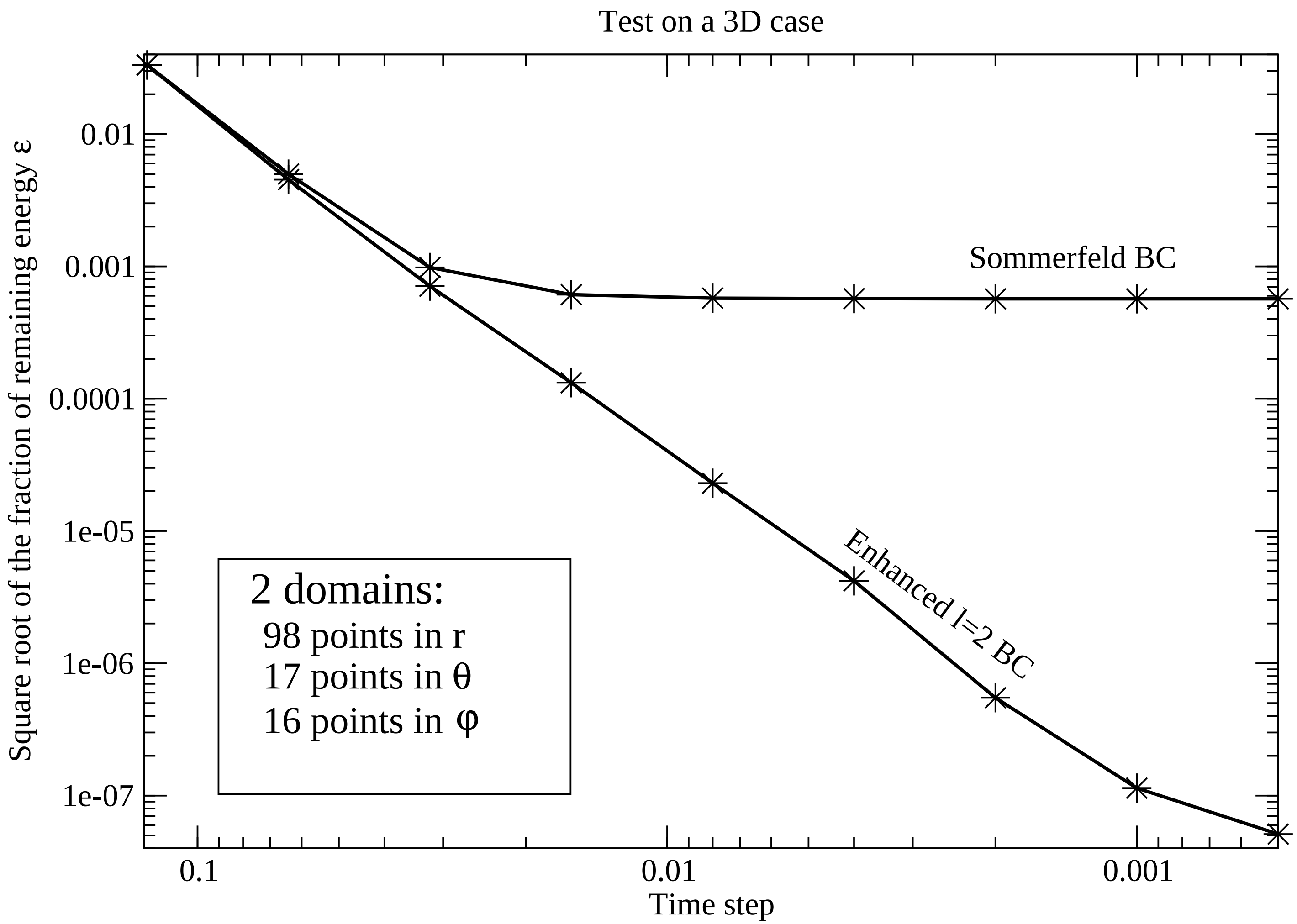
Non reflecting BC up to $\ell = 2$

Method of Bayliss & Turkel [Comm. Pure Appl. Math. **33**, 707 (1980)]:

$$
\begin{aligned}
B_1 u &:= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial r} + \frac{u}{r} \\
B_2 u &:= \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{3}{r} \right) B_1 u \\
B_3 u &:= \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{5}{r} \right) B_2 u
\end{aligned}
$$

Boundary condition : $B_3 u|_{r=R} = 0$.
$\Rightarrow$ ensures that spherical harmonics with $\ell = 0$, $\ell = 1$ and $\ell = 2$ are perfectly outgoing.
This is important for gravitational waves.

Comparison with Sommerfeld boundary condition



[Novak & Bonazzola, gr-qc/0203102]

Numerical implementation: LORENE

Langage **O**bjet pour la **RE**lativite **N**ameriqu**E**

A library of C++ classes devoted to multi-domain spectral methods, with adaptive spherical coordinates.

- 1997 : start of Lorene
- 1999 : Accurate models of rapidly rotating strange quark stars
- 1999 : Neutron star binaries on closed circular orbits
- 2001 : Public domain (GPL), Web page: `http://www.lorene.obspm.fr`
- 2001 : Black hole binaries on closed circular orbits
- 2002 : 3-D wave equation with non-reflecting boundary conditions
- 2002 : Maclaurin-Jacobi bifurcation point in general relativity

2

Applications to general relativity

Rotating relativistic stars

Spacetime metric :

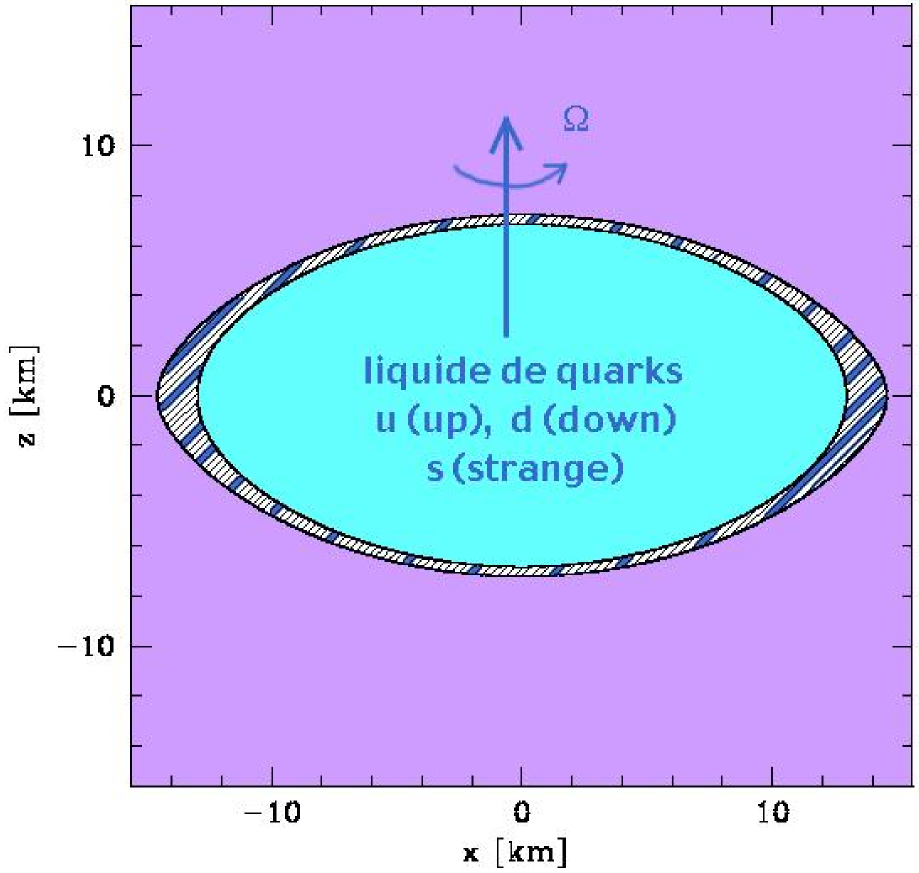
$$g_{\mu\nu}\, dx^\mu dx^\nu = -N^2\, \mathrm{d}t^2 + B^2 r^2 \sin^2\theta (\mathrm{d}\varphi - N^\varphi \mathrm{d}t)^2 + A^2(\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2)$$

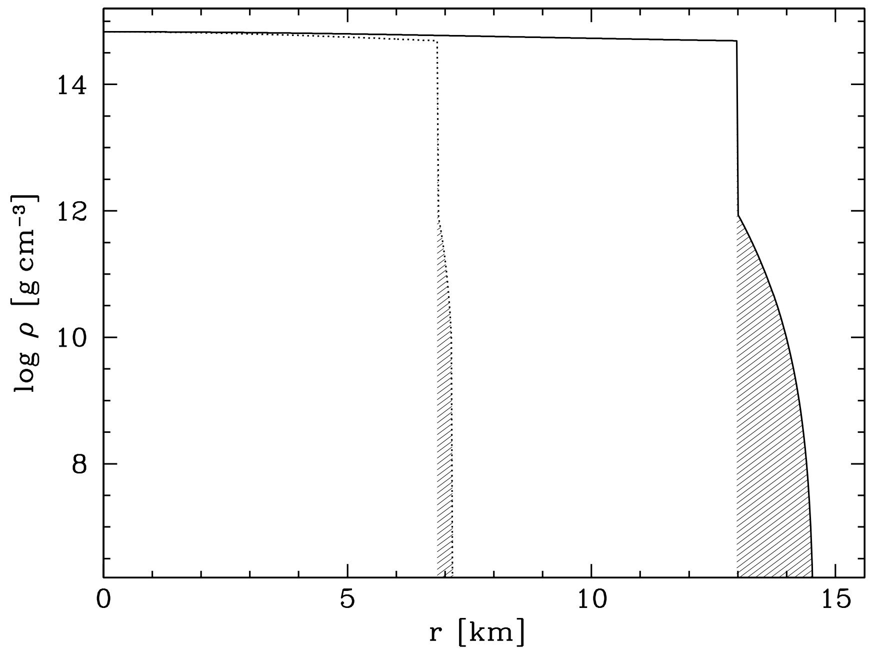
Einstein equations:

$$\begin{aligned}
\Delta_3\, \nu &= 4\pi A^2 (E + 3p + (E+p)U^2) + \frac{B^2 r^2 \sin^2\theta}{2N^2} (\partial N^\varphi)^2 - \partial\nu\, \partial(\nu + \beta) \\
\tilde{\Delta}_3\, (N^\varphi r \sin\theta) &= -16\pi \frac{N A^2}{B} (E+p)U - r\sin\theta\, \partial N^\varphi\, \partial(3\beta - \nu) \\
\Delta_2\, [(NB-1)\, r\sin\theta] &= 16\pi N A^2 B p r \sin\theta \\
\Delta_2\, \zeta &= 8\pi A^2\, [P + (E+p)U^2] + \frac{3B^2 r^2 \sin^2\theta}{4N^2} (\partial N^\varphi)^2 - (\partial\nu)^2\ ,
\end{aligned}$$

with $\nu := \ln N$, $\quad \zeta := \ln(AN)$, $\quad \beta := \ln B$.

Strange quark stars

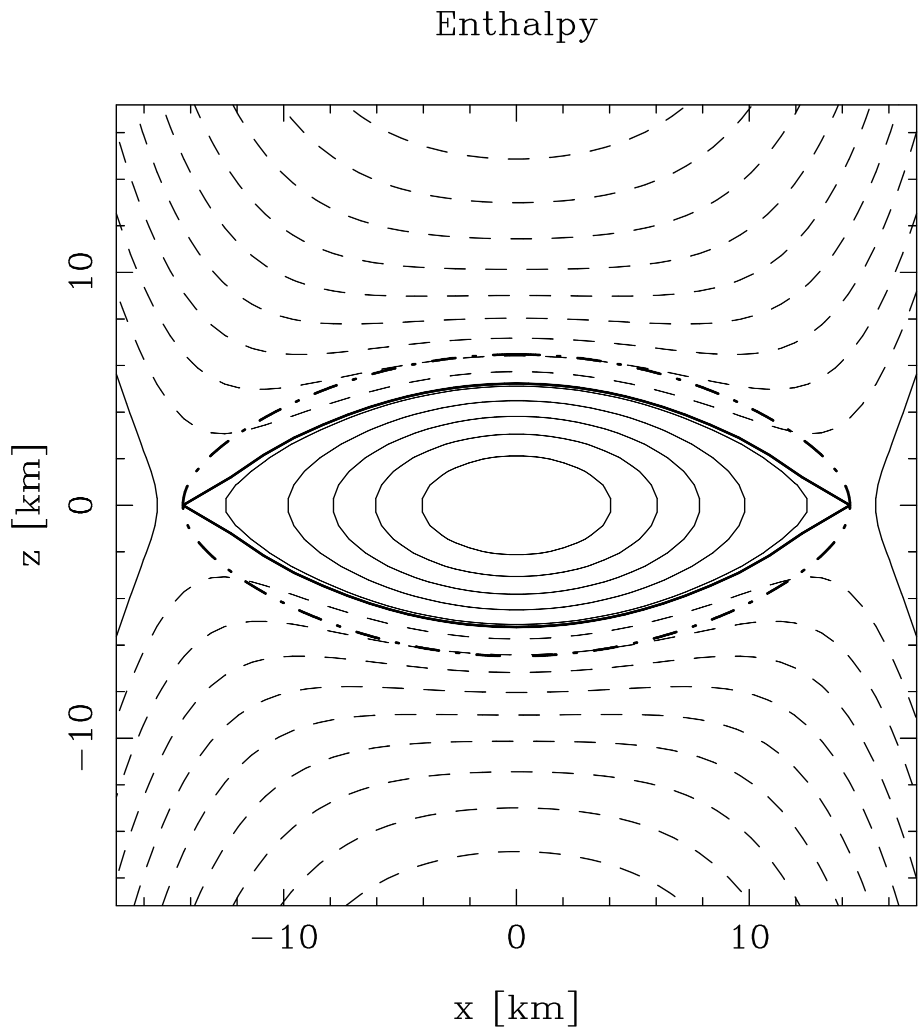




EOS: $B = 56\ \mathrm{MeV\,fm^{-3}}$, $\alpha_\mathrm{s} = 0.2$, $m_\mathrm{s} = 200\ \mathrm{MeV}\,c^{-2}$
star: $M_\mathrm{B} = 1.63\,M_\odot$, $f = 1210$ Hz.

[from Zdunik, Haensel, Gourgoulhon, A&A **372**, 535 (2001)]

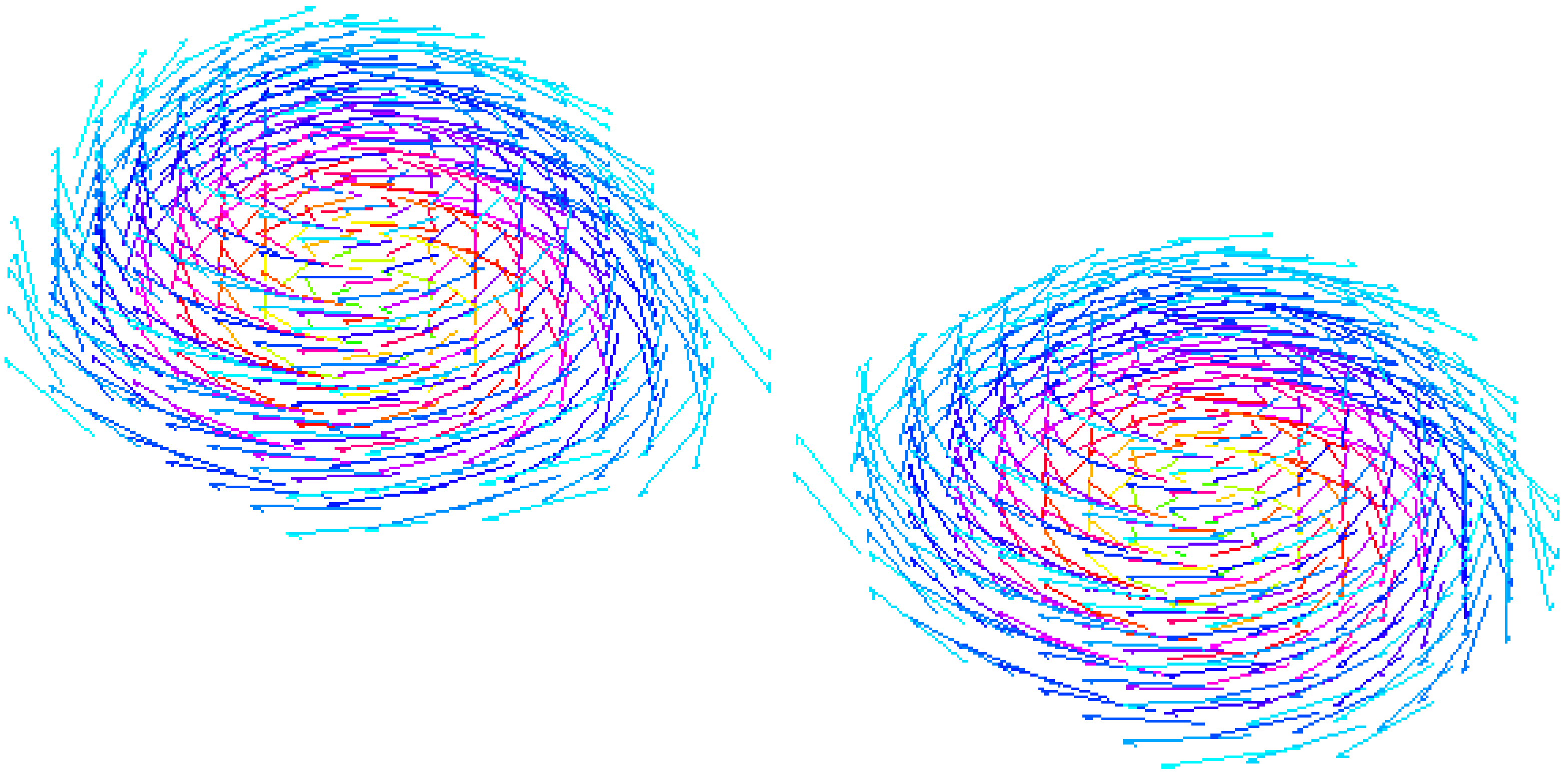
Maximally rotating strange quark stars



[from Gourgoulhon et al., A&A **349**, 851 (1999)]

Minimal rotation period (for $m_\mathrm{s} = 0$ and $\alpha_\mathrm{s} = 0$): $P_\mathrm{min} = 0.634\, {B_{60}}^{-1/2}$ ms

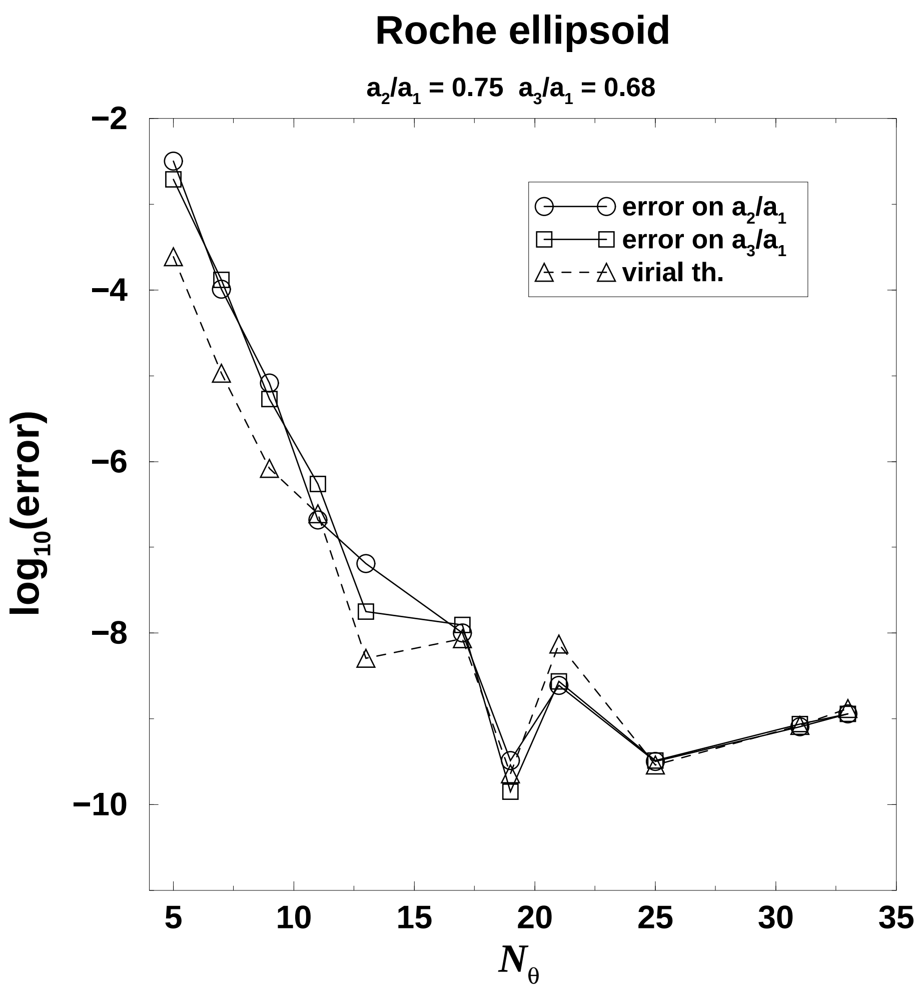
Binary neutron stars



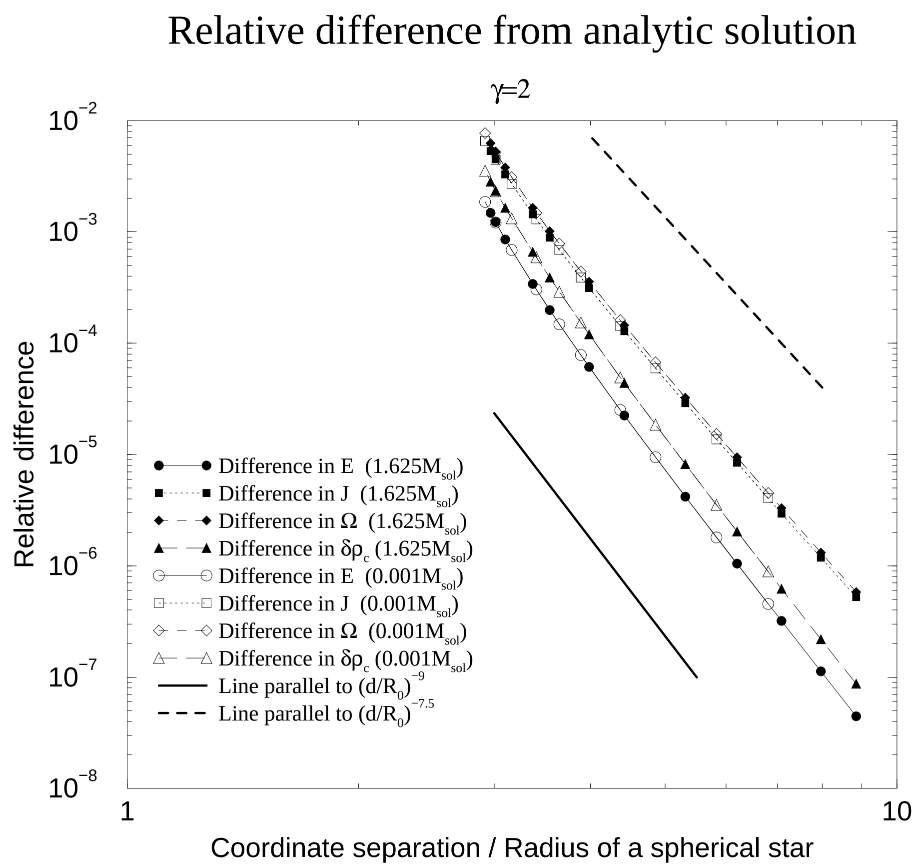
Velocity field w.r.t. co-orbiting frame for irrotational binaries

[from Gourgoulhon, Grandclément, Taniguchi, Marck & Bonazzola, Phys. Rev. D **63**, 064029 (2001)]

Comparison with analytical solutions



Difference w.r.t. the Roche solution



Difference w.r.t. Taniguchi & Nakamura approxim. solution

Binary black holes in circular orbits

Framework: 3+1 formalism with maximal slicing and helical symmetry

Isenberg-Wilson-Mathews approximation: conformally flat spatial metric: $\boldsymbol{\gamma} = \Psi^4 \boldsymbol{f}$

$\Rightarrow$ spacetime metric : $ds^2 = -N^2 dt^2 + \Psi^4 f_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$

Amounts to solve 5 of the 10 Einstein equations (one more than IVP !) :

$$\Delta\Psi = -\frac{\Psi^5}{8}\hat{A}_{ij}\hat{A}^{ij} \qquad \text{(Lichnerowicz equation)} \qquad \textit{(Hamiltonian constraint)}$$

$$\Delta\beta^i + \frac{1}{3}\bar{D}^i\bar{D}_j\beta^j = 2\hat{A}^{ij}\left(\bar{D}_j N - 6N\bar{D}_j \ln\Psi\right) \qquad \textit{(momentum constraint)}$$

$$\Delta N = N\Psi^4\hat{A}_{ij}\hat{A}^{ij} - 2\bar{D}_j \ln\Psi\, \bar{D}^j N \qquad \textit{(trace of } \frac{\partial K_{ij}}{\partial t} = \cdots \textit{)}$$

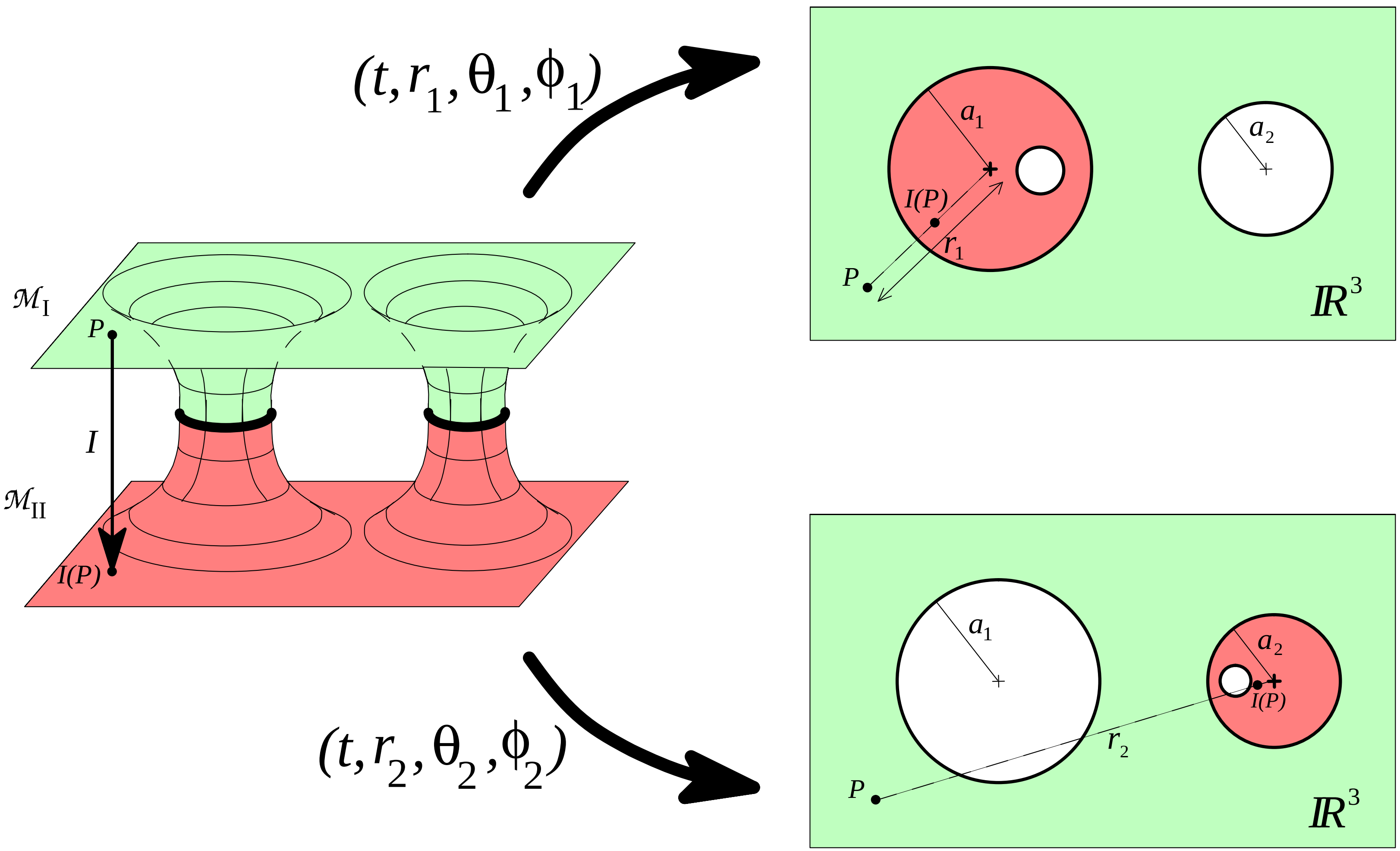
with $\hat{A}_{ij} := \Psi^{-4}K_{ij}$ and $\hat{A}^{ij} := \Psi^4 K^{ij}$

Kinematical relation between $\boldsymbol{\gamma}$ and $\boldsymbol{K}$:

$$\hat{A}^{ij} = \frac{1}{2N}(L\beta)^{ij} \text{ with } (L\beta)^{ij} := \bar{D}^i\beta^j + \bar{D}^j\beta^i - \frac{2}{3}\bar{D}_k\beta^k\, f^{ij} \qquad \textit{(traceless part)}$$

$$\bar{D}_i\beta^i = -6\beta^i\bar{D}_i \ln\Psi \qquad \textit{(trace part)}$$

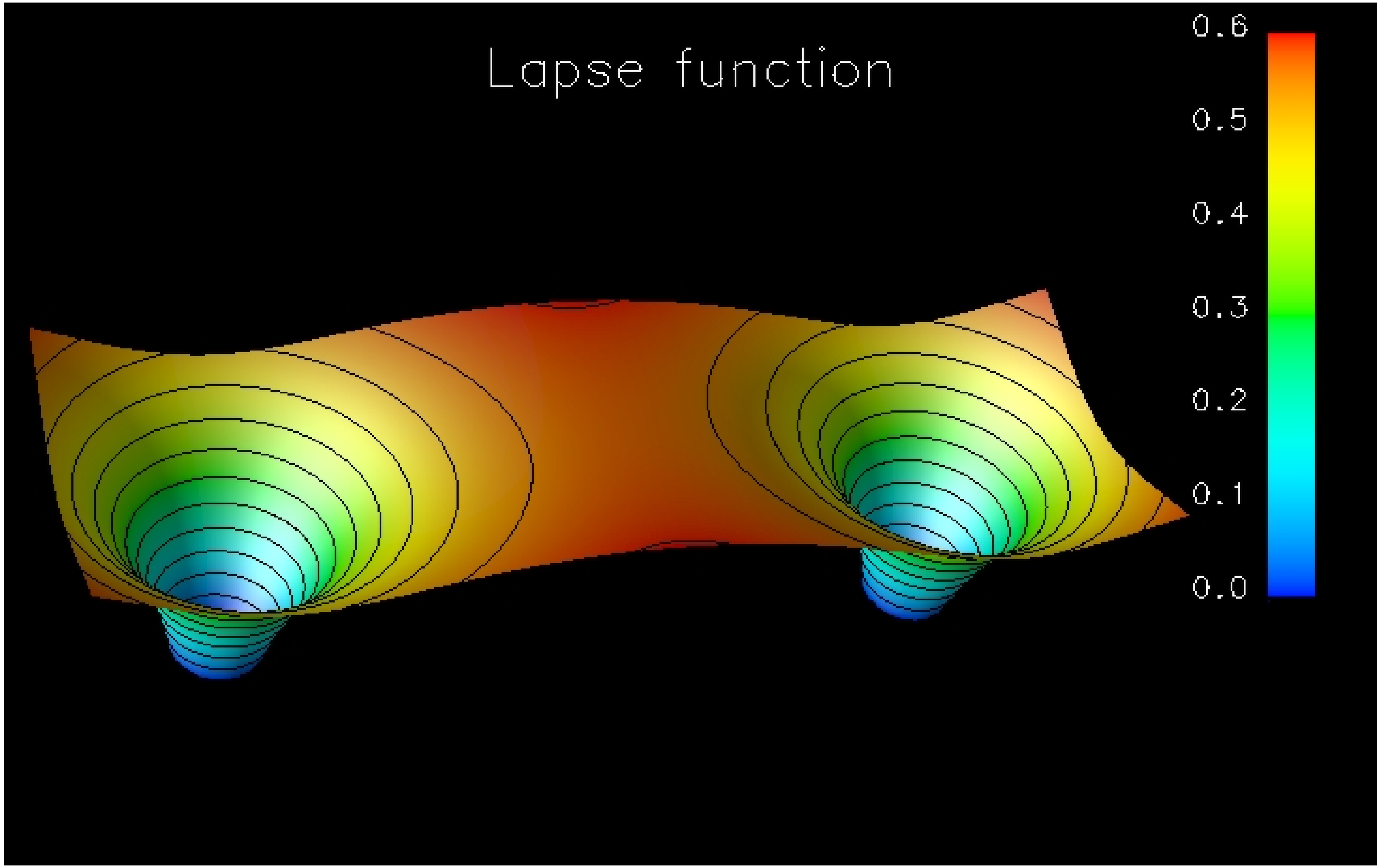
Spacetime manifold



[from Gourgoulhon, Grandclément & Bonazzola, Phys. Rev. D **65**, 044020 (2002)]

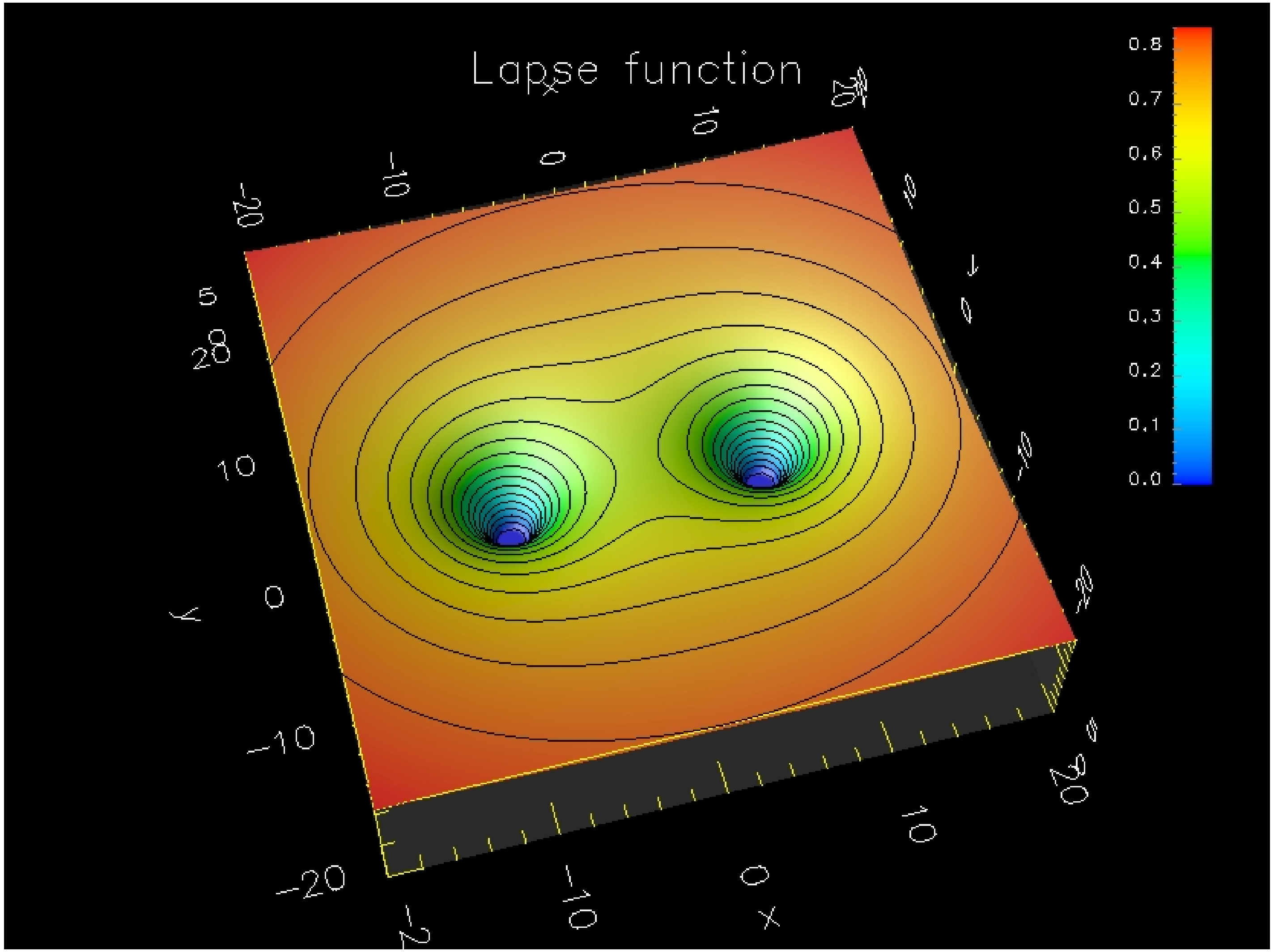
Numerical results

ISCO configuration



[from Grandclément, Gourgoulhon & Bonazzola, Phys. Rev. D **65**, 044021 (2002)]

ISCO configuration



[from Grandclément, Gourgoulhon, Bonazzola, PRD **65**, 044021 (2002)]

Spectral methods developed in other relativity groups

- **Cornell group**: Black holes
- **Bartnik:** quasi-spherical slicing
- **Carsten Gundlach:** apparent horizon finder
- **Jörg Frauendiener:** conformal field equations
- **Jena group:** extremely precise models of rotating stars