

Black holes: from event horizons to trapping horizons

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2 April 2009

Plan

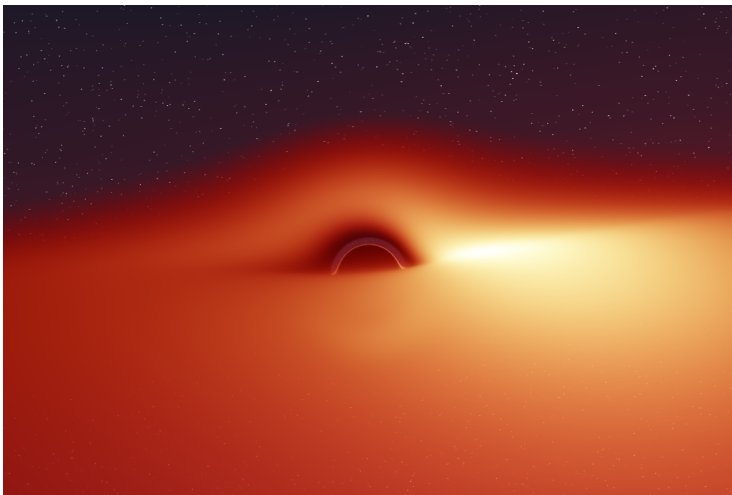
- 1 Concept of black hole and event horizon
- 2 Local approaches to black holes
- 3 Viscous fluid analogy
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- 5 Applications to numerical relativity
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What is a black hole ?

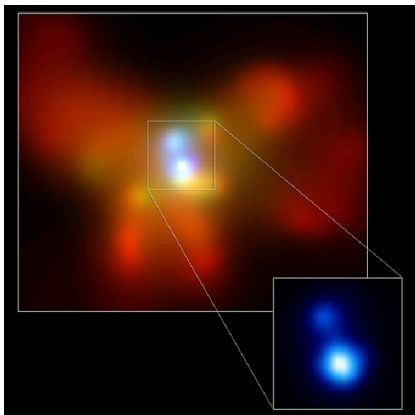
... for the astrophysicist: a very deep gravitational potential well



[J.A. Marck, CQG 13, 393 (1996)]

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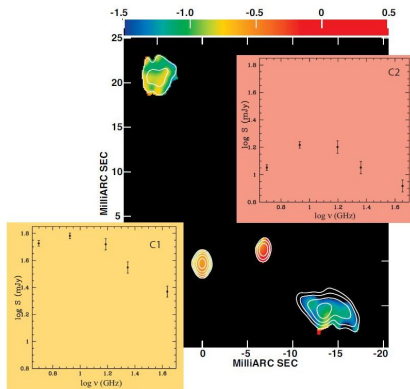
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Binary BH in galaxy NGC 6240

$d = 1.4$ kpc

[Komossa et al., ApJ 582, L15 (2003)]

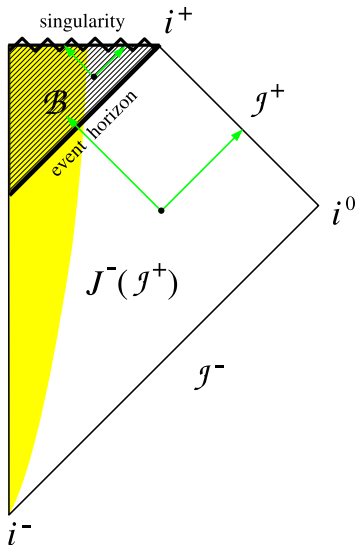


Binary BH in radio galaxy 0402+379

$d = 7.3$ pc

[Rodriguez et al., ApJ 646, 49 (2006)]

What is a black hole ?



... for the mathematical physicist:

$$\mathcal{B} := \mathcal{M} - J^-(\mathcal{I}^+)$$

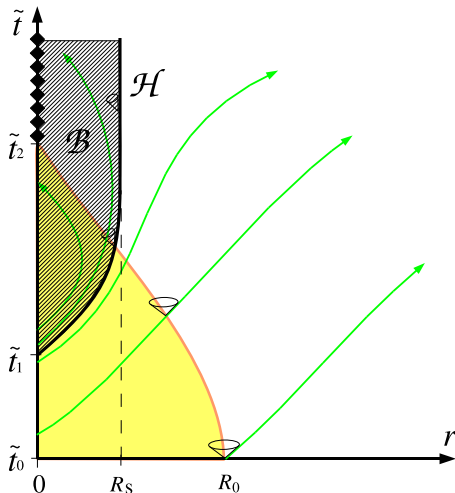
i.e. the region of spacetime where light rays cannot escape to infinity

- (\mathcal{M}, g) = asymptotically flat manifold
- \mathcal{I}^+ = future null infinity
- $J^-(\mathcal{I}^+)$ = causal past of \mathcal{I}^+

event horizon: $\mathcal{H} := \partial J^-(\mathcal{I}^+)$
(boundary of $J^-(\mathcal{I}^+)$)

\mathcal{H} smooth $\implies \mathcal{H}$ null hypersurface

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Drawbacks of the classical definition

- not applicable in **cosmology**, for in general (\mathcal{M}, g) is not asymptotically flat

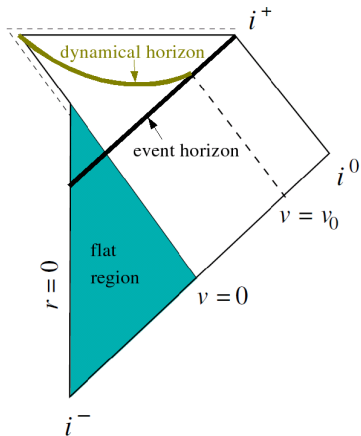
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Determination of $J^-(\mathcal{I}^+)$ requires the knowledge of the entire future null infinity. Moreover this is *not locally linked with the notion of strong gravitational field*:



Example of event horizon in a **flat** region of spacetime:

Vaidya metric, describing incoming radiation from infinity:

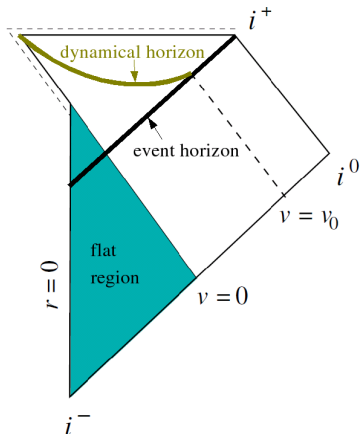
$$ds^2 = - \left(1 - \frac{2m(v)}{r} \right) dv^2 + 2dv dr + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)$$

$$\text{with } \begin{aligned} m(v) &= 0 && \text{for } v < 0 \\ dm/dv &> 0 && \text{for } 0 \leq v \leq v_0 \\ m(v) &= M_0 && \text{for } v > v_0 \end{aligned}$$

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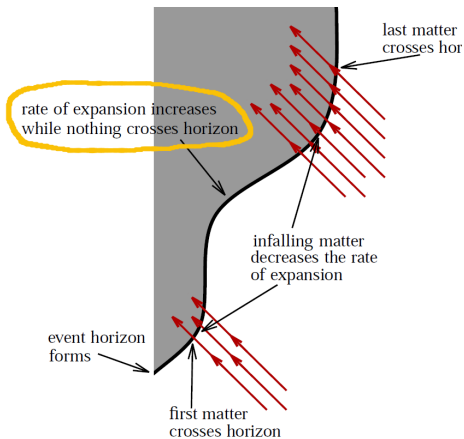
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\Rightarrow no local physical experiment whatsoever can locate the event horizon

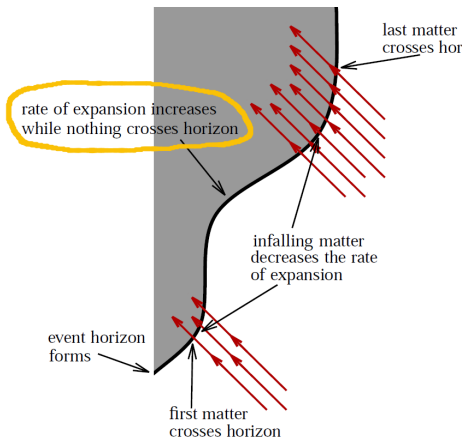
Another non-local feature: teleological nature of event horizons



The classical black hole boundary, i.e. the **event horizon**, responds in advance to what will happen in the future.

[Booth, *Can. J. Phys.* **83**, 1073 (2005)]

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The classical black hole boundary, i.e. the **event horizon**, responds in advance to what will happen in the future.

To deal with black holes as ordinary physical objects, a **local** definition would be desirable

→ quantum gravity, numerical relativity

[Booth, Can. J. Phys. **83**, 1073 (2005)]

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Local characterizations of black holes

Recently a **new paradigm** appeared in the theoretical approach of black holes: instead of *event horizons*, black holes are described by

- **trapping horizons** (Hayward 1994)
- **isolated horizons** (Ashtekar et al. 1999)
- **dynamical horizons** (Ashtekar and Krishnan 2002)
- **slowly evolving horizons** (Booth and Fairhurst 2004)

All these concepts are **local** and are based on the notion of **trapped surfaces**

What is a trapped surface ?

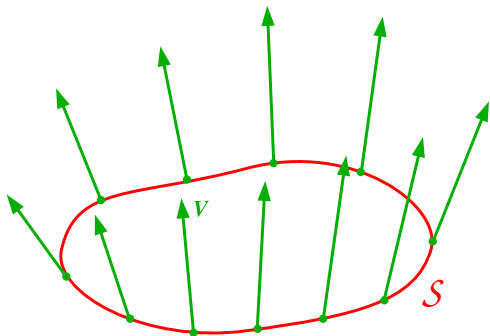
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- 1 Consider a spacelike 2-surface \mathcal{S}
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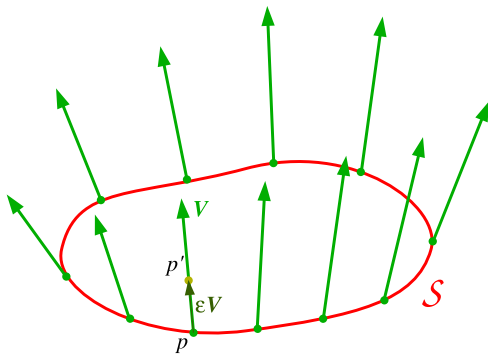
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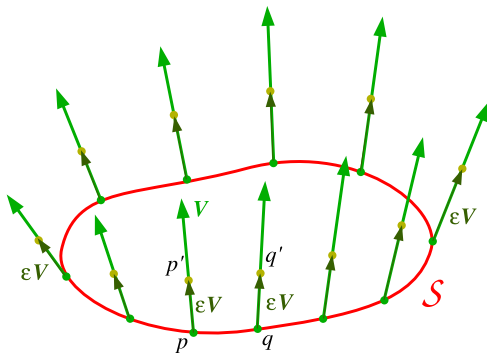
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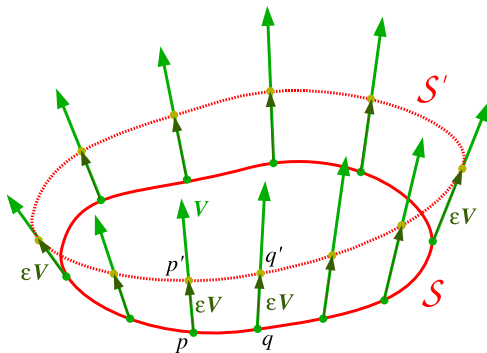
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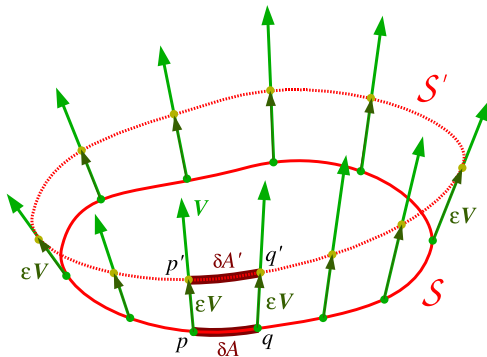
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- 5 This defines a new surface \mathcal{S}' (Lie dragging)

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At each point, the **expansion of \mathcal{S} along v** is defined from the relative change in

the area element δA :

$$\theta^{(v)} := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \frac{\delta A' - \delta A}{\delta A} = \mathcal{L}_v \ln \sqrt{q} = q^{\mu\nu} \nabla_\mu v_\nu$$

What is a trapped surface ?

2/ The definition

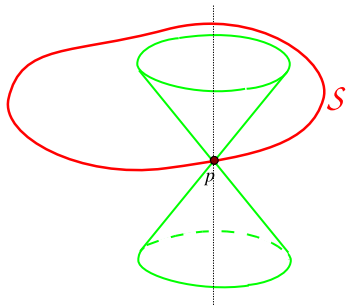
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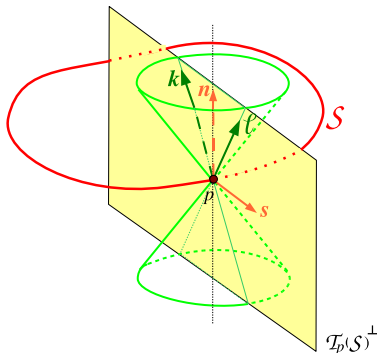


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Being spacelike, \mathcal{S} lies outside the light cone

\exists two future-directed null directions orthogonal to \mathcal{S} :

ℓ = outgoing, expansion $\theta^{(\ell)}$

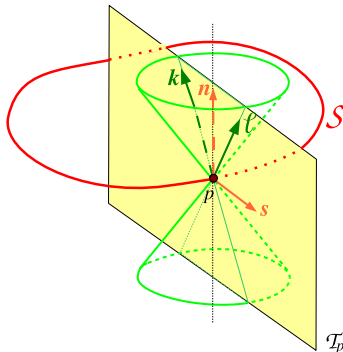
k = ingoing, expansion $\theta^{(k)}$

In flat space, $\theta^{(k)} < 0$ and $\theta^{(\ell)} > 0$

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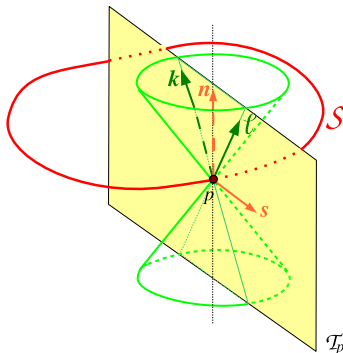
\mathcal{S} is **marginally trapped** $\iff \theta^{(k)} < 0$ and $\theta^{(\ell)} = 0$

[Penrose 1965]

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trapped surface = **local** concept characterizing very strong gravitational fields

Link with apparent horizons

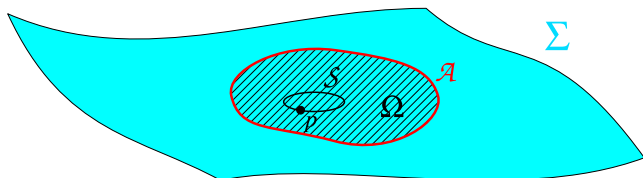
A closed spacelike 2-surface \mathcal{S} is said to be **outer trapped** (resp. **marginally outer trapped (MOTS)**) iff [Hawking & Ellis 1973]

- the notions of *interior* and *exterior* of \mathcal{S} can be defined (for instance spacetime asymptotically flat) $\Rightarrow \ell$ is chosen to be the *outgoing* null normal and k to be the *ingoing* one
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Σ : spacelike hypersurface extending to spatial infinity (Cauchy surface)

outer trapped region of Σ : $\Omega =$ set of points $p \in \Sigma$ through which there is an outer trapped surface \mathcal{S} lying in Σ

apparent horizon in Σ : $\mathcal{A} =$ connected component of the boundary of Ω

Proposition [Hawking & Ellis 1973]: \mathcal{A} smooth $\implies \mathcal{A}$ is a MOTS

Connection with singularities and black holes

Proposition [Penrose (1965)]:

provided that the weak energy condition holds,

\exists a trapped surface $\mathcal{S} \implies \exists$ a singularity in (\mathcal{M}, g) (in the form of a future inextendible null geodesic)

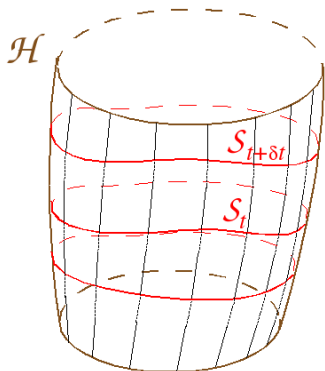
Proposition [Hawking & Ellis (1973)]:

provided that the cosmic censorship conjecture holds,

\exists a trapped surface $\mathcal{S} \implies \exists$ a black hole \mathcal{B} and $\mathcal{S} \subset \mathcal{B}$

Local definitions of “black holes”

A hypersurface \mathcal{H} of (\mathcal{M}, g) is said to be

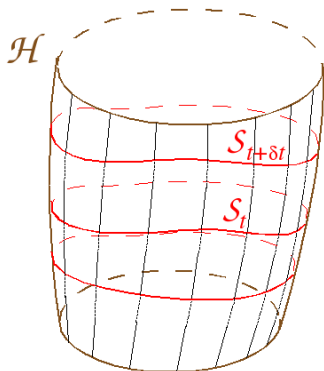


- a **future outer trapping horizon (FOTH)** iff
 - \mathcal{H} foliated by marginally trapped 2-surfaces ($\theta^{(k)} < 0$ and $\theta^{(\ell)} = 0$)
 - $\mathcal{L}_k \theta^{(\ell)} < 0$ (locally outermost trapped surf.)

[Hayward, PRD **49**, 6467 (1994)]

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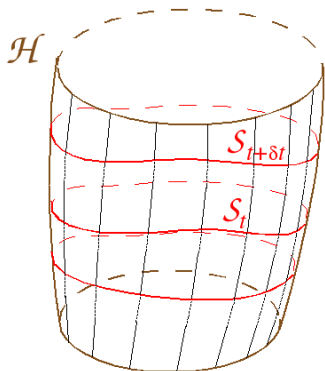
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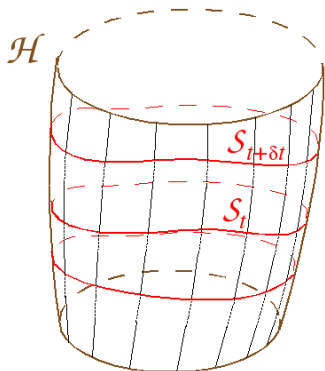
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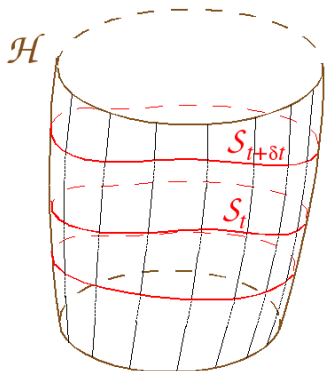


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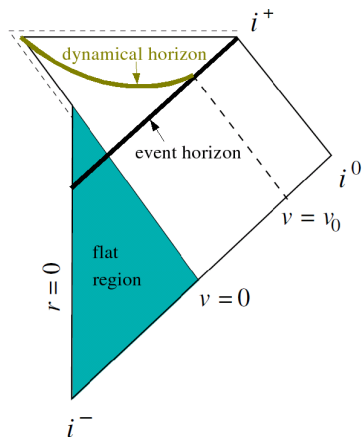


BH in equilibrium = IH
(e.g. Kerr)

BH out of equilibrium = DH
generic BH = FOTH

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Example: Vaidya spacetime



- The **event horizon** crosses the flat region
- The **dynamical horizon** lies entirely outside the flat region

[Ashtekar & Krishnan, LRR 7, 10 (2004)]

Dynamics of these new horizons

The *trapping horizons* and *dynamical horizons* have their **own dynamics**, ruled by Einstein equations.

In particular, one can establish for them

- existence and (partial) uniqueness theorems
[Andersson, Mars & Simon, PRL **95**, 111102 (2005)],
[Ashtekar & Galloway, Adv. Theor. Math. Phys. **9**, 1 (2005)]
- first and second laws of black hole mechanics
[Ashtekar & Krishnan, PRD **68**, 104030 (2003)], [Hayward, PRD **70**, 104027 (2004)]
- a viscous fluid bubble analogy (“membrane paradigm”, as for the event horizon)
[EG, PRD **72**, 104007 (2005)], [EG & Jaramillo, PRD **74**, 087502 (2006)]

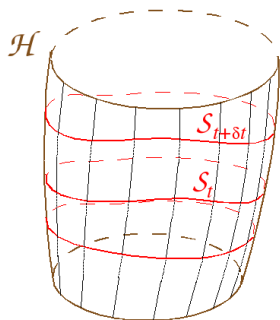
Foliation of a hypersurface by spacelike 2-surfaces

hypersurface \mathcal{H} = submanifold of spacetime (\mathcal{M}, g) of codimension 1

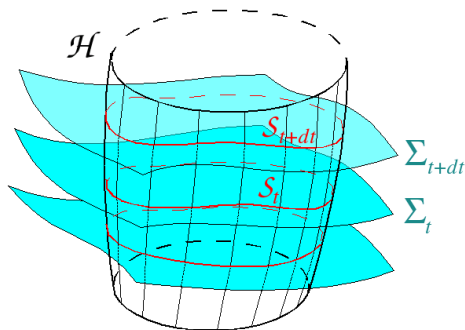
\mathcal{H} can be $\begin{cases} \text{spacelike} \\ \text{null} \\ \text{timelike} \end{cases}$

$$\mathcal{H} = \bigcup_{t \in \mathbb{R}} \mathcal{S}_t$$

\mathcal{S}_t = spacelike 2-surface



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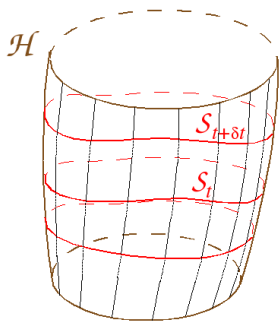
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\Leftarrow 3+1 perspective

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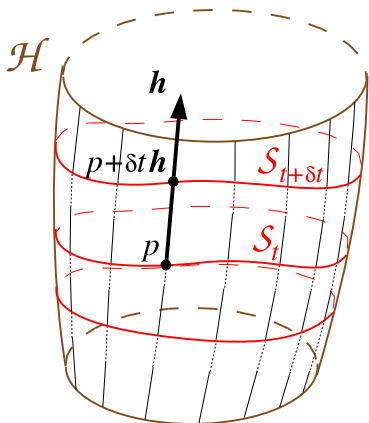
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intrinsic viewpoint adopted here (i.e. not relying on extra-structure such as a 3+1 foliation)

q : induced metric on \mathcal{S}_t (positive definite)

\mathcal{D} : connection associated with q

Evolution vector on the horizon

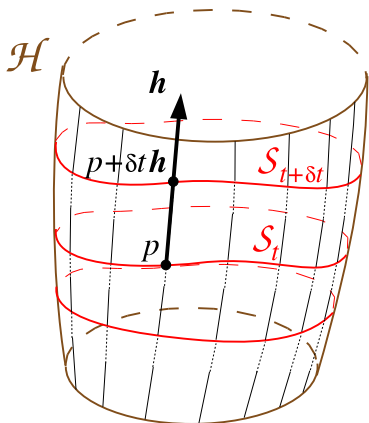


Vector field h on \mathcal{H} defined by

- (i) h is tangent to \mathcal{H}
- (ii) h is orthogonal to \mathcal{S}_t
- (iii) $\mathcal{L}_h t = h^\mu \partial_\mu t = \langle dt, h \rangle = 1$

NB: (iii) \implies the 2-surfaces \mathcal{S}_t are Lie-dragged by h

Evolution vector on the horizon



Vector field h on \mathcal{H} defined by

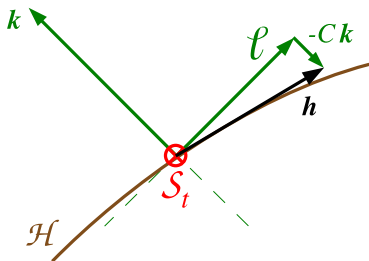
- (i) h is tangent to \mathcal{H}
- (ii) h is orthogonal to S_t
- (iii) $\mathcal{L}_h t = h^\mu \partial_\mu t = \langle dt, h \rangle = 1$

NB: (iii) \implies the 2-surfaces S_t are Lie-dragged by h

Define $C := \frac{1}{2} h \cdot h$

\mathcal{H} is spacelike	\iff	$C > 0$	\iff	h is spacelike
\mathcal{H} is null	\iff	$C = 0$	\iff	h is null
\mathcal{H} is timelike	\iff	$C < 0$	\iff	h is timelike.

Normal null frame associated with the evolution vector



The foliation $(S_t)_{t \in \mathbb{R}}$ entirely fixes the ambiguities in the choice of the null normal frame (ℓ, k) , via the evolution vector h : there exists a **unique normal null frame** (ℓ, k) such that

$$h = \ell - Ck \quad \text{and} \quad \ell \cdot k = -1$$

Normal fundamental form: $\Omega^{(\ell)} := -k \cdot \nabla_{\bar{q}} \ell$ or $\Omega_{\alpha}^{(\ell)} := -k_{\mu} \nabla_{\nu} \ell^{\mu} q^{\nu}_{\alpha}$

Evolution of h along itself: $\nabla_h h = \kappa \ell + (C\kappa - \mathcal{L}_h C)k - \mathcal{D}C$

NB: null limit : $C = 0, h = \ell \implies \nabla_{\ell} \ell = \kappa \ell \implies \kappa = \text{surface gravity}$

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- 1 Concept of black hole and event horizon
- 2 Local approaches to black holes
- 3 Viscous fluid analogy**
- 4 Angular momentum and area evolution laws
- 5 Applications to numerical relativity
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Concept of black hole viscosity

- **Hartle and Hawking (1972, 1973)**: introduced the concept of **black hole viscosity** when studying the response of the *event horizon* to external perturbations
- **Damour (1979)**: 2-dimensional **Navier-Stokes** like equation for the event horizon \implies *shear viscosity* and *bulk viscosity*
- **Thorne and Price (1986)**: **membrane paradigm** for black holes

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Shall we restrict the analysis to the event horizon ?

Can we extend the concept of viscosity to the local characterizations of black hole recently introduced, i.e. **future outer trapping horizons** and **dynamical horizons** ?

NB: *event horizon* = null hypersurface
future outer trapping horizon = null or spacelike hypersurface
dynamical horizon = spacelike hypersurface

Navier-Stokes equation in Newtonian fluid dynamics

$$\rho \left(\frac{\partial v^i}{\partial t} + v^j \nabla_j v^i \right) = -\nabla^i P + \mu \Delta v^i + \left(\zeta + \frac{\mu}{3} \right) \nabla^i (\nabla_j v^j) + f^i$$

or, in terms of fluid momentum density $\pi_i := \rho v_i$,

$$\frac{\partial \pi_i}{\partial t} + v^j \nabla_j \pi_i + \theta \pi_i = -\nabla_i P + 2\mu \nabla^j \sigma_{ij} + \zeta \nabla_i \theta + f_i$$

where θ is the fluid expansion:

$$\theta := \nabla_j v^j$$

and σ_{ij} the velocity shear tensor:

$$\sigma_{ij} := \frac{1}{2} (\nabla_i v_j + \nabla_j v_i) - \frac{1}{3} \theta \delta_{ij}$$

P is the pressure, μ the shear viscosity, ζ the bulk viscosity and f_i the density of external forces

Original Damour-Navier-Stokes equation

Hyp: \mathcal{H} = null hypersurface (particular case: black hole **event horizon**)

Then $\mathbf{h} = \boldsymbol{\ell}$ ($C = 0$)

Damour (1979) has derived from **Einstein equation** the relation

$${}^S\mathcal{L}_{\boldsymbol{\ell}}\boldsymbol{\Omega}^{(\ell)} + \theta^{(\ell)}\boldsymbol{\Omega}^{(\ell)} = \mathcal{D}\kappa - \mathcal{D} \cdot \boldsymbol{\sigma}^{(\ell)} + \frac{1}{2}\mathcal{D}\theta^{(\ell)} + 8\pi\bar{q}^*T \cdot \boldsymbol{\ell}$$

or equivalently

$${}^S\mathcal{L}_{\boldsymbol{\ell}}\boldsymbol{\pi} + \theta^{(\ell)}\boldsymbol{\pi} = -\mathcal{D}P + 2\mu\mathcal{D} \cdot \boldsymbol{\sigma}^{(\ell)} + \zeta\mathcal{D}\theta^{(\ell)} + \mathbf{f} \quad (*)$$

with $\boldsymbol{\pi} := -\frac{1}{8\pi}\boldsymbol{\Omega}^{(\ell)}$ momentum surface density

$P := \frac{\kappa}{8\pi}$ pressure

$\mu := \frac{1}{16\pi}$ shear viscosity

$\zeta := -\frac{1}{16\pi}$ bulk viscosity

$\mathbf{f} := -\bar{q}^*T \cdot \boldsymbol{\ell}$ external force surface density (T = stress-energy tensor)

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(*) is identical to a 2-dimensional Navier-Stokes equation

Original Damour-Navier-Stokes equation (con't)

Introducing a coordinate system (t, x^1, x^2, x^3) such that

- t is compatible with ℓ : $\mathcal{L}_\ell t = 1$
- \mathcal{H} is defined by $x^1 = \text{const}$, so that $x^a = (x^2, x^3)$ are coordinates spanning \mathcal{S}_t

then

$$\ell = \frac{\partial}{\partial t} + \mathbf{V}$$

with \mathbf{V} tangent to \mathcal{S}_t : velocity of \mathcal{H} 's null generators with respect to the coordinates x^a [Damour, PRD 18, 3598 (1978)].

Then

$$\theta^{(\ell)} = \mathcal{D}_a V^a + \frac{\partial}{\partial t} \ln \sqrt{q} \quad q := \det q_{ab}$$

$$\sigma_{ab}^{(\ell)} = \frac{1}{2} (\mathcal{D}_a V_b + \mathcal{D}_b V_a) - \frac{1}{2} \theta^{(\ell)} q_{ab} + \frac{1}{2} \frac{\partial q_{ab}}{\partial t}$$

◀ compare

Negative bulk viscosity of event horizons

From the Damour-Navier-Stokes equation, $\zeta = -\frac{1}{16\pi} < 0$

This negative value would yield to a *dilation or contraction instability* in an ordinary fluid

It is in agreement with the tendency of a null hypersurface to continually contract or expand

The event horizon is stabilized by the **teleological condition** imposing its expansion to vanish in the far future (equilibrium state reached)

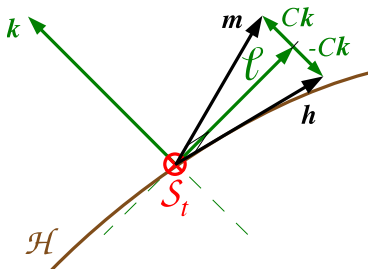
Generalization to the non-null case

Starting remark: in the null case (event horizon), ℓ plays two different roles:

- evolution vector along \mathcal{H} (e.g. term ${}^S\mathcal{L}_\ell$)
- normal to \mathcal{H} (e.g. term $\bar{q}^*T \cdot \ell$)

When \mathcal{H} is no longer null, these two roles have to be taken by two different vectors:

- **evolution vector**: obviously h
- **vector normal to \mathcal{H}** : a natural choice is $m := \ell + Ck$



Generalized Damour-Navier-Stokes equation

From the contracted Ricci identity applied to the vector m and projected onto \mathcal{S}_t : $(\nabla_\mu \nabla_\nu m^\mu - \nabla_\nu \nabla_\mu m^\mu) q^\nu{}_\alpha = R_{\mu\nu} m^\mu q^\nu{}_\alpha$ and using Einstein equation to express $R_{\mu\nu}$, one gets an evolution equation for $\Omega^{(\ell)}$ along \mathcal{H} :

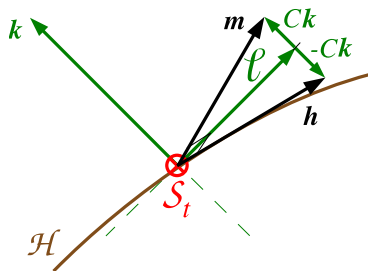
$$\mathcal{L}_h \Omega^{(\ell)} + \theta^{(h)} \Omega^{(\ell)} = \mathcal{D}\kappa - \mathcal{D} \cdot \sigma^{(m)} + \frac{1}{2} \mathcal{D}\theta^{(m)} - \theta^{(k)} \mathcal{D}C + 8\pi \vec{q}^* T \cdot m$$

- $\Omega^{(\ell)}$: normal fundamental form of \mathcal{S}_t associated with null normal ℓ
- $\theta^{(h)}$, $\theta^{(m)}$ and $\theta^{(k)}$: expansion scalars of \mathcal{S}_t along the vectors h , m and k respectively
- \mathcal{D} : covariant derivative within (\mathcal{S}_t, q)
- κ : component of $\nabla_h h$ along ℓ
- $\sigma^{(m)}$: shear tensor of \mathcal{S}_t along the vector m
- C : half the scalar square of h

Null limit (event horizon)

If \mathcal{H} is a null hypersurface,

$$h = m = \ell \quad \text{and} \quad C = 0$$



and we recover the original Damour-Navier-Stokes equation:

$${}^S\mathcal{L}_\ell \Omega^{(\ell)} + \theta^{(\ell)} \Omega^{(\ell)} = \mathcal{D}_\kappa - \mathcal{D} \cdot \sigma^{(\ell)} + \frac{1}{2} \mathcal{D} \theta^{(\ell)} + 8\pi \bar{q}^* T \cdot \ell$$

Case of future trapping horizons

Definition [Hayward, PRD 49, 6467 (1994)] :

\mathcal{H} is a **future trapping horizon** iff $\theta^{(\ell)} = 0$ and $\theta^{(k)} < 0$.

The generalized Damour-Navier-Stokes equation reduces then to

$${}^S\mathcal{L}_h \Omega^{(\ell)} + \theta^{(h)} \Omega^{(\ell)} = \mathcal{D}\kappa - \mathcal{D} \cdot \sigma^{(m)} - \frac{1}{2} \mathcal{D}\theta^{(h)} - \theta^{(k)} \mathcal{D}C + 8\pi \bar{q}^* T \cdot m$$

[EG, PRD 72, 104007 (2005)]

NB: Notice the change of sign in the $-\frac{1}{2} \mathcal{D}\theta^{(h)}$ term with respect to the original Damour-Navier-Stokes equation

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The explanation: it is $\theta^{(m)}$ which appears in the general equation and

$$\theta^{(m)} + \theta^{(h)} = 2\theta^{(\ell)} \implies \begin{cases} \text{event horizon } (m = h) & : \theta^{(m)} = \theta^{(\ell)} \\ \text{trapping horizon } (\theta^{(\ell)} = 0) & : \theta^{(m)} = -\theta^{(h)} \end{cases}$$

Viscous fluid form

$${}^S\mathcal{L}_h \pi + \theta^{(h)} \pi = -\mathcal{D}P + \frac{1}{8\pi} \mathcal{D} \cdot \sigma^{(m)} + \zeta \mathcal{D}\theta^{(h)} + f$$

with $\pi := -\frac{1}{8\pi} \Omega^{(\ell)}$ momentum surface density

$P := \frac{\kappa}{8\pi}$ pressure

$\frac{1}{8\pi} \sigma^{(m)}$ shear stress tensor

$\zeta := \frac{1}{16\pi}$ bulk viscosity

$f := -\bar{q}^* T \cdot m + \frac{\theta^{(k)}}{8\pi} \mathcal{D}C$ external force surface density

Similar to the Damour-Navier-Stokes equation for an event horizon, except

- the **Newtonian-fluid** relation between *stress* and *strain* does not hold: $\sigma^{(m)}/8\pi \neq 2\mu\sigma^{(h)}$, rather $\sigma^{(m)}/8\pi = [\sigma^{(h)} + 2C\sigma^{(k)}]/8\pi$

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- **positive bulk viscosity**

This positive value of bulk viscosity shows that FOTHs and DHs behave as “ordinary” physical objects, in perfect agreement with their **local nature**

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Angular momentum of trapping horizons

Definition [Booth & Fairhurst, CQG 22, 4545 (2005)]: Let φ be a vector field on \mathcal{H} which

- is tangent to \mathcal{S}_t
- has closed orbits
- has vanishing divergence with respect to the induced metric: $\mathcal{D} \cdot \varphi = 0$ (weaker than being a Killing vector of $(\mathcal{S}_t, \mathbf{q})$!)

For dynamical horizons, $\theta^{(h)} \neq 0$ and there is a unique choice of φ as the generator (conveniently normalized) of the curves of constant $\theta^{(h)}$

[Hayward, PRD 74, 104013 (2006)]

The *generalized angular momentum associated with φ* is then defined by

$$J(\varphi) := -\frac{1}{8\pi} \oint_{\mathcal{S}_t} \langle \Omega^{(\ell)}, \varphi \rangle s_{\epsilon},$$

Remark 1: does not depend upon the choice of null vector ℓ , thanks to the divergence-free property of φ

Remark 2:

- coincides with **Ashtekar & Krishnan's** definition for a dynamical horizon
- coincides with **Brown-York** angular momentum if \mathcal{H} is timelike and φ a Killing vector

Angular momentum flux law

Under the supplementary hypothesis that φ is transported along the evolution vector \mathbf{h} : $\mathcal{L}_{\mathbf{h}} \varphi = 0$, the generalized Damour-Navier-Stokes equation leads to

$$\frac{d}{dt} J(\varphi) = - \oint_{S_t} \mathbf{T}(m, \varphi) \cdot \mathbf{s}_\epsilon - \frac{1}{16\pi} \oint_{S_t} \left[\boldsymbol{\sigma}^{(m)} : \mathcal{L}_\varphi \mathbf{q} - 2\theta^{(k)} \varphi \cdot \mathcal{D}\mathbf{C} \right] \cdot \mathbf{s}_\epsilon$$

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Two interesting limiting cases:

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Two interesting limiting cases:

- \mathcal{H} = null hypersurface : $C = 0$ and $\mathbf{m} = \boldsymbol{\ell}$:

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i.e. Eq. (6.134) of the *Membrane Paradigm* book (Thorne, Price & MacDonald 1986)

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- \mathcal{H} = future trapping horizon :

$$\frac{d}{dt} J(\varphi) = - \oint_{S_t} \mathbf{T}(\mathbf{m}, \varphi) \cdot \mathbf{s}_\epsilon - \frac{1}{16\pi} \oint_{S_t} \boldsymbol{\sigma}^{(m)} : \mathcal{L}_\varphi \mathbf{q} \cdot \mathbf{s}_\epsilon$$

Area evolution law for an event horizon

$A(t)$: area of the 2-surface \mathcal{S}_t ; ${}^s\epsilon$: volume element of \mathcal{S}_t ; $\bar{\kappa}(t) := \frac{1}{A(t)} \int_{\mathcal{S}_t} \kappa {}^s\epsilon$

Integrating the null Raychaudhuri equation on \mathcal{S}_t , one gets

$$\frac{d^2 A}{dt^2} - \bar{\kappa} \frac{dA}{dt} = - \int_{\mathcal{S}_t} \left[8\pi T(\ell, \ell) + \sigma^{(\ell)} : \sigma^{(\ell)} - \frac{(\theta^{(\ell)})^2}{2} + (\bar{\kappa} - \kappa)\theta^{(\ell)} \right] {}^s\epsilon \quad (1)$$

[Damour, 1979]

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Simplified analysis : assume $\bar{\kappa} = \text{const} > 0$:

- Cauchy problem \implies diverging solution of the homogeneous equation:

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$$\frac{dA}{dt} = \alpha \exp(\bar{\kappa}t) \quad !$$

- correct treatment: impose $\frac{dA}{dt} = 0$ at $t = +\infty$ (teleological !)

$$\frac{dA}{dt} = \int_t^{+\infty} D(u) e^{\bar{\kappa}(t-u)} du \quad D(t) : \text{r.h.s. of Eq. (1)}$$

Non causal evolution

Area evolution law for a dynamical horizon

Dynamical horizon : $C > 0$; $\kappa' := \kappa - \mathcal{L}_h \ln C$; $\bar{\kappa}'(t) := \frac{1}{A(t)} \int_{S_t} \kappa' s_\epsilon$

From the (m, h) component of Einstein equation, one gets

$$\frac{d^2 A}{dt^2} + \bar{\kappa}' \frac{dA}{dt} = \int_{S_t} \left[8\pi T(m, h) + \sigma^{(h)} : \sigma^{(m)} + \frac{(\theta^{(h)})^2}{2} + (\bar{\kappa}' - \kappa') \theta^{(h)} \right] s_\epsilon \quad (2)$$

[EG & Jaramillo, PRD **74**, 087502 (2006)]

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[EG & Jaramillo, PRD **74**, 087502 (2006)]

Simplified analysis : assume $\bar{\kappa}' = \text{const} > 0$

(OK for small departure from equilibrium [Booth & Fairhurst, PRL **92**, 011102 (2004)]):

Standard Cauchy problem :

$$\frac{dA}{dt} = \left. \frac{dA}{dt} \right|_{t=0} + \int_0^t D(u) e^{\bar{\kappa}'(u-t)} du \quad D(t) : \text{r.h.s. of Eq. (2)}$$

Causal evolution, in agreement with local nature of dynamical horizons

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Applications to numerical relativity

- **Initial data:** isolated horizons (helical symmetry)

[EG, Grandclément & Bonazzola, PRD **65**, 044020 (2002)]

[Grandclément, EG & Bonazzola, PRD **65**, 044021 (2002)]

[Cook & Pfeiffer, PRD **70**, 104016 (2004)]

- **A posteriori analysis:** estimating mass, linear and angular momentum of formed black holes

[Schnetter, Krishnan & Beyer, PRD **74**, 024028 (2006)]

[Cook & Whiting, PRD **76**, 041501 (2007)]

[Krishnan, Lousto & Zlochower, PRD **76**, 081501(R) (2007)]

- **Numerical construction of spacetime:** inner boundary conditions for a constrained scheme with “black hole excision”

[Jaramillo, EG, Cordero-Carrión, & J.M. Ibáñez, PRD **77**, 047501 (2008)]

[Vasset, Novak & Jaramillo, arXiv:0901??]

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Review articles

- A. Ashtekar and B. Krishnan : *Isolated and dynamical horizons and their applications*, Living Rev. Relativity **7**, 10 (2004) ;
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- I. Booth : *Black hole boundaries*, Canadian J. Phys. **83**, 1073 (2005) ;
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- E.ourgoulhon and J.L. Jaramillo : *A 3+1 perspective on null hypersurfaces and isolated horizons*, Phys. Rep. **423**, 159 (2006) ;
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