(Numerical and) symbolic general relativity

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Computational tools developed at LUTH

- LORENE: library for solving PDE equations with spectral methods in spherical coordinates
  https://lorene.obspm.fr/ [C++]
- CoCoNut: GR-hydro code for 3D gravitational collapse [with U. Valencia]
  https://www.uv.es/coconut/ [C++]
- Kadath: library for solving PDE equations with spectral methods (generic coordinates)
  https://kadath.obspm.fr/ [C++]
- Gyoto: code for geodesic computation (ray-tracing) [with LESIA]
  https://gyoto.obspm.fr/ [C++, Python]
- CompOSE: Data base of nuclear matter equations of state
  https://compose.obspm.fr/
- SageManifolds: differential geometry and tensor calculus with SageMath
  https://sagemanifolds.obspm.fr/ [Python]

All these tools are free software (GPL)

SageMath in a few words

SageMath (*nickname:* Sage) is a **free open-source** computer algebra system

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SageMath is based on Python

- no need to learn any specific syntax to use it
- Python is a powerful *object oriented language*, with a neat syntax
- SageMath benefits from the Python ecosystem (e.g. Jupyter notebook)

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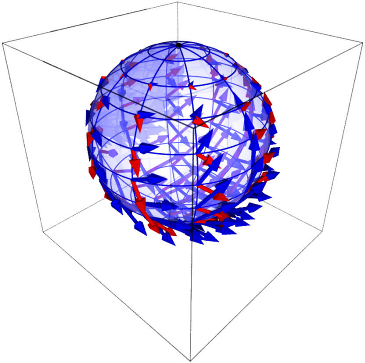
SageMath is developed by an enthusiastic community

- mostly composed of mathematicians
- welcoming newcomers

Tensor calculus with SageMath

SageManifolds project: extends `SageMath` towards **differential geometry** and **tensor calculus**



Stereographic-coordinate frame on $\mathbb{S}^2$

- `https://sagemanifolds.obspm.fr`
- fully included in `SageMath` (after review process)
- $\sim$ 15 contributors (developers and reviewers) cf. `https://sagemanifolds.obspm.fr/authors.html`
- dedicated mailing list
- help: `https://ask.sagemath.org`

Everybody is very welcome to contribute
$\Longrightarrow$ visit `https://sagemanifolds.obspm.fr/contrib.html`

Current status

Already present (SageMath 8.9):

- differentiable manifolds: tangent spaces, vector frames, tensor fields, curves, pullback and pushforward operators, submanifolds
- standard tensor calculus (tensor product, contraction, symmetrization, etc.), even on non-parallelizable manifolds, and with all monoterm tensor symmetries taken into account
- Lie derivatives of tensor fields
- differential forms: exterior and interior products, exterior derivative, Hodge duality
- multivector fields: exterior and interior products, Schouten-Nijenhuis bracket
- affine connections (curvature, torsion)
- pseudo-Riemannian metrics
- computation of geodesics (numerical integration)

Current status

Already present *(cont'd)*:

- some plotting capabilities (charts, points, curves, vector fields)
- parallelization (on tensor components) of CPU demanding computations
- extrinsic geometry of pseudo-Riemannian submanifolds
- tensor series expansions

Future prospects:

- more symbolic backends (Giac, FriCAS, ...)
- more graphical outputs
- symplectic forms, fibre bundles, spinors, integrals on submanifolds, variational calculus, etc.
- connection with numerical relativity: use SageMath to explore numerically-generated spacetimes

The kerrgeodesic_gw package

`kerrgeodesic_gw`: SageMath package implementing computations of gravitational waveforms, energy fluxes and inspiralling time for bodies on circular orbits around a Kerr black hole, as well as SNR in LISA detector.
Gravitational waves computed via the theory of perturbations of Kerr metric (Teukolsky 1973, Detweiler 1978)

`kerrgeodesic_gw` is

- entirely open-source:
  `https://github.com/BlackHolePerturbationToolkit/kerrgeodesic_gw`
- distributed via PyPi (Python Package Index):
  `https://pypi.org/project/kerrgeodesic-gw/`
  so that the installation in SageMath is very easy:
  `sage -pip install kerrgeodesic_gw`
- part of the Black Hole Perturbation Toolkit:
  `http://bhptoolkit.org/`

Example 1: timelike orbits in Kerr spacetime

https://nbviewer.jupyter.org/github/BlackHolePerturbationToolkit/kerrgeodesic_gw/blob/master/Notebooks/geod_Kerr.ipynb

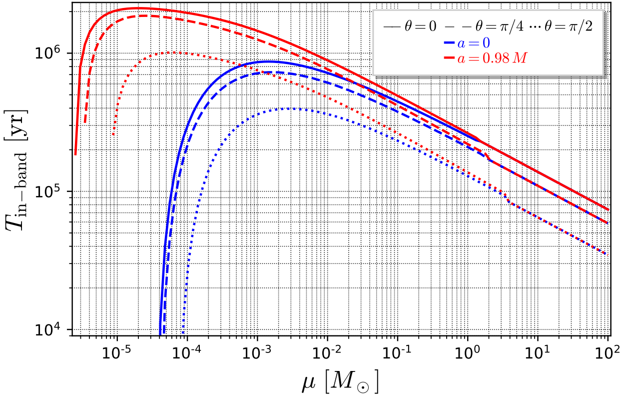
Example 2: gravitational waves from circular orbits around a Kerr black hole

https://nbviewer.jupyter.org/github/BlackHolePerturbationToolkit/kerrgeodesic_gw/blob/master/Notebooks/grav_waves_circular.ipynb

Application: Gravitational waves from bodies orbiting the Galactic Center black hole and their detectability by LISA

[Gourgoulhon, Le Tiec, Vincent & Warburton, A&A **627**, A92 (2019)]

Time in LISA band with $\mathrm{SNR}_{1\,\mathrm{yr}} \geq 10$ for an inspiralling compact object



μ: mass of the inspiralling compact object

Primordial BHs with $1M_\oplus \leq \mu \leq 5M_{\mathrm{Jup}}$ spend more than 10^6 yr in LISA band with $\mathrm{SNR}_{1\,\mathrm{yr}} \geq 10$

[Gourgoulhon, Le Tiec, Vincent & Warburton, A&A **627**, A92 (2019)]

Time in LISA band $\mathrm{SNR}_{1\,\mathrm{yr}} \geq 10$ for brown dwarfs and main-sequence stars

Results for

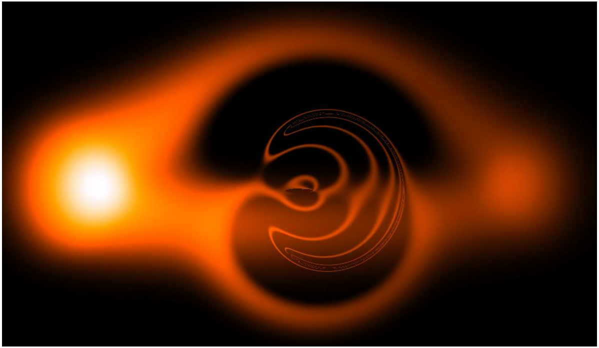
- inclination angle $\theta = 0$
- BH spin $a = 0$ (outside parentheses) and $a = 0.98M$ (inside parentheses)

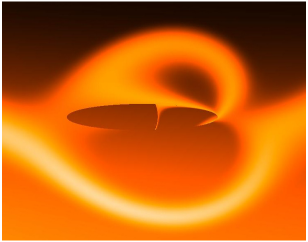
	brown dwarf	red dwarf	Sun-type	$2.4\,M_\odot$-star
$\mu/M_\odot$	0.062	0.20	1	2.40
$\rho/\rho_\odot$	$131.$	18.8	1	0.367
$r_{0,\mathrm{max}}/M$	28.2 (28.0)	35.0 (34.9)	47.1 (47.0)	55.6 (55.6)
$f_{m=2}(r_{0,\mathrm{max}})$ $[\mathrm{mHz}]$	0.105 (0.106)	0.076 (0.076)	0.049 (0.049)	0.038 (0.038)
r_{Roche}/M	7.31 (6.93)	13.3 (13.0)	34.2 (34.1)	47.6 (47.5)
$T^{\mathrm{ins}}_{\text{in-band}}$ $[10^5\ \mathrm{yr}]$	4.98 (5.55)	3.72 (3.99)	1.83 (1.89)	0.938 (0.945)

Brown dwarfs stay for $\sim 5 \times 10^5$ yr in LISA band

Example 3: naked rotating wormhole

Regular (singularity-free) spacetime with wormhole topology ($\mathbb{R}^2 \times \mathbb{S}^2$), sustained by exotic matter, asymptotically close a to Kerr spacetime with a naked singularity ($a > M$) and surrounded by an accretion torus





zoom on the central region

[Lamy, Gourgoulhon, Paumard & Vincent, CQG **35**, 115009 (2018)]

- Derivation of the geodesic equation: `SageMath`
- Integration of the geodesic equation: `Gyoto`