Black holes: new approaches and the generalized Damour-Navier-Stokes equation

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based on a collaboration with José Luis Jaramillo

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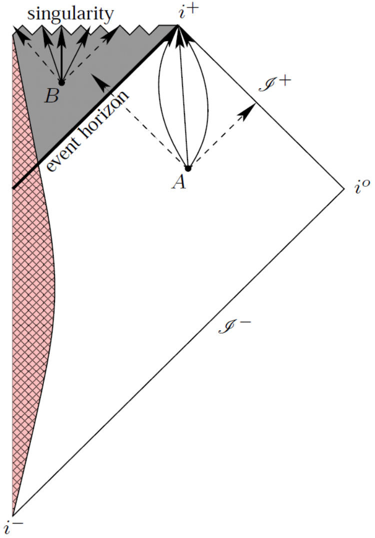
Plan

Outline

Classical definition of a black hole



[from Booth, gr-qc/0508107]

black hole [e.g. Wald (1984)]:

$$\mathcal{B} := \mathscr{M} - J^-(\mathscr{I}^+)$$

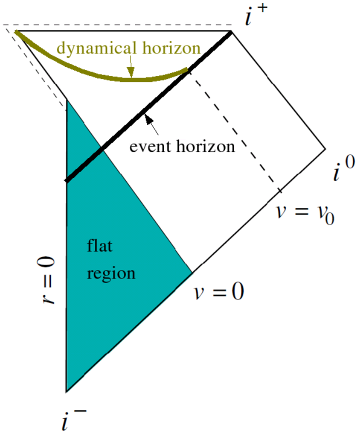
- $\mathscr{M}$ = asymptotically flat manifold
- $\mathscr{I}^+$ = future null infinity
- $J^-(\mathscr{I}^+)$ = causal past of $\mathscr{I}^+$

event horizon: $\mathcal{H} := \dot{J}^-(\mathscr{I}^+)$
(boundary of $J^-(\mathscr{I}^+)$)

$\mathcal{H}$ smooth $\Longrightarrow$ $\mathcal{H}$ null hypersurface

This is a highly non-local definition !

The determination of the boundary of $J^-(\mathscr{I}^+)$ requires the knowledge of the entire future null infinity. Moreover this is not locally linked with the notion of strong gravitational field:



Example of event horizon in a **flat** region of spacetime:

Vaidya metric, describing incoming radiation from infinity:

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right) dv^2 + 2 dv\, dr + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

with $m(v) = 0$ for $v < 0$

$dm/dv > 0$ for $0 \le v \le v_0$

$m(v) = M_0$ for $v > v_0$

[Ashtekar & Krishnan, LRR **7**, 10 (2004)]

Outline

Trapped surfaces

Local concepts characterizing very strong gravitational fields:

- trapped surfaces: introduced in 1965 by Penrose
- outer trapped surfaces and related notion of apparent horizon introduced in 1973 by Hawking and Ellis.

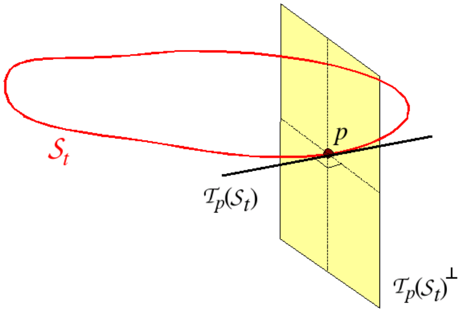
Closed spacelike surfaces

$\mathcal{S}$: closed (i.e. compact without boundary) spacelike 2-dimensional surface embedded in spacetime $(\mathscr{M}, \boldsymbol{g})$

$\mathcal{S}$ spacelike $\iff$ metric $\boldsymbol{q}$ induced by $\boldsymbol{g}$ is positive definite

$\boldsymbol{q}$ not degenerate $\Longrightarrow$ orthogonal decomposition of the tangent space at any $p \in \mathscr{M}$:

$$\mathcal{T}_p(\mathscr{M}) = \mathcal{T}_p(\mathcal{S}) \oplus \mathcal{T}_p(\mathcal{S})^{\perp}$$



$\boldsymbol{q}$: induced metric on $\mathcal{S}$, components: $q_{\alpha\beta}$

$\vec{\boldsymbol{q}}$: orthogonal projector onto $\mathcal{S}$, components: $q^{\alpha}{}_{\beta}$

Projection operator $\vec{q}^*$

$\boldsymbol{A}$: tensor of covariance type (m, n)
$\vec{\boldsymbol{q}}^* \boldsymbol{A}$: tensor of same covariance type, defined by

$$(\vec{q}^* A)^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n} := q^{\alpha_1}{}_{\mu_1} \cdots q^{\alpha_m}{}_{\mu_m} q^{\nu_1}{}_{\beta_1} \cdots q^{\nu_n}{}_{\beta_n} A^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n}$$

Remark: for a vector: $\vec{\boldsymbol{q}}^* \boldsymbol{v} = \vec{\boldsymbol{q}}(\boldsymbol{v})$
for a 1-form, $\vec{\boldsymbol{q}}^* \boldsymbol{\omega} = \boldsymbol{\omega} \circ \vec{\boldsymbol{q}}$

Definition: a tensor $\boldsymbol{A}$ is *tangent to* $\mathcal{S}$ iff $\vec{\boldsymbol{q}}^* \boldsymbol{A} = \boldsymbol{A}$.

Expansion and shear along normal vectors

Let $\boldsymbol{v}$ be a vector field on $\mathscr{M}$, defined at least at $\mathcal{S}$ and everywhere normal to $\mathcal{S}$.
NB: $\boldsymbol{v}$ is not assumed to be null

Deformation tensor of $\mathcal{S}$ along v: $\boldsymbol{\Theta}^{(v)} := \vec{\boldsymbol{q}}^{*} \boldsymbol{\nabla} \underline{\boldsymbol{v}}$ or $\Theta^{(v)}_{\alpha\beta} := \nabla_\nu v_\mu \, q^\mu{}_\alpha q^\nu{}_\beta$

$\boldsymbol{v}$ normal to a 2-surface $(\mathcal{S}) \Longrightarrow \boldsymbol{\Theta}^{(v)}$ is a symmetric bilinear form

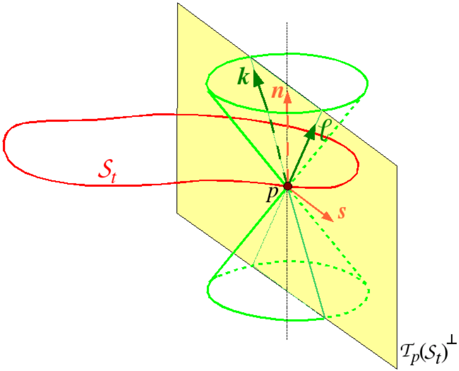
Prop: $\boldsymbol{\Theta}^{(v)} = \dfrac{1}{2} \vec{\boldsymbol{q}}^{*} \boldsymbol{\mathcal{L}}_{\boldsymbol{v}} \, \boldsymbol{q}$

Decomposition into traceless part (**shear** $\boldsymbol{\sigma}^{(v)}$) and trace part (**expansion** $\theta^{(v)}$):

$\boldsymbol{\Theta}^{(v)} = \boldsymbol{\sigma}^{(v)} + \dfrac{1}{2} \theta^{(v)} \, \boldsymbol{q}$ with $\theta^{(v)} := q^{\mu\nu} \Theta^{(v)}_{\mu\nu} = \boldsymbol{\mathcal{L}}_{\boldsymbol{v}} \ln \sqrt{q}$, $q := \det q_{ab}$

Prop: $\boldsymbol{\mathcal{L}}_{\boldsymbol{v}} \, {}^{\mathcal{S}}\boldsymbol{\epsilon} = \theta^{(v)} \, {}^{\mathcal{S}}\boldsymbol{\epsilon}$ with ${}^{\mathcal{S}}\boldsymbol{\epsilon}$ surface element of $(\mathcal{S}, \boldsymbol{q})$: ${}^{\mathcal{S}}\boldsymbol{\epsilon} = \sqrt{q} \, \mathbf{d}x^2 \wedge \mathbf{d}x^3$
$\Longrightarrow$ hence the name *expansion*

Null frames normal to $\mathcal{S}$ and trapping of light rays

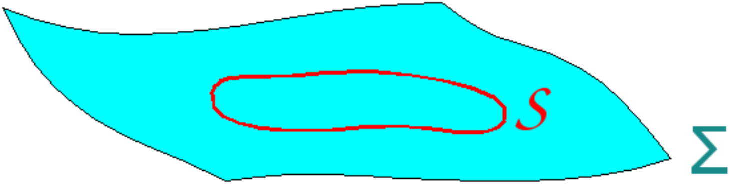


$\exists$ two future-directed null directions orthogonal to $\mathcal{S}$, generated by a pair linearly independent future-directed null vectors $(\boldsymbol{\ell}, \boldsymbol{k})$:
$\boldsymbol{\ell} \cdot \boldsymbol{\ell} = 0, \quad \boldsymbol{k} \cdot \boldsymbol{k} = 0, \quad \boldsymbol{\ell} \cdot \boldsymbol{k} =: -e^{\sigma}$

- $\mathcal{S}$ is trapped $\iff$ $\theta^{(\boldsymbol{k})} \leq 0$ and $\theta^{(\boldsymbol{\ell})} \leq 0$
  $\mathcal{S}$ is marginally trapped $\iff$ $\theta^{(\boldsymbol{k})} \leq 0$ and $\theta^{(\boldsymbol{\ell})} = 0$ (or vice-versa)
- $\mathcal{S}$ is outer trapped $\iff$ $\boldsymbol{\ell}$ is *outgoing*[1] and $\theta^{(\boldsymbol{\ell})} \leq 0$
  $\mathcal{S}$ is marginally outer trapped (MOTS) $\iff$ $\boldsymbol{\ell}$ is *outgoing* and $\theta^{(\boldsymbol{\ell})} = 0$

[1] requires assumption of asymptotic flatness

Apparent horizon



Σ: spacelike hypersurface extending to spatial infinity (Cauchy surface)

outer trapped region of Σ : Ω = set of points $p \in \Sigma$ through which there is a outer trapped surface $\mathcal{S}$ lying in Σ

apparent horizon in Σ: $\mathcal{A}$ = connected component of the boundary of Ω

Prop. [Hawking & Ellis (1973)]: $\mathcal{A}$ smooth $\Longrightarrow$ $\mathcal{A}$ is a MOTS

NB [Eardley, PRD **57**, 2299 (1998)] : $\mathcal{A}$ is not necessarily smooth

Connection with singularities and black holes

Prop. [Penrose (1965)]: provided the weak energy condition holds,
$\exists$ a trapped surface $\mathcal{S} \Longrightarrow \exists$ a singularity in $(\mathscr{M}, \boldsymbol{g})$ (in the form of a future inextendible null geodesic)

Prop. [Hawking & Ellis (1973)]: provided the cosmic censorship conjecture holds, any apparent horizon $\mathcal{A}$ is contained in a black hole

Local definitions of "black holes"

A hypersurface $\mathcal{H}$ of $(\mathscr{M}, \boldsymbol{g})$ is said to be

- a **future outer trapping horizon (FOTH)** [Hayward, PRD **49**, 6467 (1994)] iff
  (i) $\mathcal{H}$ foliated by marginally trapped 2-surfaces ($\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} = 0$)
  (ii) $\boldsymbol{\mathcal{L}}_{\boldsymbol{k}}\, \theta^{(\boldsymbol{\ell})} < 0$ (assuming $\mathcal{H}$ is member of a dual-null foliation)
- a **dynamical horizon** [Ashtekar & Krishnan, PRL **89** 261101 (2002)] iff
  (i) $\mathcal{H}$ is foliated by marginally trapped 2-surfaces
  (ii) $\mathcal{H}$ is spacelike
- a **non-expanding horizon** [Hájiček (1973)] iff
  (i) $\mathcal{H}$ is null (null normal $\boldsymbol{\ell}$)
  (ii) $\mathcal{H}$ has the $\mathbb{R} \times \mathbb{S}^2$ topology
  (iii) $\theta^{(\boldsymbol{\ell})} = 0$
  (iv) the null dominant energy condition holds at $\mathcal{H}$
- an **isolated horizon** [Ashtekar, Beetle & Fairhurst, CQG **16**, L1 (1999)] iff
  (i) $\mathcal{H}$ is a non-expanding horizon
  (ii) $\mathcal{H}$'s full geometry is not evolving along the null generators: $[\boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, , \hat{\boldsymbol{\nabla}}] = 0$

BH in equilibrium: NEH, IH, BH out of equilibrium: DH, generic BH: FOTH

Outline

Concept of black hole viscosity

- Hartle (1973): introduced the concept of **black hole viscosity** when studying the response of the *event horizon* to external perturbations
- Damour (1979): 2-dimensional **Navier-Stokes** like equation for the event horizon $\Longrightarrow$ *shear viscosity* and *bulk viscosity*
- Thorne and Price (1986): **membrane paradigm** for black holes

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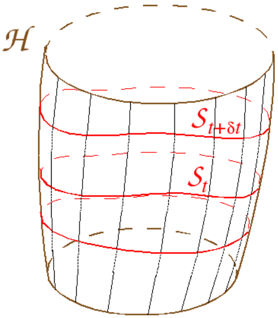
Shall we restrict the analysis to the event horizon ?

Can we extend the concept of viscosity to the local characterizations of black hole recently introduced, i.e. future outer trapping horizons and dynamical horizons ?

NB: *event horizon* = null hypersurface
future outer trapping horizon = null or spacelike hypersurface
dynamical horizon = spacelike hypersurface

Outline

Foliation of a hypersurface by spacelike 2-surfaces



hypersurface $\mathcal{H}$ = submanifold of spacetime $(\mathscr{M}, \boldsymbol{g})$ of codimension 1

$\mathcal{H}$ can be $\begin{cases} \text{spacelike} \\ \text{null} \\ \text{timelike} \end{cases}$

$$\mathcal{H} = \bigcup_{t\in\mathbb{R}} \mathcal{S}_t$$

$\mathcal{S}_t$ = spacelike 2-surface

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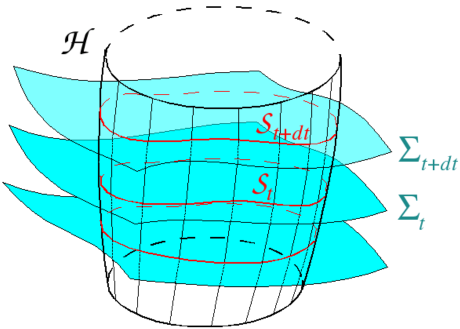

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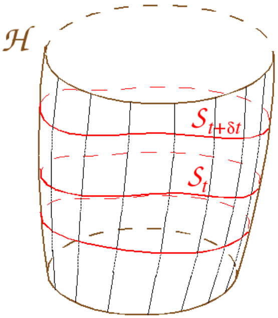
$\Longleftarrow$ 3+1 perspective

Foliation of a hypersurface by spacelike 2-surfaces



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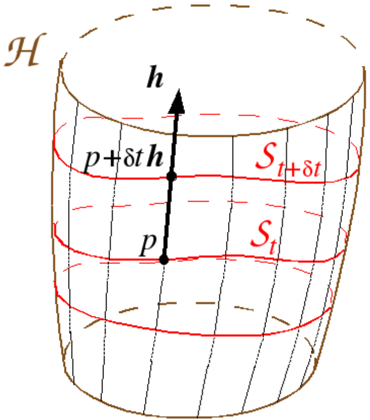
intrinsic viewpoint adopted here (i.e. not relying on extra-structure such as a 3+1 foliation)

Evolution vector



Vector field $\boldsymbol{h}$ on $\mathcal{H}$ defined by

- (i) $\boldsymbol{h}$ is tangent to $\mathcal{H}$
- (ii) $\boldsymbol{h}$ is orthogonal to $\mathcal{S}_t$
- (iii) $\mathcal{L}_{\boldsymbol{h}}\, t = h^\mu \partial_\mu t = \langle \mathbf{d}t, \boldsymbol{h} \rangle = 1$

NB: (iii) $\Longrightarrow$ the 2-surfaces $\mathcal{S}_t$ are Lie-dragged by $\boldsymbol{h}$

Lie derivatives along $\boldsymbol{h}$

Since the 2-surfaces $\mathcal{S}_t$ are Lie-dragged by $\boldsymbol{h}$, so are their tangent vectors:

$$\forall \boldsymbol{v} \in \mathcal{T}(\mathcal{S}_t), \; \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{v} \in \mathcal{T}(\mathcal{S}_t)$$

i.e. $\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}$ = internal operator on $\mathcal{T}(\mathcal{S}_t)$

Extension to 1-forms in $\mathcal{T}^*(\mathcal{S}_t)$:

$$\forall \boldsymbol{v} \in \mathcal{T}(\mathcal{S}_t), \quad \langle \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{\omega}, \boldsymbol{v}\rangle := \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \langle \boldsymbol{\omega}, \boldsymbol{v}\rangle - \langle \boldsymbol{\omega}, \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{v}\rangle.$$

Extension to any tensor $\boldsymbol{A}$ tangent to $\mathcal{S}_t$ by tensor products

Definition:

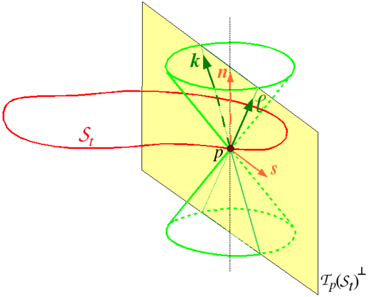
$$^{\mathcal{S}}\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{A} := \vec{\boldsymbol{q}}^{*} \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{A} = \vec{\boldsymbol{q}}^{*} \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \vec{\boldsymbol{q}}^{*} \boldsymbol{A}$$

Norm of $\boldsymbol{h}$ and type of $\mathcal{H}$

Definition: $C := \frac{1}{2}\boldsymbol{h}\cdot\boldsymbol{h}$

$$
\begin{aligned}
&\mathcal{H} \text{ is spacelike} && \iff && C > 0 && \iff && \boldsymbol{h} \text{ is spacelike} \\
&\mathcal{H} \text{ is null} && \iff && C = 0 && \iff && \boldsymbol{h} \text{ is null} \\
&\mathcal{H} \text{ is timelike} && \iff && C < 0 && \iff && \boldsymbol{h} \text{ is timelike.}
\end{aligned}
$$

Frames normal to $\mathcal{S}_t$



Two natural types of choice for a vector basis of $\mathcal{T}_p(\mathcal{S}_t)^\perp$:

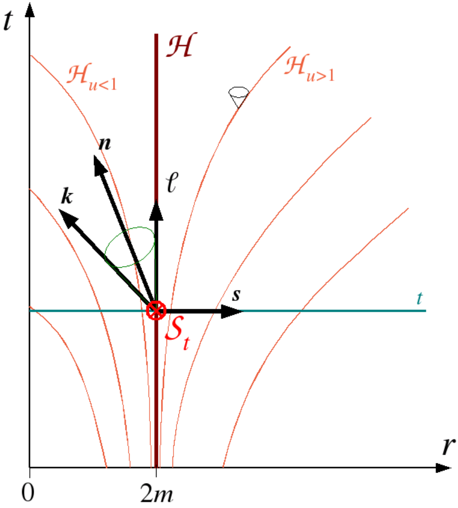
1. an orthonormal basis $(\boldsymbol{n}, \boldsymbol{s})$ ($\boldsymbol{n}$ = timelike, $\boldsymbol{s}$ = spacelike):
   $\boldsymbol{n}\cdot\boldsymbol{n} = -1, \quad \boldsymbol{s}\cdot\boldsymbol{s} = 1, \quad \boldsymbol{n}\cdot\boldsymbol{s} = 0$
2. a pair linearly independent future-directed null vectors $(\boldsymbol{\ell}, \boldsymbol{k})$:
   $\boldsymbol{\ell}\cdot\boldsymbol{\ell} = 0, \quad \boldsymbol{k}\cdot\boldsymbol{k} = 0, \quad \boldsymbol{\ell}\cdot\boldsymbol{k} =: -e^{\sigma}$

Degrees of freedom:

1. boost : $\begin{cases} \boldsymbol{n}' = \cosh\eta\, \boldsymbol{n} + \sinh\eta\, \boldsymbol{s} \\ \boldsymbol{s}' = \sinh\eta\, \boldsymbol{n} + \cosh\eta\, \boldsymbol{s} \end{cases}, \quad \eta \in \mathbb{R}$
2. rescaling : $\begin{cases} \boldsymbol{\ell}' = \lambda\, \boldsymbol{\ell}, & \lambda > 0 \\ \boldsymbol{k}' = \mu\, \boldsymbol{k}, & \mu > 0 \end{cases}$

Orthogonal projector: $\vec{\boldsymbol{q}} = \mathbf{1} + \langle \underline{\boldsymbol{n}}, .\rangle\, \boldsymbol{n} - \langle \underline{\boldsymbol{s}}, .\rangle\, \boldsymbol{s} = \mathbf{1} + e^{-\sigma}\langle \underline{\boldsymbol{k}}, .\rangle\, \boldsymbol{\ell} + e^{-\sigma}\langle \underline{\boldsymbol{\ell}}, .\rangle\, \boldsymbol{k}$

Example of normal frames



$\mathcal{H}$ = event horizon of Schwarzschild black hole
$\mathcal{S}_t$ = slice of constant Eddington-Finkelstein time

Second fundamental tensor of $\mathcal{S}_t$

Tensor $\boldsymbol{\mathcal{K}}$ of type $(1, 2)$ relating the covariant derivative of a vector tangent to $\mathcal{S}_t$ taken by the spacetime connection $\boldsymbol{\nabla}$ to that taken by the connection $\boldsymbol{\mathcal{D}}$ in $\mathcal{S}_t$ compatible with the induced metric $\boldsymbol{q}$:

$$\forall(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{T}(\mathcal{S}_t)^2, \quad \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} = \boldsymbol{\mathcal{D}}_{\boldsymbol{u}}\boldsymbol{v} + \boldsymbol{\mathcal{K}}(\boldsymbol{u}, \boldsymbol{v})$$

Prop:
$\mathcal{K}^{\alpha}{}_{\beta\gamma} = \nabla_{\mu} q^{\alpha}{}_{\nu}\, q^{\mu}{}_{\beta} q^{\nu}{}_{\gamma}$

$\mathcal{K}^{\alpha}{}_{\beta\gamma} = n^{\alpha}\, \Theta^{(\boldsymbol{n})}_{\beta\gamma} - s^{\alpha}\, \Theta^{(\boldsymbol{s})}_{\beta\gamma} = e^{-\sigma} \left(k^{\alpha}\, \Theta^{(\boldsymbol{\ell})}_{\beta\gamma} + \ell^{\alpha}\, \Theta^{(\boldsymbol{k})}_{\beta\gamma} \right)$

Remark: for a hypersurface of normal $\boldsymbol{n}$ and extrinsic curvature $\boldsymbol{K}$,
$\mathcal{K}^{\alpha}{}_{\beta\gamma} = -n^{\alpha} K_{\beta\gamma}$

Normal fundamental forms

Extrinsic geometry of $\mathcal{S}_t$ not entirely specified by $\mathcal{K}$ (contrary to the hypersurface case)

$\mathcal{K}$ involves only the deformation tensors $\mathbf{\Theta}^{(\cdot)}$ of the normals to $\mathcal{S}_t \Longrightarrow \mathcal{K}$ encodes only the part of the variation of $\mathcal{S}_t$'s normals which is parallel to $\mathcal{S}_t$

Variation of the two normals with respect to each other: encoded by the **normal fundamental forms** (also called *external rotation coefficients* or *connection on the normal bundle*, or if $\mathcal{H}$ is null, *Hájíček 1-form*):

1. $\boldsymbol{\Omega}^{(\boldsymbol{n})} := \boldsymbol{s} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{n}$ or $\Omega^{(\boldsymbol{n})}_\alpha := s_\mu \nabla_\nu n^\mu \, q^\nu{}_\alpha$
   $\boldsymbol{\Omega}^{(\boldsymbol{s})} := \boldsymbol{n} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{s}$
2. $\boldsymbol{\Omega}^{(\boldsymbol{\ell})} := \dfrac{1}{\boldsymbol{k}\cdot\boldsymbol{\ell}}\, \boldsymbol{k} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{\ell}$ or $\Omega^{(\boldsymbol{\ell})}_\alpha := \dfrac{1}{k_\rho \ell^\rho} k_\mu \nabla_\nu \ell^\mu \, q^\nu{}_\alpha$
   $\boldsymbol{\Omega}^{(\boldsymbol{k})} := \dfrac{1}{\boldsymbol{k}\cdot\boldsymbol{\ell}}\, \boldsymbol{\ell} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{k}$

Basic properties of the normal fundamental forms

From the definition: $\mathbf{\Omega}^{(s)} = -\mathbf{\Omega}^{(n)}$ and $\mathbf{\Omega}^{(k)} = -\mathbf{\Omega}^{(\ell)} + \mathcal{D}\sigma$

Relation between the (n, s)-type and the (ℓ, k)-type:
$\mathbf{\Omega}^{(\ell)} = \mathbf{\Omega}^{(n)} \quad [\ell = n + s]$ and $\mathbf{\Omega}^{(k)} = -\mathbf{\Omega}^{(n)} \quad [k = n - s]$

The normal fundamental forms are not unique

(contrary to the second fundamental tensor $\mathcal{K}$)
Dependence of the normal frame

1. $(n, s) \mapsto (n', s') \Longrightarrow \boxed{\mathbf{\Omega}^{(n')} = \mathbf{\Omega}^{(n)} + \mathcal{D}\eta}$
2. $(\ell, k) \mapsto (\ell', k') \Longrightarrow \boxed{\mathbf{\Omega}^{(\ell')} = \mathbf{\Omega}^{(\ell)} + \mathcal{D} \ln \lambda}$

"Surface-gravity" 1-forms

If the vector fields $(\boldsymbol{\ell}, \boldsymbol{k})$ are extended away from $\mathcal{S}_t$, define the 1-form

$$\boldsymbol{\kappa}^{(\boldsymbol{\ell})} := \frac{1}{\boldsymbol{k}\cdot\boldsymbol{\ell}}\,\boldsymbol{k}\cdot\boldsymbol{\nabla}_{\boldsymbol{p}}\,\boldsymbol{\ell} \quad \text{or} \quad \kappa^{(\ell)}_{\alpha} := \frac{1}{k_\rho \ell^\rho} k_\mu \nabla_\nu \ell^\mu \, p^\nu{}_\alpha$$

where $\boldsymbol{p}$ is the orthogonal projector complementary to $\vec{\boldsymbol{q}}$: $\mathbf{1} = \vec{\boldsymbol{q}} + \boldsymbol{p}$.

NB: Since $\boldsymbol{p}$ is a projector in a direction transverse to $\mathcal{S}_t$, the 1-form $\boldsymbol{\kappa}^{(\boldsymbol{\ell})}$ is not intrinsic to the 2-surface $\mathcal{S}_t$: it depends on the choice of $\boldsymbol{\ell}$ away from $\mathcal{S}_t$

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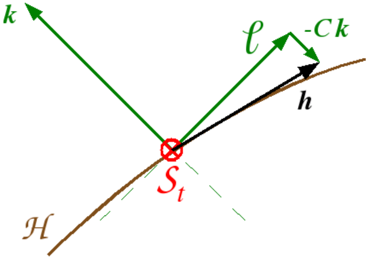
If $\boldsymbol{\ell}$ is extended along one of the two families of light rays emanating radially from $\mathcal{S}_t$, then $\boldsymbol{\ell}$ is pre-geodesic: $\boldsymbol{\nabla}_{\boldsymbol{\ell}}\,\boldsymbol{\ell} = \nu_{(\boldsymbol{\ell})}\,\boldsymbol{\ell}$, with the *inaffinity parameter* (*surface gravity* if $\boldsymbol{\ell} =$ null Killing vector of Kerr spacetime) given by the 1-form $\boldsymbol{\kappa}^{(\boldsymbol{\ell})}$ applied to $\boldsymbol{\ell}$:

$$\nu_{(\boldsymbol{\ell})} = \langle \boldsymbol{\kappa}^{(\boldsymbol{\ell})}, \boldsymbol{\ell} \rangle$$

Normal null frame associated with the evolution vector



The foliation $(\mathcal{S}_t)_{t\in\mathbb{R}}$ entirely fixes the ambiguities in the choice of the null normal frame $(\boldsymbol{\ell}, \boldsymbol{k})$, via the evolution vector $\boldsymbol{h}$: there exists a unique normal null frame $(\boldsymbol{\ell}, \boldsymbol{k})$ such that

$$\boldsymbol{h} = \boldsymbol{\ell} - C\boldsymbol{k} \qquad \text{and} \qquad \boldsymbol{\ell}\cdot\boldsymbol{k} = -1$$

Outline

Navier-Stokes equation in Newtonian fluid dynamics

$$\rho \left(\frac{\partial v^i}{\partial t} + v^j \nabla_j v^i \right) = -\nabla^i P + \mu \Delta v^i + \left(\zeta + \frac{\mu}{3} \right) \nabla^i (\nabla_j v^j) + f^i$$

or, in terms of fluid momentum density $\pi_i := \rho v_i$,

$$\boxed{\frac{\partial \pi_i}{\partial t} + v^j \nabla_j \pi_i + \theta \pi_i = -\nabla_i P + 2\mu \nabla^j \sigma_{ij} + \zeta \nabla_i \theta + f_i}$$

where θ is the fluid expansion:

$$\theta := \nabla_j v^j$$

and σ_{ij} the velocity shear tensor:

$$\sigma_{ij} := \frac{1}{2} (\nabla_i v_j + \nabla_j v_i) - \frac{1}{3} \theta \, \delta_{ij}$$

P is the pressure, μ the shear viscosity, ζ the bulk viscosity and f_i the density of external forces

Original Damour-Navier-Stokes equation

Hyp: $\mathcal{H}$ = null hypersurface (particular case: black hole **event horizon**)
Then $\boldsymbol{h} = \boldsymbol{\ell}$ ($C = 0$) reminder

Damour (1979) has derived from Einstein equation the relation

$$\boxed{{}^{\mathcal{S}}\!\boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, \boldsymbol{\Omega}^{(\ell)} + \theta^{(\ell)}\boldsymbol{\Omega}^{(\ell)} = \boldsymbol{\mathcal{D}}\nu_{(\ell)} - \boldsymbol{\mathcal{D}}\cdot\vec{\boldsymbol{\sigma}}^{(\ell)} + \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\ell)} + 8\pi\vec{\boldsymbol{q}}^{*}\boldsymbol{T}\cdot\boldsymbol{\ell}}$$

or equivalently

$$\boxed{{}^{\mathcal{S}}\!\boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, \boldsymbol{\pi} + \theta^{(\ell)}\boldsymbol{\pi} = -\boldsymbol{\mathcal{D}}P + 2\mu\boldsymbol{\mathcal{D}}\cdot\vec{\boldsymbol{\sigma}}^{(\ell)} + \zeta\boldsymbol{\mathcal{D}}\theta^{(\ell)} + \boldsymbol{f}}$$

with $\boldsymbol{\pi} := -\frac{1}{8\pi}\boldsymbol{\Omega}^{(\ell)}$ momentum surface density

$P := \frac{\nu_{(\ell)}}{8\pi}$ pressure

$\mu := \frac{1}{16\pi}$ shear viscosity

$\zeta := -\frac{1}{16\pi}$ bulk viscosity

$\boldsymbol{f} := -\vec{\boldsymbol{q}}^{*}\boldsymbol{T}\cdot\boldsymbol{\ell}$ external force surface density ($\boldsymbol{T}$ = stress-energy tensor)

Original Damour-Navier-Stokes equation (con't)

Introducing a coordinate system (t, x^1, x^2, x^3) such that

- t is compatible with $\boldsymbol{\ell}$: $\boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, t = 1$
- $\mathcal{H}$ is defined by $x^1 = \mathrm{const}$, so that $x^a = (x^2, x^3)$ are coordinates spanning $\mathcal{S}_t$

then

$$\boldsymbol{\ell} = \frac{\partial}{\partial t} + \boldsymbol{V}$$

with $\boldsymbol{V}$ tangent to $\mathcal{S}_t$: velocity of $\mathcal{H}$'s null generators with respect to the coordinates x^a [Damour 1978].

Then

$$\theta^{(\boldsymbol{\ell})} = \mathcal{D}_a V^a + \frac{\partial}{\partial t} \ln \sqrt{q} \qquad q := \det q_{ab}$$

$$\sigma^{(\boldsymbol{\ell})}_{ab} = \frac{1}{2}\left(\mathcal{D}_a V_b + \mathcal{D}_b V_a\right) - \frac{1}{2}\theta^{(\boldsymbol{\ell})}\, q_{ab} + \frac{1}{2}\frac{\partial q_{ab}}{\partial t}$$

compare

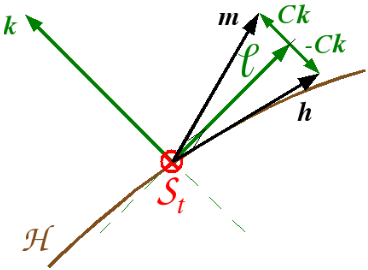
Generalization to the non-null case

Starting remark: in the null case, ℓ plays two different roles:

- evolution vector along $\mathcal{H}$ (e.g. term ${}^{\mathcal{S}}\!\boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}$)
- normal to $\mathcal{H}$ (e.g. term $\vec{q}^{*}\boldsymbol{T} \cdot \boldsymbol{\ell}$)

When $\mathcal{H}$ is no longer null, these two roles have to be taken by two different vectors:

- evolution vector: obviously $\boldsymbol{h}$ reminder
- vector normal to $\mathcal{H}$: a natural choice is $\boxed{\boldsymbol{m} := \boldsymbol{\ell} + C\boldsymbol{k}}$



Generalized Damour-Navier-Stokes equation

Starting point of the calculation: contracted Ricci identity applied to the vector $\boldsymbol{m}$ and projected onto $\mathcal{S}_t$:

$$(\nabla_\mu \nabla_\nu m^\mu - \nabla_\nu \nabla_\mu m^\mu)\, q^\nu{}_\alpha = R_{\mu\nu} m^\mu q^\nu{}_\alpha$$

Final result:

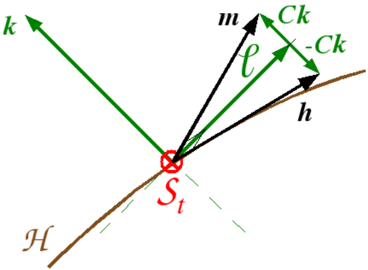
$$\,^{\mathcal{S}}\!\mathcal{L}_{\boldsymbol{h}}\, \boldsymbol{\Omega}^{(\ell)} + \theta^{(\boldsymbol{h})}\, \boldsymbol{\Omega}^{(\ell)} = \boldsymbol{\mathcal{D}}\langle \boldsymbol{\kappa}^{(\ell)}, \boldsymbol{h}\rangle - \boldsymbol{\mathcal{D}} \cdot \vec{\boldsymbol{\sigma}}^{(\boldsymbol{m})} + \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{m})} - \theta^{(\boldsymbol{k})}\boldsymbol{\mathcal{D}}C + 8\pi \vec{\boldsymbol{q}}^{*}\boldsymbol{T} \cdot \boldsymbol{m}$$

- $\boldsymbol{\Omega}^{(\ell)}$: normal fundamental form of $\mathcal{S}_t$ associated with null normal $\boldsymbol{\ell}$ reminder
- $\theta^{(\boldsymbol{h})}$, $\theta^{(\boldsymbol{m})}$ and $\theta^{(\boldsymbol{k})}$: expansion scalars of $\mathcal{S}_t$ along the vectors $\boldsymbol{h}$, $\boldsymbol{m}$ and $\boldsymbol{k}$ respectively reminder
- $\boldsymbol{\mathcal{D}}$: covariant derivative within $(\mathcal{S}_t, \boldsymbol{q})$
- $\boldsymbol{\kappa}^{(\ell)}$: "surface-gravity" 1-form associated with the null vector $\boldsymbol{\ell}$ reminder
- $\boldsymbol{\sigma}^{(\boldsymbol{m})}$: shear tensor of $\mathcal{S}_t$ along the vector $\boldsymbol{m}$ reminder
- C : half the scalar square of $\boldsymbol{h}$ reminder

Null limit

In the null limit,

$$\boldsymbol{h} = \boldsymbol{m} = \boldsymbol{\ell} \qquad \text{and} \qquad C = 0$$



and we recover the original Damour-Navier-Stokes equation:

$$^{\mathcal{S}}\mathcal{L}_{\boldsymbol{\ell}}\, \boldsymbol{\Omega}^{(\ell)} + \theta^{(\ell)} \boldsymbol{\Omega}^{(\ell)} = \boldsymbol{\mathcal{D}} \nu_{(\ell)} - \boldsymbol{\mathcal{D}} \cdot \vec{\boldsymbol{\sigma}}^{(\ell)} + \frac{1}{2} \boldsymbol{\mathcal{D}} \theta^{(\ell)} + 8\pi \vec{\boldsymbol{q}}^{*} \boldsymbol{T} \cdot \boldsymbol{\ell}$$

Behavior under a change of normal fundamental form

$\boldsymbol{\ell} \mapsto \boldsymbol{\ell}' = \lambda\boldsymbol{\ell} \Longrightarrow \boldsymbol{\Omega}^{(\boldsymbol{\ell}')} = \boldsymbol{\Omega}^{(\boldsymbol{\ell})} + \boldsymbol{\mathcal{D}} \ln \lambda$ and $\boldsymbol{\kappa}^{(\boldsymbol{\ell}')} = \boldsymbol{\kappa}^{(\boldsymbol{\ell})} + \boldsymbol{\nabla}_{\boldsymbol{p}} \ln \lambda$

$\Longrightarrow$ generalized Damour-Navier-Stokes equation:

$$\begin{aligned} {}^{\mathcal{S}}\!\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{\Omega}^{(\boldsymbol{\ell}')} + \theta^{(\boldsymbol{h})}\, \boldsymbol{\Omega}^{(\boldsymbol{\ell}')} &= \boldsymbol{\mathcal{D}}\langle \boldsymbol{\kappa}^{(\boldsymbol{\ell}')}, \boldsymbol{h}\rangle - \boldsymbol{\mathcal{D}} \cdot \vec{\boldsymbol{\sigma}}^{(\boldsymbol{m})} + \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{m})} + \theta^{(\boldsymbol{\ell})}\boldsymbol{\mathcal{D}} \ln \lambda \\ &\quad -\theta^{(\boldsymbol{k})}\left(\boldsymbol{\mathcal{D}}C + C\boldsymbol{\mathcal{D}} \ln \lambda\right) + 8\pi \vec{\boldsymbol{q}}^{*}\boldsymbol{T} \cdot \boldsymbol{m} \end{aligned}$$

Choice: $\boldsymbol{\ell}' = \tilde{\boldsymbol{\ell}}$ = null *geodesic* vector along the light rays emanating radially from $\mathcal{S}_t$ ($\mathbf{d}\underline{\tilde{\boldsymbol{\ell}}} = 0$), then $\boldsymbol{\mathcal{D}}C + C\boldsymbol{\mathcal{D}} \ln \lambda = 0$ and the equation reduces to

$$\boxed{{}^{\mathcal{S}}\!\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{\Omega}^{(\tilde{\boldsymbol{\ell}})} + \theta^{(\boldsymbol{h})}\, \boldsymbol{\Omega}^{(\tilde{\boldsymbol{\ell}})} = \boldsymbol{\mathcal{D}}\langle \boldsymbol{\kappa}^{(\tilde{\boldsymbol{\ell}})}, \boldsymbol{h}\rangle - \boldsymbol{\mathcal{D}} \cdot \vec{\boldsymbol{\sigma}}^{(\boldsymbol{m})} + \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{m})} + \theta^{(\boldsymbol{\ell})}\boldsymbol{\mathcal{D}} \ln \lambda + 8\pi \vec{\boldsymbol{q}}^{*}\boldsymbol{T} \cdot \boldsymbol{m}}$$

Application to future trapping horizons

Definition (Hayward 1994) : $\mathcal{H}$ is a **future trapping horizon** iff $\theta^{(\boldsymbol{\ell})} = 0$ and $\theta^{(\boldsymbol{k})} < 0$.

The generalized Damour-Navier-Stokes equation reduces then to

$$ {}^{\mathcal{S}}\!\mathcal{L}_{\boldsymbol{h}}\, \boldsymbol{\Omega}^{(\tilde{\ell})} + \theta^{(\boldsymbol{h})}\, \boldsymbol{\Omega}^{(\tilde{\ell})} = \boldsymbol{\mathcal{D}}\langle \boldsymbol{\kappa}^{(\tilde{\ell})}, \boldsymbol{h}\rangle - \boldsymbol{\mathcal{D}} \cdot \vec{\boldsymbol{\sigma}}^{(\boldsymbol{m})} + \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{m})} + 8\pi \vec{\boldsymbol{q}}^{*}\boldsymbol{T}\cdot\boldsymbol{m} $$

NB: It has exactly the same structure than Damour's original equation ◂ reminder: apart from substitutions of $\boldsymbol{\ell}$ by either $\boldsymbol{h}$ or $\boldsymbol{m}$, it does not contain any extra term

Outline

Generalized angular momentum

Definition [Booth & Fairhurst, gr-qc/0505049)]: Let φ be a vector field on $\mathcal{H}$ which

- is tangent to $\mathcal{S}_t$
- has closed orbits
- has vanishing divergence with respect to the induced metric: $\boldsymbol{\mathcal{D}} \cdot \varphi = 0$

The *generalized angular momentum associated with* φ is then defined by

$$J(\varphi) := -\frac{1}{8\pi} \oint_{\mathcal{S}_t} \langle \boldsymbol{\Omega}^{(\ell)}, \varphi \rangle \, {}^{\mathcal{S}}\!\epsilon,$$

Remark 1: does not depend upon the choice of null vector ℓ, thanks to the divergence-free property of φ

Remark 2:

- coincides with Ashtekar & Krishnan's definition for a dynamical horizon
- coincides with Brown-York angular momentum if $\mathcal{H}$ is timelike and φ a Killing vector

Angular momentum flux law

Under the supplementary hypothesis that φ is transported along the evolution vector $\boldsymbol{h}$: $\mathcal{L}_{\boldsymbol{h}}\,\boldsymbol{\varphi} = 0$, the generalized Damour-Navier-Stokes equation leads to

$$\frac{d}{dt}J(\varphi) = -\oint_{\mathcal{S}_t} \boldsymbol{T}(\boldsymbol{m},\boldsymbol{\varphi})\,{}^{\mathcal{S}}\!\boldsymbol{\epsilon} - \frac{1}{16\pi}\oint_{\mathcal{S}_t}\left[\vec{\boldsymbol{\sigma}}^{(\boldsymbol{m})} : \mathcal{L}_{\boldsymbol{\varphi}}\,\boldsymbol{q} - 2\theta^{(\boldsymbol{k})}\boldsymbol{\varphi}\cdot\boldsymbol{\mathcal{D}}C\right]{}^{\mathcal{S}}\!\boldsymbol{\epsilon}$$

- $\mathcal{H}$ = null hypersurface : $C = 0$ and $\boldsymbol{m} = \boldsymbol{\ell}$:

$$\frac{d}{dt}J(\varphi) = -\oint_{\mathcal{S}_t} \boldsymbol{T}(\boldsymbol{\ell},\boldsymbol{\varphi})\,{}^{\mathcal{S}}\!\boldsymbol{\epsilon} - \frac{1}{16\pi}\oint_{\mathcal{S}_t}\vec{\boldsymbol{\sigma}}^{(\boldsymbol{\ell})} : \mathcal{L}_{\boldsymbol{\varphi}}\,\boldsymbol{q}\,{}^{\mathcal{S}}\!\boldsymbol{\epsilon}$$

  i.e. Eq. (6.134) of the *Membrane Paradigm* book (Thorne, Price & MacDonald 1986)

- $\mathcal{H}$ = future trapping horizon :

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Two interesting limiting cases:

- $\mathcal{H}$ = null hypersurface : $C = 0$ and $\boldsymbol{m} = \boldsymbol{\ell}$:

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  [I. Booth : *Black hole boundaries*, Can. J. Phys., in press, gr-qc/0508107]
- Generalized Damour-Navier-Stokes equation:
  [E. Gourgoulhon : *Generalized Damour-Navier-Stokes equation applied to trapping horizons*, Phys. Rev. D **72**, 104007 (2005)]