Magnetized binary compact objects

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Plan

1. Bekenstein-Oron formulation of relativistic ideal MHD
2. Quasistationary evolution of a magnetized binary system
3. Computing equilibrium configurations

Outline

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2. Quasistationary evolution of a magnetized binary system
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Variational principle

Bekenstein & Oron (2000)'s action for a magnetized perfect fluid with infinite conductivity:

$$S(g_{\alpha\beta}, N^\alpha, s, \gamma, A_\alpha) = S_{\rm grav} + S_{\rm fluid} + S_{\rm MHD}$$

- $N^\alpha = nu^\alpha$ baryon number 4-current
- s : entropy per baryon
- γ : Lin vorticity function (if absent, the theory describes only potential flows)
- A_α : electromagnetic 4-potential: $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$
- $S_{\rm grav} = \dfrac{1}{16\pi} \displaystyle\int R\sqrt{-g}\, d^4x$: Hilbert-Einstein action
- $S_{\rm fluid} = \displaystyle\int \left[-\varepsilon(s,n) + \phi\nabla_\alpha N^\alpha + \chi\nabla_\alpha(sN^\alpha) + \lambda\nabla_\alpha(\gamma N^\alpha)\right]\sqrt{-g}\, d^4x$: perfect fluid action [Schutz 1970]
- $S_{\rm MHD} = \displaystyle\int \left(-\frac{1}{16\pi} F_{\alpha\beta}F^{\alpha\beta} - q^\alpha F_{\alpha\beta} N^\beta\right)\sqrt{-g}\, d^4x$: ideal MHD action [Bekenstein & Oron 2000]

Lagrange multipliers: ϕ, χ, λ, q^α

Equations of motion (1/2)

- Variation w.r.t. $\phi \Longrightarrow$ baryon number conservation :

$$\boxed{\nabla_\alpha N^\alpha = 0} \qquad (1)$$

- Variation w.r.t. $\chi \Longrightarrow \nabla_\alpha (s N^\alpha) = 0 \Longrightarrow \boxed{u^\alpha \nabla_\alpha s = 0}$ (adiabatic flow)
- Variation w.r.t $N^\alpha \Longrightarrow h u_\alpha = \nabla_\alpha \phi + s \nabla_\alpha \chi + \gamma \nabla_\alpha \lambda - F_{\alpha\beta} q^\beta$
  where h is the enthalpy per baryon: $h := \dfrac{\varepsilon + p}{n}$
- Variation w.r.t. $q^\alpha \Longrightarrow F_{\alpha\beta} N^\beta = 0 \Longrightarrow \boxed{F_{\alpha\beta} u^\beta = 0}$ (infinite conductivity)
- Variation w.r.t. $A_\alpha \Longrightarrow$ Maxwell equation $\boxed{\nabla_\beta F^{\alpha\beta} = 4\pi j^\alpha}$ with

$$\boxed{j^\alpha := \nabla_\beta (N^\alpha q^\beta - N^\beta q^\alpha)} \qquad (2)$$

(1) and (2) $\Longrightarrow \boxed{\nabla_\alpha j^\alpha = 0}$ (conservation of electric charge)

Equations of motion (2/2)

Combining various equations resulting from the variational principle we get the MHD Euler equation:

$$(\varepsilon + p)u^\beta \nabla_\beta u_\alpha = -(\delta^\beta{}_\alpha + u_\alpha u^\beta)\nabla_\beta p + F_{\alpha\beta} j^\beta$$

which can be put in the equivalent form

$$\vec{\boldsymbol{u}} \cdot \boldsymbol{\Omega} = T\mathbf{d}s \qquad (3)$$

where T is the fluid temperature and $\boldsymbol{\Omega}$ is the **generalized vorticity 2-form**, i.e. the exterior derivative of the **generalized momentum 1-form** $\boldsymbol{w}$:

$$\boldsymbol{w} := h\,\underline{\boldsymbol{u}} + \boldsymbol{F} \cdot \vec{\boldsymbol{q}} \qquad \text{and} \qquad \boldsymbol{\Omega} := \mathbf{d}\boldsymbol{w}$$

Components: $w_\alpha = hu_\alpha + F_{\alpha\beta} q^\beta$ and $\Omega_{\alpha\beta} = \partial_\alpha w_\beta - \partial_\beta w_\alpha$

(3) is the ideal MHD generalization of the pure-hydrodynamics equation of motion in *canonical form* [Synge 1937], [Lichnerowicz 1941], [Taub 1959], [Carter 1979]

MHD Kelvin's theorem

$\mathcal{C}(\tau)$: closed contour dragged along by the fluid (proper time τ)

Magnetized fluid circulation around $\mathcal{C}(\tau)$: $C(\tau) := \oint_{\mathcal{C}(\tau)} \boldsymbol{w}$

Variation of the circulation as the contour is dragged by the fluid:

$$\frac{dC}{d\tau} = \frac{d}{d\tau} \oint_{\mathcal{C}(\tau)} \boldsymbol{w} = \oint_{\mathcal{C}(\tau)} \mathcal{L}_{\vec{\boldsymbol{u}}}\, \boldsymbol{w}$$

By virtue of Cartan's identity, $\mathcal{L}_{\vec{\boldsymbol{u}}}\, \boldsymbol{w} = \vec{\boldsymbol{u}} \cdot \mathbf{d}\boldsymbol{w} + \mathbf{d}(\vec{\boldsymbol{u}} \cdot \boldsymbol{w}) = \vec{\boldsymbol{u}} \cdot \boldsymbol{\Omega} - \mathbf{d}h$

Thanks to the e.o.m. (3) we get $\mathcal{L}_{\vec{\boldsymbol{u}}}\, \boldsymbol{w} = T\mathbf{d}s - \mathbf{d}h$

Since $\mathcal{C}(\tau)$ is closed, $\oint_{\mathcal{C}(\tau)} \mathbf{d}h = 0$

Hence

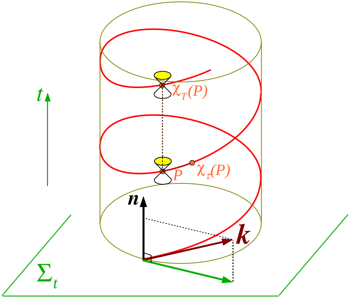
$$\boxed{\frac{dC}{d\tau} = \oint_{\mathcal{C}(\tau)} T\mathbf{d}s}$$

If $T = \mathrm{const}$ or $s = \mathrm{const}$ on $\mathcal{C}(\tau)$, then C is conserved

Bekenstein & Oron's generalisation of Kelvin's theorem

Outline

Equilibrium configuration



Hypothesis:

Magnetized binary system **in equilibrium** on a **circular orbit**

Geometrical translation:

Einstein-Maxwell spacetime $(\mathscr{M}, \boldsymbol{g}, \boldsymbol{F})$ with **helical symmetry** : $\exists$ a vector field $\vec{\boldsymbol{k}}$ of helical type such that

- $\mathcal{L}_{\vec{k}}\, \boldsymbol{g} = 0$ (1)

  $\iff \nabla_\alpha k_\beta + \nabla_\beta k_\alpha = 0$ ($\vec{\boldsymbol{k}}$ = Killing vector)
- $\mathcal{L}_{\vec{k}}\, \boldsymbol{F} = 0$ (2)

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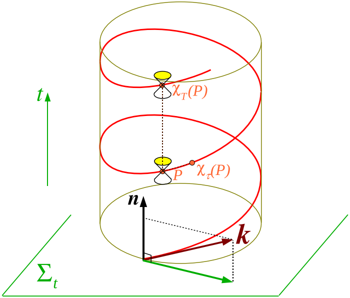

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(1) and (2) are approximations of actual binary spacetimes:

- (1) does not take into account outgoing *gravitational radiation*
- (2) does not allow for outgoing *electromagnetic radiation*

Modeling the slow evolution of the system (inspiral)

Sequence

$$\mathcal{Q}(\lambda) = (\mathscr{M}, \boldsymbol{g}(\lambda), \vec{\boldsymbol{u}}(\lambda), n(\lambda), s(\lambda), \boldsymbol{A}(\lambda))$$

of equilibrium magnetized perfect fluid spacetimes such that $\mathcal{Q}(\lambda)$ and $\mathcal{Q}(\lambda + d\lambda)$ are related by an evolution obeying Einstein-Maxwell equations, baryon number conservation and infinite conductivity

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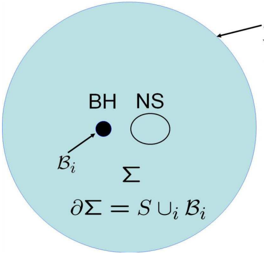
Eulerian change of a quantity f: $\delta f := \dfrac{df}{d\lambda}$

Lagrangian displacement $\vec{\boldsymbol{\xi}}$: vector joining a fluid element at some point P in configuration $\mathcal{Q}(\lambda)$ to the same fluid element in $\mathcal{Q}(\lambda + d\lambda)$

Lagrangian change of a quantity f : $\Delta f = (\delta + \mathcal{L}_{\vec{\boldsymbol{\xi}}}) f$

Noether charge



Family of Noether charges:

$$Q(\lambda) = -\frac{1}{8\pi}\oint_S \left(\nabla^\alpha k^\beta + V^\alpha k^\beta - V^\beta k^\alpha\right) dS_{\alpha\beta}$$

with $V^\alpha(\lambda)$ vector field satisfying

$$\frac{1}{\sqrt{-g}}\frac{d}{d\lambda}\left(\sqrt{-g}V^\alpha\right) = 8\pi\Theta^\alpha$$

$$\Theta^\alpha := \frac{1}{16\pi}\left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu}\right)\nabla_\beta \frac{dg_{\mu\nu}}{d\lambda} + (\varepsilon + p)(g^{\alpha\beta} + u^\alpha u^\beta)\xi_\beta - \frac{1}{4\pi}F^{\alpha\beta}\frac{dA_\beta}{d\lambda} + A_\beta(j^\alpha\xi^\beta - j^\beta\xi^\alpha)$$

V^α can be chosen to make $Q(\lambda)$ finite

$\vec{k}$ Killing vector $\Longrightarrow$ $Q(\lambda)$ is independent of the choice of the 2-surface S

Variation of the Noether charge

If evolution $\mathcal{Q}(\lambda) \rightarrow \mathcal{Q}(\lambda + d\lambda)$ perserves the baryon number, the entropy, the magnetized fluid circulation and the magnetic flux, then, from the equations of motion listed in Part 1,

$$\delta Q = \sum_a \left(\frac{\kappa_a}{8\pi} \delta \mathcal{A}_a + \Phi_a \delta q_a \right) \quad (4)$$

where

- $\sum_a$ is the sum over the black holes (if any)
- κ_a is the surface gravity of BH no. a : $\nabla_{\vec{k}} \vec{k} = \kappa_a \vec{k}$
- $\mathcal{A}_a$ is the area of BH no. a
- q_a is the total electric charge of BH no. a
- Φ_a is the (constant) electric potential of BH no. a:
  $\Phi_a = - A_\alpha k^\alpha |_{\mathcal{B}_a} = \text{const}$

Equation (4) generalizes to the single symmetry case a relation obtained previously by [Carter 1979] in the stationary and axisymmetric case

Generalized first law of thermodynamics

Assume the metric is **asymptotically flat** (Isenber-Wilson-Mathews approximation, 2-PN approximation, waveless approximation,...)
Then $\vec{\boldsymbol{k}}$ is related to two **asymptotically Killing vectors** $\vec{\boldsymbol{t}}$ (timelike) and $\vec{\boldsymbol{\varphi}}$ (spacelike) by

$$\vec{\boldsymbol{k}} = \vec{\boldsymbol{t}} + \Omega\vec{\boldsymbol{\varphi}}, \qquad \Omega = \mathrm{const}$$

and one can define

- the ADM mass M
- the total angular momentum J

Then one can show

$$\delta Q = \delta M - \Omega\,\delta J \tag{5}$$

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Combining (4) and (5), we get

First law of thermodynamics for a magnetized binary system

$$\boxed{\delta M = \Omega\,\delta J + \sum_a \left(\frac{\kappa_a}{8\pi} \delta \mathcal{A}_a + \Phi_a \delta q_a \right)} \qquad (6)$$

Generalizes the law obtained by [Friedman, Uryu & Shibata 2002] to the MHD case

Outline

Irrotational magnetized binaries

Zero-temperature limit of the MHD Euler equation (3) :

$$\boxed{\vec{\boldsymbol{u}} \cdot \boldsymbol{\Omega} = 0} \tag{7}$$

with $\boldsymbol{\Omega} := \mathbf{d}\boldsymbol{w}$ and $\boldsymbol{w} := h\,\underline{\boldsymbol{u}} + \boldsymbol{F} \cdot \vec{\boldsymbol{q}}$

Let us define a **irrotational magnetized flow** as a flow for which $\boxed{\boldsymbol{\Omega} = 0}$
Then there exists (locally) a scalar field Φ such that $\boxed{\boldsymbol{w} = \mathbf{d}\Phi}$

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Motivations for computing irrotational magnetized NS binaries:

- the MHD Euler equation (7) is automatically satisfied
- if the NS have initial low spin, the nuclear matter viscosity is by far too low to synchronize the spins with the orbital frequency $\Longrightarrow$ assuming irrotationality all along the evolutionary sequence is a very good approximation

Equations for irrotational magnetized binaries

The fluid 4-velocity is

$$u^\alpha = \frac{1}{h}\left(\nabla^\alpha \Phi - F^{\alpha\beta} q_\beta\right)$$

The normalization relation $u_\alpha u^\alpha = -1$ then leads to

$$h^2 = -\left(\nabla_\alpha \Phi - F_{\alpha\beta} q^\beta\right)\left(\nabla^\alpha \Phi - F^{\alpha\beta} q_\beta\right)$$

and the baryon number conservation $\nabla_\alpha (n u^\alpha) = 0$ is equivalent to

$$\nabla_\alpha \left[\frac{n}{h}\left(\nabla^\alpha \Phi - F^{\alpha\beta} q_\beta\right)\right] = 0$$

Scheme to compute irrotational magnetized binaries

Choose some EOS $\varepsilon = \varepsilon(h)$, $p = p(h)$, $n = n(h)$ and some vector field $\vec{q}$

Then *iteratively*

1. solve
$$\nabla_\alpha \left[\frac{n}{h} \left(\nabla^\alpha \Phi - F^{\alpha\beta} q_\beta \right) \right] = 0$$
to get Φ
2. compute the enthalpy via $h^2 = - \left(\nabla_\alpha \Phi - F_{\alpha\beta} q^\beta \right) \left(\nabla^\alpha \Phi - F^{\alpha\beta} q_\beta \right)$
3. compute ε, p and n via the EOS
4. solve the Maxwell equation (in Lorenz gauge)
$$\nabla_\beta \nabla^\beta A^\alpha - R^\alpha{}_\beta A^\beta = 4\pi \nabla_\beta (n\, q^\alpha u^\beta - n\, q^\beta u^\alpha)$$
to get A_α and $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$
5. solve the Einstein equations

The magnetic field configuration is specified by $\vec{q}$

Conclusion and perspectives

- We are studying *self-consistent models* of magnetized NS-NS and NS-BH systems within the hypothesis of helical symmetry
- We are using the Bekenstein-Oron formulation of relativistic ideal MHD
- We have derived a relation of the type 'first law of thermodynamics' governing the slow inspiral phase
- For irrotational binaries, we have derived some integration scheme
- There remains to perform the numerical implementation