Symbolic tensor calculus on manifolds

Éric Gourgoulhon

Laboratoire Univers et Théories (LUTH)
CNRS / Observatoire de Paris / Université Paris Diderot
Université Paris Sciences et Lettres
92190 Meudon, France

http://sagemanifolds.obspm.fr/jncf2018/

Journées Nationales de Calcul Formel
Centre International de Rencontres Mathématiques
Luminy, Marseille, France
22-26 January 2018

Outline

http://sagemanifolds.obspm.fr/jncf2018/

Outline

What is tensor calculus on manifolds?

By *tensor calculus* it is usually meant

- arithmetics of tensor fields
- tensor product, contraction
- (anti)symmetrization
- Lie derivative along a vector field
- pullback and pushforward associated to a smooth manifold map
- exterior calculus on differential forms
- ...

What is tensor calculus on manifolds?

By *tensor calculus* it is usually meant

- arithmetics of tensor fields
- tensor product, contraction
- (anti)symmetrization
- Lie derivative along a vector field
- pullback and pushforward associated to a smooth manifold map
- exterior calculus on differential forms
- ...

On pseudo-Riemannian manifolds:

- musical isomorphisms
- Levi-Civita connection
- curvature tensor
- Hodge duality
- ...

A few words about history

Symbolic tensor calculus is almost as old as computer algebra:

- Computer algebra system started to be developed in the 1960's; for instance `Macsyma` (to become `Maxima` in 1998) was initiated in 1968 at MIT

A few words about history

Symbolic tensor calculus is almost as old as computer algebra:

- Computer algebra system started to be developed in the 1960's; for instance `Macsyma` (to become `Maxima` in 1998) was initiated in 1968 at MIT
- In 1965, J.G. Fletcher developed the `GEOM` program, to compute the Riemann tensor of a given metric

A few words about history

Symbolic tensor calculus is almost as old as computer algebra:

- Computer algebra system started to be developed in the 1960's; for instance Macsyma (to become Maxima in 1998) was initiated in 1968 at MIT
- In 1965, J.G. Fletcher developed the GEOM program, to compute the Riemann tensor of a given metric
- In 1969, during his PhD under Pirani supervision, Ray d'Inverno wrote ALAM (Atlas Lisp Algebraic Manipulator) and used it to compute the Riemann tensor of Bondi metric. The original calculations took Bondi and his collaborators 6 months to go. The computation with ALAM took 4 minutes and yielded to the discovery of 6 errors in the original paper [J.E.F. Skea, *Applications of SHEEP* (1994)]

A few words about history

Symbolic tensor calculus is almost as old as computer algebra:

- Computer algebra system started to be developed in the 1960's; for instance `Macsyma` (to become `Maxima` in 1998) was initiated in 1968 at MIT
- In 1965, J.G. Fletcher developed the `GEOM` program, to compute the Riemann tensor of a given metric
- In 1969, during his PhD under Pirani supervision, Ray d'Inverno wrote `ALAM (Atlas Lisp Algebraic Manipulator)` and used it to compute the Riemann tensor of Bondi metric. The original calculations took Bondi and his collaborators 6 months to go. The computation with ALAM took 4 minutes and yielded to the discovery of 6 errors in the original paper [J.E.F. Skea, *Applications of SHEEP* (1994)]
- Since then, many software tools for tensor calculus have been developed... A rather exhaustive list: `http://www.xact.es/links.html`

Tensor calculus software

Packages for general purpose computer algebra systems

- xAct free package for Mathematica [J.-M. Martin-Garcia]
- Ricci free package for Mathematica [J.L. Lee]
- MathTensor package for Mathematica [S.M. Christensen & L. Parker]
- GRTensor III package for Maple [P. Musgrave, D. Pollney & K. Lake]
- DifferentialGeometry included in Maple [I.M. Anderson & E.S. Cheb-Terrab]
- Atlas 2 for Maple and Mathematica
- SageManifolds included in SageMath

Standalone applications

- SHEEP, Classi, STensor, based on Lisp, developed in 1970's and 1980's (free) [R. d'Inverno, I. Frick, J. Åman, J. Skea, et al.]
- Cadabra (free) [K. Peeters]
- Redberry (free) [D.A. Bolotin & S.V. Poslavsky]

cf. the complete list at http://www.xact.es/links.html

Tensor calculus software

Two types of **tensor computations**:

Abstract calculus (index manipulations)

- xAct/xTensor
- MathTensor
- Ricci
- Cadabra
- Redberry

Component calculus (explicit computations)

- xAct/xCoba
- Atlas 2
- DifferentialGeometry
- SageManifolds

The purpose of this lecture

Present a symbolic tensor calculus method that

- runs on fully specified smooth manifolds (described by an atlas)

The purpose of this lecture

Present a symbolic tensor calculus method that

- runs on fully specified smooth manifolds (described by an atlas)
- is not limited to a single coordinate chart or vector frame

The purpose of this lecture

Present a symbolic tensor calculus method that

- runs on fully specified smooth manifolds (described by an atlas)
- is not limited to a single coordinate chart or vector frame
- runs even on non-parallelizable manifolds

The purpose of this lecture

Present a symbolic tensor calculus method that

- runs on fully specified smooth manifolds (described by an atlas)
- is not limited to a single coordinate chart or vector frame
- runs even on non-parallelizable manifolds
- is independent of the symbolic engine (e.g. *Pynac/Maxima*, *SymPy*,...) used to perform calculus at the level of coordinate expressions

The purpose of this lecture

Present a symbolic tensor calculus method that

- runs on fully specified smooth manifolds (described by an atlas)
- is not limited to a single coordinate chart or vector frame
- runs even on non-parallelizable manifolds
- is independent of the symbolic engine (e.g. *Pynac/Maxima*, *SymPy*,...) used to perform calculus at the level of coordinate expressions

The purpose of this lecture

Present a symbolic tensor calculus method that

- runs on fully specified smooth manifolds (described by an atlas)
- is not limited to a single coordinate chart or vector frame
- runs even on non-parallelizable manifolds
- is independent of the symbolic engine (e.g. *Pynac/Maxima*, *SymPy*,...) used to perform calculus at the level of coordinate expressions

with some details of its implementation in SageMath, which has been performed via the *SageManifolds project*:

http://sagemanifolds.obspm.fr

by these contributors:

http://sagemanifolds.obspm.fr/authors.html

Outline

Topological manifold

Definition

Let $\mathbb{K}$ be a topological field. Given an integer $n \geq 1$, a ***topological manifold of dimension*** n ***over*** $\mathbb{K}$ is a topological space M obeying the following properties:

1. M is a Hausdorff (separated) space
2. M has a *countable base*: there exists a countable family $(U_k)_{k\in\mathbb{N}}$ of open sets of M such that any open set of M can be written as the union of some members of this family.
3. Around each point of M, there exists a neighbourhood which is homeomorphic to an open subset of $\mathbb{K}^n$.

SageMath implementation

See the online worksheet

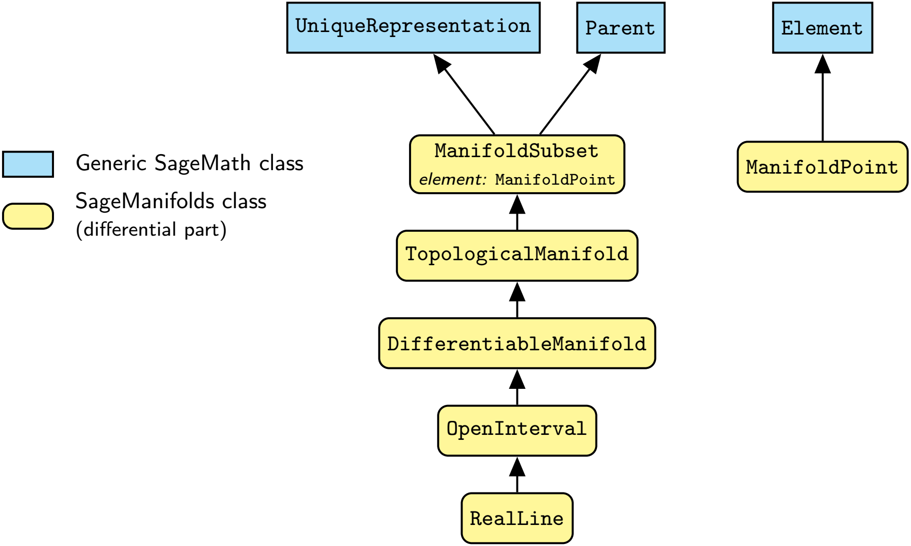
http://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Worksheets/JNCF2018/jncf18_scalar.ipynb

On CoCalc:

https://cocalc.com/share/e3c2938e-d8b0-4efd-8503-cdb313ffead9/SageManifolds/Worksheets/JNCF2018/jncf18_scalar.ipynb?viewer=share

Direct links available at
http://sagemanifolds.obspm.fr/jncf2018/

Manifold classes



Coordinate charts

Property 3 of manifold definition $\Longrightarrow$ labeling points by coordinates $(x^\alpha)_{\alpha\in\{0,\ldots,n-1\}} \in \mathbb{K}^n$.

Definition

Let M be a topological manifold of dimension n over $\mathbb{K}$ and $U \subset M$ be an open set. A ***coordinate chart*** (or simply a ***chart***) on U is a homeomorphism

$$\begin{array}{cccc} X: & U \subset M & \longrightarrow & X(U) \subset \mathbb{K}^n \\ & p & \longmapsto & (x^0, \ldots, x^{n-1}). \end{array}$$

Coordinate charts

Property 3 of manifold definition $\Longrightarrow$ labeling points by coordinates $(x^\alpha)_{\alpha\in\{0,\ldots,n-1\}} \in \mathbb{K}^n$.

Definition

Let M be a topological manifold of dimension n over $\mathbb{K}$ and $U \subset M$ be an open set. A ***coordinate chart*** (or simply a ***chart***) on U is a homeomorphism

$$\begin{array}{rccl} X: & U \subset M & \longrightarrow & X(U) \subset \mathbb{K}^n \\ & p & \longmapsto & (x^0, \ldots, x^{n-1}). \end{array}$$

In general, more than one chart is required to cover the entire manifold:

Examples:

- at least 2 charts are necessary to cover the n-dimensional sphere $\mathbb{S}^n$ $(n \geq 1)$ and the torus $\mathbb{T}^2$
- at least 3 charts are necessary to cover the real projective plane $\mathbb{RP}^2$

Atlas

Definition

An ***atlas*** on M is a set of pairs $(U_i, X_i)_{i\in I}$, where I is a set, U_i an open subset of M and X_i a chart on U_i, such that the union of all U_i's covers M:

$$\bigcup_{i\in I} U_i = M.$$

Smooth manifolds

For manifolds, the concept of differentiability is defined from the smooth structure of $\mathbb{K}^n$, via an atlas:

Definition

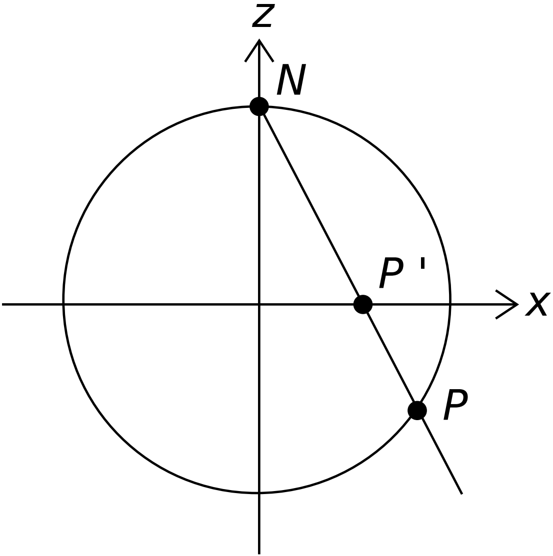
A ***smooth manifold*** over $\mathbb{K}$ is a topological manifold M equipped with an atlas $(U_i, X_i)_{i\in I}$ such that for any non-empty intersection $U_i \cap U_j$, the map

$$X_i \circ X_j^{-1} : X_j(U_i \cap U_j) \subset \mathbb{K}^n \longrightarrow X_i(U_i \cap U_j) \subset \mathbb{K}^n$$

is smooth (i.e. C^∞).

The map $X_i \circ X_j^{-1}$ is called a ***transition map*** or a ***change of coordinates***.

Stereographic coordinates



Outline

Scalar fields

Definition

Given a smooth manifold M over a topological field $\mathbb{K}$, a ***scalar field*** (also called a ***scalar-valued function***) on M is a smooth map

$$\begin{array}{rccc} f: & M & \longrightarrow & \mathbb{K} \\ & p & \longmapsto & f(p) \end{array}$$

Scalar fields

Definition

Given a smooth manifold M over a topological field $\mathbb{K}$, a ***scalar field*** (also called a ***scalar-valued function***) on M is a smooth map

$$\begin{array}{rccc} f: & M & \longrightarrow & \mathbb{K} \\ & p & \longmapsto & f(p) \end{array}$$

A scalar field has different coordinate representations F, $\hat{F}$, etc. in different charts X, $\hat{X}$, etc. defined on M:

$$f(p) = F(\underbrace{x^1, \ldots, x^n}_{\substack{\text{coord. of } p \\ \text{in chart } X}}) = \hat{F}(\underbrace{\hat{x}^1, \ldots, \hat{x}^n}_{\substack{\text{coord. of } p \\ \text{in chart } \hat{X}}}) = \ldots$$

$F : \mathrm{Im}\, X \to \mathbb{K}$ is called a ***chart function*** associated to X.

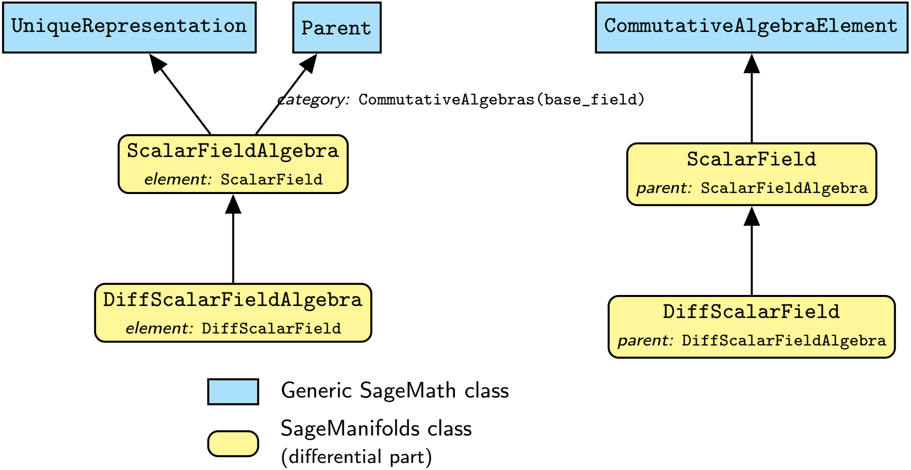
Scalar field algebra

The set $C^\infty(M)$ of scalar fields on M has naturally the structure of a ***commutative algebra over*** $\mathbb{K}$

1. it is clearly a vector space over $\mathbb{K}$
2. it is endowed with a commutative ring structure by pointwise multiplication:

$$\forall f, g \in C^\infty(M), \quad \forall p \in M, \quad (f.g)(p) := f(p)g(p)$$

Scalar field classes



Outline

Tangent vectors

Definition

Let M be a smooth manifold of dimension n over the topological field $\mathbb{K}$ and $C^\infty(M)$ the algebra of scalar fields on M. For $p \in M$, a ***tangent vector at*** p is a map

$$\boldsymbol{v} : C^\infty(M) \longrightarrow \mathbb{K}$$

that is $\mathbb{K}$-linear and such that

$$\forall f, g \in C^\infty(M), \quad \boldsymbol{v}(fg) = \boldsymbol{v}(f)g(p) + f(p)\boldsymbol{v}(g)$$

Because of the above property, one says that $\boldsymbol{v}$ is a ***derivation at*** p.

Tangent vectors

Definition

Let M be a smooth manifold of dimension n over the topological field $\mathbb{K}$ and $C^\infty(M)$ the algebra of scalar fields on M. For $p \in M$, a ***tangent vector at*** p is a map

$$\boldsymbol{v} : C^\infty(M) \longrightarrow \mathbb{K}$$

that is $\mathbb{K}$-linear and such that

$$\forall f, g \in C^\infty(M), \quad \boldsymbol{v}(fg) = \boldsymbol{v}(f)g(p) + f(p)\boldsymbol{v}(g)$$

Because of the above property, one says that $\boldsymbol{v}$ is a ***derivation at*** p.

Proposition

The set T_pM of all tangent vectors at p is a vector space of dimension n over $\mathbb{K}$; it is called the ***tangent space to*** M ***at*** p.

SageMath implementation

See the online worksheet

http://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Worksheets/JNCF2018/jncf18_vector.ipynb

On CoCalc:

https://cocalc.com/share/e3c2938e-d8b0-4efd-8503-cdb313ffead9/SageManifolds/Worksheets/JNCF2018/jncf18_vector.ipynb?viewer=share

Direct links available at
http://sagemanifolds.obspm.fr/jncf2018/

Tangent bundle

Definition

The ***tangent bundle*** of M is the disjoint union of the tangent spaces at all points of M:

$$TM = \coprod_{p\in M} T_pM$$

Elements of TM are usually denoted by $(p, \boldsymbol{v})$, with $\boldsymbol{v} \in T_pM$. The tangent bundle is canonically endowed with the ***projection map***:

$$\begin{array}{cccc} \pi : & TM & \longrightarrow & M \\ & (p, \boldsymbol{v}) & \longmapsto & p \end{array}$$

Tangent bundle

Definition

The ***tangent bundle*** of M is the disjoint union of the tangent spaces at all points of M:

$$TM = \coprod_{p \in M} T_p M$$

Elements of TM are usually denoted by $(p, \boldsymbol{v})$, with $\boldsymbol{v} \in T_p M$. The tangent bundle is canonically endowed with the ***projection map***:

$$\begin{array}{cccc} \pi : & TM & \longrightarrow & M \\ & (p, \boldsymbol{v}) & \longmapsto & p \end{array}$$

The tangent bundle inherits some manifold structure from M:

Proposition

TM is a smooth manifold of dimension $2n$ over $\mathbb{K}$ ($n = \dim M$).

Vector fields

Definition

A ***vector field*** on M is a continuous right-inverse of the projection map, i.e. a map

$$\begin{array}{rccc} \boldsymbol{v}: & M & \longrightarrow & TM \\ & p & \longmapsto & \boldsymbol{v}|_p \end{array}$$

such that $\pi \circ \boldsymbol{v} = \mathrm{Id}_M$. In other words, we have

$$\forall p \in M, \quad \boldsymbol{v}|_p \in T_pM.$$

Set of vector fields

The set $\mathfrak{X}(M)$ of all vector fields on M is endowed with two algebraic structures:

1. $\mathfrak{X}(M)$ is an infinite-dimensional vector space over $\mathbb{K}$, the scalar multiplication $\mathbb{K} \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $(\lambda, \boldsymbol{v}) \mapsto \lambda \boldsymbol{v}$ being defined by

$$\forall p \in M, \quad (\lambda \boldsymbol{v})|_p = \lambda \boldsymbol{v}|_p \,,$$

2. $\mathfrak{X}(M)$ is a module over the commutative algebra $C^\infty(M)$ the scalar multiplication $C^\infty(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $(f, \boldsymbol{v}) \mapsto f\boldsymbol{v}$ being defined by

$$\forall p \in M, \quad (f\boldsymbol{v})|_p = f(p)\boldsymbol{v}|_p \,,$$

   the right-hand side involving the scalar multiplication by $f(p) \in \mathbb{K}$ in the vector space T_pM.

$\mathfrak{X}(M)$ as a $C^\infty(M)$-module

Case where $\mathfrak{X}(M)$ is a free module

$\mathfrak{X}(M)$ is a ***free module*** over $C^\infty(M)$ $\iff$ $\mathfrak{X}(M)$ admits a basis

If this occurs, then $\mathfrak{X}(M)$ is actually a ***free module of finite rank*** over $C^\infty(M)$ and $\mathrm{rank}\,\mathfrak{X}(M) = \dim M = n$.
One says then that M is a ***parallelizable*** manifold.
A basis $(\boldsymbol{e}_a)_{1\leq a\leq n}$ of $\mathfrak{X}(M)$ is called a ***vector frame***; for any $p \in M$, $(\boldsymbol{e}_a|_p)_{1\leq a\leq n}$ is a basis of the tangent vector space T_pM.
Basis expansion[1]:

$$\forall \boldsymbol{v} \in \mathfrak{X}(M), \quad \boldsymbol{v} = v^a \boldsymbol{e}_a, \quad \text{with } v^a \in C^\infty(M) \tag{1}$$

At each point $p \in M$, Eq. (1) gives birth to an identity in the tangent space T_pM:

$$\boldsymbol{v}|_p = v^a(p)\, \boldsymbol{e}_a|_p\,, \quad \text{with } v^a(p) \in \mathbb{K},$$

which is nothing but the expansion of the tangent vector $\boldsymbol{v}|_p$ on the basis $(\boldsymbol{e}_a|_p)_{1\leq a\leq n}$ of the vector space T_pM.

[1] Einstein's convention for summation on repeated indices is assumed.

Parallelizable manifolds

M is **parallelizable** $\iff$ $\mathfrak{X}(M)$ is a free $C^\infty(M)$-module of rank n
$\iff$ M admits a global vector frame
$\iff$ the tangent bundle is trivial: $TM \simeq M \times \mathbb{K}^n$

Parallelizable manifolds

M is **parallelizable** $\iff$ $\mathfrak{X}(M)$ is a free $C^\infty(M)$-module of rank n
$\iff$ M admits a global vector frame
$\iff$ the tangent bundle is trivial: $TM \simeq M \times \mathbb{K}^n$

Examples of parallelizable manifolds

- $\mathbb{R}^n$ (global coordinate chart $\Rightarrow$ global vector frame)
- the circle $\mathbb{S}^1$ (*rem:* no global coordinate chart)
- the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$
- the 3-sphere $\mathbb{S}^3 \simeq \mathrm{SU}(2)$, as any Lie group
- the 7-sphere $\mathbb{S}^7$
- any orientable 3-manifold (Steenrod theorem)

Parallelizable manifolds

M is **parallelizable** $\iff$ $\mathfrak{X}(M)$ is a free $C^\infty(M)$-module of rank n
$\iff$ M admits a global vector frame
$\iff$ the tangent bundle is trivial: $TM \simeq M \times \mathbb{K}^n$

Examples of parallelizable manifolds

- $\mathbb{R}^n$ (global coordinate chart $\Rightarrow$ global vector frame)
- the circle $\mathbb{S}^1$ (*rem:* no global coordinate chart)
- the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$
- the 3-sphere $\mathbb{S}^3 \simeq \mathrm{SU}(2)$, as any Lie group
- the 7-sphere $\mathbb{S}^7$
- any orientable 3-manifold (Steenrod theorem)

Examples of non-parallelizable manifolds

- the sphere $\mathbb{S}^2$ (hairy ball theorem!) and any n-sphere $\mathbb{S}^n$ with $n \notin \{1, 3, 7\}$
- the real projective plane $\mathbb{RP}^2$

SageMath implementation of vector fields

Choice of the $C^\infty(M)$-module point of view for $\mathfrak{X}(M)$, instead of the infinite-dimensional $\mathbb{K}$-vector space one

$\Longrightarrow$ implementation advantages:

- reduction to finite-dimensional structures: free $C^\infty(U)$-modules of rank n on parallelizable open subsets $U \subset M$
- for tensor calculus on each parallelizable open set U, use of exactly the same `FiniteRankFreeModule` code as for the tangent spaces

SageMath implementation of vector fields

Choice of the $C^\infty(M)$-module point of view for $\mathfrak{X}(M)$, instead of the infinite-dimensional $\mathbb{K}$-vector space one

$\Longrightarrow$ implementation advantages:

- reduction to finite-dimensional structures: free $C^\infty(U)$-modules of rank n on parallelizable open subsets $U \subset M$
- for tensor calculus on each parallelizable open set U, use of exactly the same `FiniteRankFreeModule` code as for the tangent spaces

Decomposition of M into parallelizable parts

Assumption: the smooth manifold M can be covered by a finite number m of parallelizable open subsets U_i $(1 \leq i \leq m)$

Example: this holds if M is compact (finite atlas)

SageMath implementation of vector fields

$$M = \bigcup_{i=1}^{m} U_i, \qquad \text{with} \quad U_i \text{ parallelizable}$$

For each i, $\mathfrak{X}(U_i)$ is a free module of rank $n = \dim M$ and is implemented in SageMath as an instance of `VectorFieldFreeModule`, which is a subclass of `FiniteRankFreeModule`.

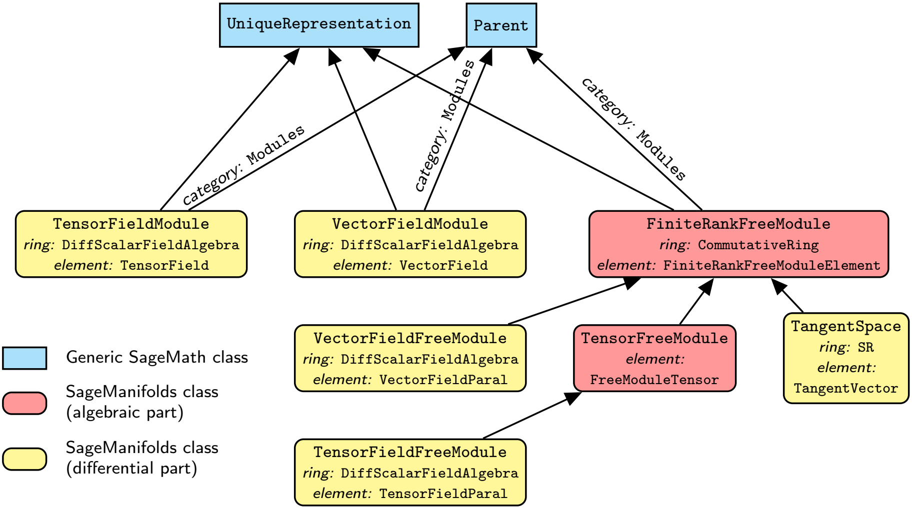
A vector field $\boldsymbol{v} \in \mathfrak{X}(M)$ is then described by its restrictions $(\boldsymbol{v}_i)_{1 \leq i \leq m}$ in each of the U_i's. Assuming that at least one vector frame is introduced in each of the U_i's, $(\boldsymbol{e}_{i,a})_{1 \leq a \leq n}$ say, the restriction $\boldsymbol{v}_i$ of $\boldsymbol{v}$ to U_i is decribed by its components v_i^a in that frame:

$$\boldsymbol{v}_i = v_i^a \, \boldsymbol{e}_{i,a}, \quad \text{with } v_i^a \in C^\infty(U_i). \tag{2}$$

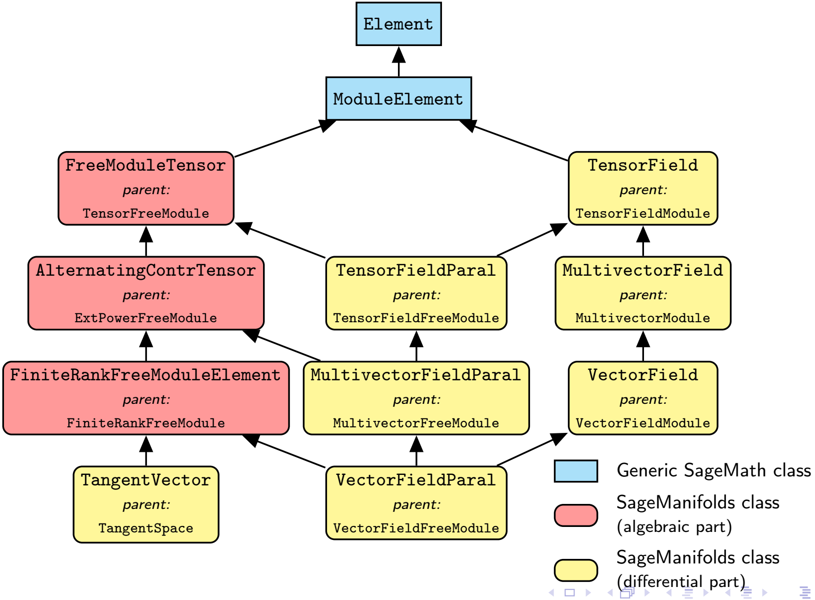
The components of $\boldsymbol{v}_i$ are stored as a *Python dictionary* whose keys are the vector frames:

$$(\boldsymbol{v}_i).\texttt{\_components} = \{(\boldsymbol{e}) : (v_i^a), \ (\hat{\boldsymbol{e}}) : (\hat{v}_i^a), \ldots\}$$

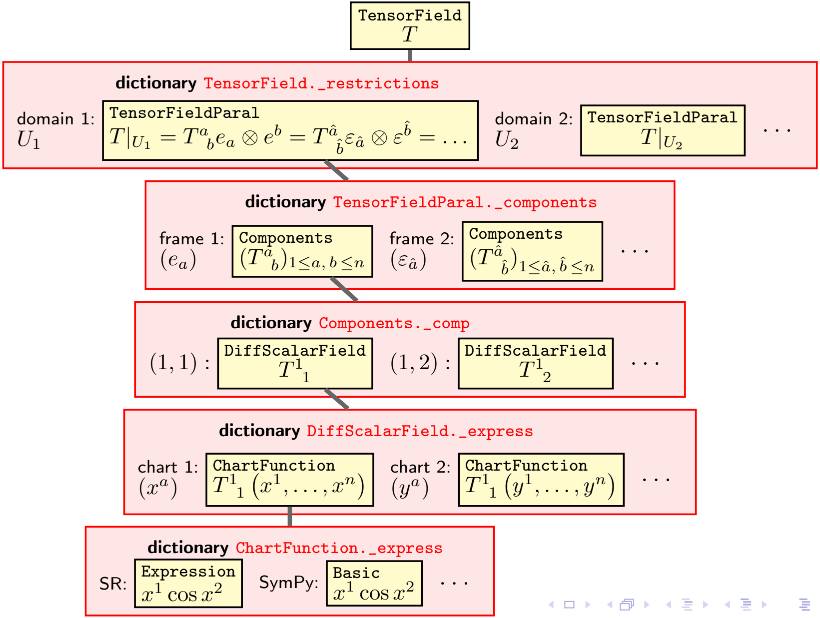
Module classes



Tensor field classes



Tensor field storage



Outline

Outline

Status of SageManifolds project

SageManifolds (http://sagemanifolds.obspm.fr/): extends SageMath towards differential geometry and tensor calculus

- $\sim$ 75,000 lines of Python code (including comments and doctests)
- submitted to SageMath community as a sequence of 31 tickets
  cf. list at https://trac.sagemath.org/ticket/18528
  $\rightarrow$ first ticket accepted in March 2015,
  the 31th one in Jan 2018
- a dozen of contributors (developers and reviewers)
  cf. http://sagemanifolds.obspm.fr/authors.html

All code is fully included in SageMath 8.1

Current status

Already present (SageMath 8.1):

- differentiable manifolds: tangent spaces, vector frames, tensor fields, curves, pullback and pushforward operators
- standard tensor calculus (tensor product, contraction, symmetrization, etc.), even on non-parallelizable manifolds
- all monoterm tensor symmetries taken into account
- Lie derivatives of tensor fields
- differential forms: exterior and interior products, exterior derivative, Hodge duality
- multivector fields: exterior and interior products, Schouten-Nijenhuis bracket
- affine connections (curvature, torsion)
- pseudo-Riemannian metrics
- computation of geodesics (numerical integration via SageMath/GSL)
- some plotting capabilities (charts, points, curves, vector fields)
- parallelization (on tensor components) of CPU demanding computations, via the Python library `multiprocessing`

Current status

Future prospects:

- more symbolic engines (Giac, FriCAS, ...)
- extrinsic geometry of pseudo-Riemannian submanifolds
- integrals on submanifolds
- more graphical outputs
- more functionalities: symplectic forms, fibre bundles, spinors, variational calculus, etc.
- connection with numerical relativity: using SageMath to explore numerically-generated spacetimes

Current status

Future prospects:

- more symbolic engines (Giac, FriCAS, ...)
- extrinsic geometry of pseudo-Riemannian submanifolds
- integrals on submanifolds
- more graphical outputs
- more functionalities: symplectic forms, fibre bundles, spinors, variational calculus, etc.
- connection with numerical relativity: using SageMath to explore numerically-generated spacetimes

Want to join the project or simply to stay tuned?

visit http://sagemanifolds.obspm.fr/
(download, documentation, example worksheets, mailing list)