Holographic computations with SageMath

Éric Gourgoulhon[1]

based on a collaboration with

Dmitry Ageev[2], Irina Ya. Aref'eva[2] and Anastasia A. Golubtsova[3]

[1] Laboratoire Univers et Théories, Observatoire de Paris, CNRS, Université PSL, Université de Paris, Meudon, France

[2] Steklov Mathematical Institute, Moscow, Russia

[3] Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, Russia

Holographic QCD at high densities, neutron stars and gravitational waves

APC, Paris
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Outline

Outline

The context: computational tools developed at LUTH

- LORENE: library for solving PDE equations with spectral methods in spherical coordinates
  https://lorene.obspm.fr/ [C++]
- CoCoNut: GR-hydro code for 3D gravitational collapse [with U. Valencia]
  https://www.uv.es/coconut/ [C++]
- Kadath: library for solving PDE equations with spectral methods (generic coordinates)
  https://kadath.obspm.fr/ [C++]
- Gyoto: code for geodesic computation (ray-tracing) [with LESIA]
  https://gyoto.obspm.fr/ [C++, Python]
- CompOSE: Data base of nuclear matter equations of state
  https://compose.obspm.fr/
- SageManifolds: differential geometry and tensor calculus with SageMath
  https://sagemanifolds.obspm.fr/ [Python]

All these tools are free software (GPL)

SageMath in a few words

SageMath (*nickname:* Sage) is a **free open-source** computer algebra system

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SageMath is based on Python

- no need to learn any specific syntax to use it
- Python is a powerful *object oriented language*, with a neat syntax
- SageMath benefits from the Python ecosystem (e.g. Jupyter notebook)

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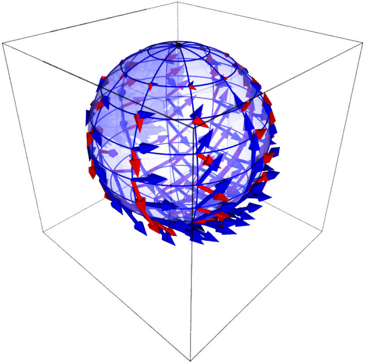
SageMath is developed by an enthusiastic community

- mostly composed of mathematicians
- welcoming newcomers

Tensor calculus with SageMath

SageManifolds project: extends SageMath towards **differential geometry** and **tensor calculus**



Stereographic-coordinate frame on $\mathbb{S}^2$

- https://sagemanifolds.obspm.fr
- fully included in SageMath (after review process)
- $\sim$ 15 contributors (developers and reviewers) cf. https://sagemanifolds.obspm.fr/authors.html
- dedicated mailing list
- help: https://ask.sagemath.org

Everybody is very welcome to contribute
$\Longrightarrow$ visit https://sagemanifolds.obspm.fr/contrib.html

Current status

Already present (SageMath 8.9):

- differentiable manifolds: tangent spaces, vector frames, tensor fields, curves, pullback and pushforward operators, submanifolds
- standard tensor calculus (tensor product, contraction, symmetrization, etc.), even on non-parallelizable manifolds, and with all monoterm tensor symmetries taken into account
- Lie derivatives of tensor fields
- differential forms: exterior and interior products, exterior derivative, Hodge duality
- multivector fields: exterior and interior products, Schouten-Nijenhuis bracket
- affine connections (curvature, torsion)
- pseudo-Riemannian metrics
- computation of geodesics (numerical integration)

Current status

Already present *(cont'd)*:

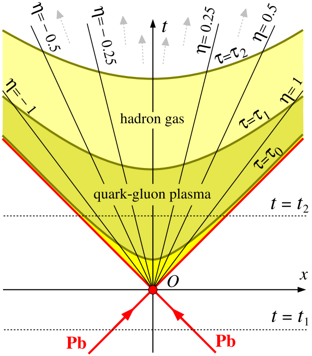
- some plotting capabilities (charts, points, curves, vector fields)
- parallelization (on tensor components) of CPU demanding computations
- extrinsic geometry of pseudo-Riemannian submanifolds
- series expansions of tensor fields
- 2 symbolic backends: Pynac/Maxima (SageMath's default) and SymPy

Future prospects:

- more symbolic backends (Giac, FriCAS, ...)
- more graphical outputs
- symplectic forms, fibre bundles, spinors, integrals on submanifolds, variational calculus, etc.
- connection with numerical relativity: use SageMath to explore numerically-generated spacetimes

Outline

Quark-gluon plasma in the gauge/gravity duality



Spacetime diagram of a heavy-ion collision (LHC)

$\tau_0 \simeq 0.2\ \mathrm{fm}/c = 6\ 10^{-25}\ \mathrm{s}$

$\tau_1 \sim 10\tau_0$

Quark-gluon plasma (QGP) in heavy-ion collisions: low-viscosity fluid with *anisotropic* pressure ($p_x < p_y$)

Thermalization of QGP $\equiv$ 5D black hole formation

Gauge theory: QCD

Gravity: 5D Lifshitz-like spacetime (*anisotropic* generalization of AdS$_5$) supported by magnetic field with formation of a black brane (Vaidya-type collapse)

Results: faster thermalization in the transversal direction; evolution of the entanglement entropy

[Aref'eva, Golubtsova & Gourgoulhon, J. High Ener. Phys. **09(2016)**, 142]

[Ageev, Aref'eva, Golubtsova & Gourgoulhon, Nucl. Phys. B **931**, 506 (2018)]

Black branes in 5D Lifshitz-like spacetimes

The SageMath notebook:
https://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Notebooks/SM_Lifshitz_black_brane.ipynb

In the nbviewer menu, click on the icon to run an interactive version on a Binder server.

Outline

Near-horizon geometry of the extremal Kerr black hole

Extremal Kerr black hole: $a = m \iff \kappa = 0$ (degenerate horizon)

Near-horizon geometry of extremal 4D Kerr is similar to $\mathrm{AdS}_2 \times \mathbb{S}^2$ geometry; it has has extended isometry group: $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)$, instead of merely $\mathbb{R} \times \mathrm{U}(1)$ for Kerr metric [Bardeen & Horowitz, PRD **60**, 104030 (1999)]

Near-horizon geometry of extremal Kerr black hole is at the basis of the **Kerr/CFT correspondence** (see [Compère, LRR **20**, 1 (2017)] for a review)

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Near-horizon geometry of extremal Kerr black hole is at the basis of the **Kerr/CFT correspondence** (see [Compère, LRR **20**, 1 (2017)] for a review)

Let us explore this geometry with a SageMath notebook:
https://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Notebooks/SM_extremal_Kerr_near_horizon.ipynb

In the nbviewer menu, click on the icon to run an interactive version on a Binder server.

Near-horizon geometry of the extremal Kerr black hole

This notebook derives the near-horizon geometry of the extremal (i.e. maximally spinning) Kerr black hole. It is based on SageMath tools developed through the SageManifolds project.

First we set up the notebook to display maths using LaTeX rendering and to perform computations in parallel on 8 threads:

In [1]:
```
%display latex
Parallelism().set(nproc=8)
```

Spacetime manifold

We declare the Kerr spacetime (or more precisely the part of it covered by Boyer-Lindquist coordinates) as a 4-dimensional Lorentzian manifold $\mathcal{M}$:

In [2]:
```
M = Manifold(4, 'M', latex_name=r'\mathcal{M}', structure='Lorentzian')
print(M)
```

```
4-dimensional Lorentzian manifold M
```

We then introduce the standard **Boyer-Lindquist coordinates** (t, r, θ, ϕ) as a chart `BL` (for *Boyer-Lindquist*) on $\mathcal{M}$:

In [3]:
```
BL.<t,r,th,ph> = M.chart(r"t r th:(0,pi):\theta ph:(0,2*pi):periodic:\phi")
print(BL); BL
```

```
Chart (M, (t, r, th, ph))
```

Out[3]: $(\mathcal{M}, (t, r, \theta, \phi))$

Metric tensor of the extremal Kerr spacetime

The metric is set by its components in the coordinate frame associated with Boyer-Lindquist coordinates, which is the current manifold's default frame:

In [4]:
```
m = var('m', domain='real')
a = m  # extremal Kerr
rho2 = r^2 + (a*cos(th))^2
Delta = r^2 -2*m*r + a^2
g = M.metric()
g[0,0] = -(1-2*m*r/rho2)
g[0,3] = -2*a*m*r*sin(th)^2/rho2
g[1,1], g[2,2] = rho2/Delta, rho2
g[3,3] = (r^2+a^2+2*m*r*(a*sin(th))^2/rho2)*sin(th)^2
g.display()
```

Out[4]:

$$g = \left(\frac{2\,mr}{m^2\cos(\theta)^2 + r^2} - 1\right) \mathrm{d}t\otimes\mathrm{d}t + \left(-\frac{2\,m^2 r\sin(\theta)^2}{m^2\cos(\theta)^2 + r^2}\right)\mathrm{d}t\otimes\mathrm{d}\phi + \left(\frac{m^2\cos(\theta)^2 + r^2}{m^2 - 2\,mr + r^2}\right)\mathrm{d}r\otimes\mathrm{d}r + \left(m^2\cos(\theta)^2 + r^2\right)\mathrm{d}\theta\otimes\mathrm{d}\theta + \left(-\frac{2\,m^2 r\sin(\theta)^2}{m^2\cos(\theta)^2 + r^2}\right)\mathrm{d}\phi\otimes\mathrm{d}t + \left(\frac{2\,m^3 r\sin(\theta)^2}{m^2\cos(\theta)^2 + r^2} + m^2 + r^2\right)\sin(\theta)^2\mathrm{d}\phi\otimes\mathrm{d}\phi$$

Check that we are dealing with a solution of the vacuum Einstein equation:

In [5]:
```
g.ricci().display()
```

Out[5]: $\mathrm{Ric}(g) = 0$

Near-horizon coordinates

Let us introduce the chart `NH` of the near-horizon coordinates $(\bar{t}, \bar{r}, \theta, \bar{\phi})$:

In [6]:
```
NH.<tb,rb,th,phb> = M.chart(r"tb:\bar{t} rb:\bar{r} th:(0,pi):\theta phb:(0,2*pi):periodic:\
print(NH)
NH
```

```
Chart (M, (tb, rb, th, phb))
```

Out[6]: $(\mathcal{M}, (\bar{t}, \bar{r}, \theta, \bar{\phi}))$

Following J. Bardeen and G. T. Horowitz, Phys. Rev. D **60**, 104030 (1999), the near-horizon coordinates $(\bar{t}, \bar{r}, \theta, \bar{\phi})$ are related to the Boyer-Lindquist coordinates by

$$\bar{t} = \epsilon t, \quad \bar{r} = \frac{r-m}{\epsilon}, \quad \theta = \theta, \quad \bar{\phi} = \phi - \frac{t}{2m},$$

where ϵ is a constant parameter. The horizon of the extremal Kerr black hole is located at $r = m$, which corresponds to $\bar{r} = 0$.

We implement the above relations as a transition map from the chart `BL` to the chart `NH`:

In [7]:
```
eps = var('eps', latex_name=r'\epsilon')
BL_to_NH = BL.transition_map(NH, [eps*t, (r-m)/eps, th, ph - t/(2*m)])
BL_to_NH.display()
```

Out[7]:

$$\begin{cases} \bar{t} & = & \epsilon t \\ \bar{r} & = & -\frac{m-r}{\epsilon} \\ \theta & = & \theta \\ \bar{\phi} & = & \phi - \frac{t}{2\,m} \end{cases}$$

The inverse relation is

In [8]:
```
BL_to_NH.inverse().display()
```

Out[8]:

$$\begin{cases} t &= \frac{\bar{t}}{\epsilon} \\ r &= \epsilon\bar{r} + m \\ \theta &= \theta \\ \phi &= \frac{2\,\epsilon m\bar{\phi} + \bar{t}}{2\,\epsilon m} \end{cases}$$

The metric components with respect the coordinates $(\bar{t}, \bar{r}, \theta, \bar{\phi})$ are computed by passing the chart `NH` to the method `display()`:

In [9]:
```
g.display(NH)
```

Out[9]:

$$g = \left(-\frac{m^2\bar{r}^2\cos(\theta)^4 - \epsilon^2\bar{r}^4 - 4\,\epsilon m\bar{r}^3 - 3\,m^2\bar{r}^2 + \left(\epsilon^2\bar{r}^4 + 4\,\epsilon m\bar{r}^3 + 6\,m^2\bar{r}^2\right)\cos(\theta)^2}{4\left(\epsilon^2 m^2\bar{r}^2 + m^4\cos(\theta)^2 + 2\,\epsilon m^3\bar{r} + m^4\right)}\right) \mathrm{d}\bar{t}\otimes\mathrm{d}\bar{t}$$

$$+ \left(-\frac{\epsilon m^2\bar{r}^2\sin(\theta)^4 - \left(\epsilon^3\bar{r}^4 + 4\,\epsilon^2 m\bar{r}^3 + 8\,\epsilon m^2\bar{r}^2 + 4\,m^3\bar{r}\right)\sin(\theta)^2}{2\left(\epsilon^2 m\bar{r}^2 + m^3\cos(\theta)^2 + 2\,\epsilon m^2\bar{r} + m^3\right)}\right) \mathrm{d}\bar{t}\otimes\mathrm{d}\bar{\phi}$$

$$+ \left(\frac{\epsilon^2\bar{r}^2 + m^2\cos(\theta)^2 + 2\,\epsilon m\bar{r} + m^2}{\bar{r}^2}\right) \mathrm{d}\bar{r}\otimes\mathrm{d}\bar{r} + \left(\epsilon^2\bar{r}^2 + m^2\cos(\theta)^2 + 2\,\epsilon m\bar{r} + m^2\right) \mathrm{d}\theta\otimes\mathrm{d}\theta$$

$$+ \left(-\frac{\epsilon m^2\bar{r}^2\sin(\theta)^4 - \left(\epsilon^3\bar{r}^4 + 4\,\epsilon^2 m\bar{r}^3 + 8\,\epsilon m^2\bar{r}^2 + 4\,m^3\bar{r}\right)\sin(\theta)^2}{2\left(\epsilon^2 m\bar{r}^2 + m^3\cos(\theta)^2 + 2\,\epsilon m^2\bar{r} + m^3\right)}\right) \mathrm{d}\bar{\phi}\otimes\mathrm{d}\bar{t}$$

$$+ \left(-\frac{\epsilon^2 m^2\bar{r}^2\sin(\theta)^4 - \left(\epsilon^4\bar{r}^4 + 4\,\epsilon^3 m\bar{r}^3 + 8\,\epsilon^2 m^2\bar{r}^2 + 8\,\epsilon m^3\bar{r} + 4\,m^4\right)\sin(\theta)^2}{\epsilon^2\bar{r}^2 + m^2\cos(\theta)^2 + 2\,\epsilon m\bar{r} + m^2}\right) \mathrm{d}\bar{\phi}\otimes\mathrm{d}\bar{\phi}$$

From now on, we use the near-horizon coordinates as the default ones on the spacetime manifold:

In [10]:
```
M.set_default_chart(NH)
M.set_default_frame(NH.frame())
```

The near-horizon metric h as the limit $\epsilon \to 0$ of the Kerr metric g

Let us define the *near-horizon metric* as the metric h on $\mathcal{M}$ that is the limit $\epsilon \to 0$ of the Kerr metric g. The limit is taken by asking for a series expansion of g with respect to ϵ up to the 0-th order (i.e. keeping only ϵ^0 terms). This is acheived via the method `truncate`:

In [11]:
```
h = M.lorentzian_metric('h')
h.set(g.truncate(eps, 0))
h.display()
```

Out[11]:

$$h = \left(-\frac{\bar{r}^2\cos(\theta)^4 + 6\,\bar{r}^2\cos(\theta)^2 - 3\,\bar{r}^2}{4\left(m^2\cos(\theta)^2 + m^2\right)}\right) d\bar{t}\otimes d\bar{t} + \left(\frac{2\,\bar{r}\sin(\theta)^2}{\cos(\theta)^2+1}\right) d\bar{t}\otimes d\bar{\phi} + \left(\frac{m^2\cos(\theta)^2 + m^2}{\bar{r}^2}\right) d\bar{r}\otimes d\bar{r} + \left(m^2\cos(\theta)^2 + m^2\right) d\theta\otimes d\theta + \left(\frac{2\,\bar{r}\sin(\theta)^2}{\cos(\theta)^2+1}\right) d\bar{\phi}\otimes d\bar{t} + \left(\frac{4\,m^2\sin(\theta)^2}{\cos(\theta)^2+1}\right) d\bar{\phi}\otimes d\bar{\phi}$$

We note that the metric h is not asymptotically flat.

Killing vectors of the near-horizon geometry

Let us first consider the vector field $\eta := \frac{\partial}{\partial \bar{\phi}}$:

In [12]:
```
eta = M.vector_field(0, 0, 0, 1, name='eta', latex_name=r'\eta')
eta.display()
```

Out[12]:

$$\eta = \frac{\partial}{\partial \bar{\phi}}$$

It is a Killing vector of the near-horizon metric, since the Lie derivative of h along η vanishes:

In [13]:
```
h.lie_derivative(eta).display()
```

Out[13]: 0

This is not surprising since the components of h are independent from $\bar{\phi}$.

Similarly, we can check that $\xi_1 := \frac{\partial}{\partial \bar{t}}$ is a Killing vector of h, reflecting the independence of the components of h from $\bar{t}$:

In [14]:
```
xi1 = M.vector_field(1, 0, 0, 0, name='xi2', latex_name=r'\xi_{1}')
xi1.display()
```

Out[14]:

$$\xi_1 = \frac{\partial}{\partial \bar{t}}$$

In [15]:
```
h.lie_derivative(xi1).display()
```

Out[15]: 0

The above two Killing vectors correspond respectively to the **axisymmetry** and the **pseudo-stationarity** of the Kerr metric. A third symmetry, which is not present in the original Kerr metric, is the invariance under the **scaling** $(\bar{t}, \bar{r}) \mapsto (\alpha\bar{t}, \bar{r}/\alpha)$, as it is clear on the metric components in Out[11]. The corresponding Killing vector is

In [16]:
```
xi2 = M.vector_field(tb, -rb, 0, 0, name='xi2', latex_name=r'\xi_{2}')
xi2.display()
```

Out[16]:
$$\xi_2 = \bar{t}\frac{\partial}{\partial\bar{t}} - \bar{r}\frac{\partial}{\partial\bar{r}}$$

In [17]:
```
h.lie_derivative(xi2).display()
```

Out[17]: 0

Finally, a fourth Killing vector is

In [18]:
```
xi3 = M.vector_field(tb^2/2 + 2*m^4/rb^2, -tb*rb, 0, -2*m^2/rb,
                     name='xi3', latex_name=r'\xi_{3}')
xi3.display()
```

Out[18]:
$$\xi_3 = \left(\frac{2\,m^4}{\bar{r}^2} + \frac{1}{2}\,\bar{t}^2\right)\frac{\partial}{\partial\bar{t}} - \bar{r}\bar{t}\frac{\partial}{\partial\bar{r}} - \frac{2\,m^2}{\bar{r}}\frac{\partial}{\partial\bar{\phi}}$$

In [19]:
```
h.lie_derivative(xi3).display()
```

Out[19]: 0

Symmetry group

We have four independent Killing vectors, η, ξ_1, ξ_2 and ξ_3, which implies that the symmetry group of the near-horizon geometry is a 4-dimensional Lie group G. Let us determine G by investigating the **structure constants** of the basis $(\eta, \xi_1, \xi_2, \xi_3)$ of the Lie algebra of G. First of all, we notice that η commutes with the other Killing vectors:

In [20]:
```
for xi in [xi1, xi2, xi3]:
    show(eta.bracket(xi).display())
```

$[\eta, \xi_1] = 0$

$[\eta, \xi_2] = 0$

$[\eta, \xi_3] = 0$

Since η generates the rotation group $\mathrm{SO}(2) = \mathrm{U}(1)$, we may write that $G = \mathrm{U}(1) \times G_3$, where G_3 is a 3-dimensional Lie group, whose generators are (ξ_1, ξ_2, ξ_3). Let us determine the structure constants of these three vectors. We have

In [21]:
```
xi1.bracket(xi2).display()
```

Out[21]:

$$[\xi_1, \xi_2] = \frac{\partial}{\partial \bar{t}}$$

In [22]:
```
xi1.bracket(xi3).display()
```

Out[22]:

$$[\xi_1, \xi_3] = \bar{t}\frac{\partial}{\partial \bar{t}} - \bar{r}\frac{\partial}{\partial \bar{r}}$$

In [23]:
```
xi2.bracket(xi3).display()
```

Out[23]:

$$[\xi_2, \xi_3] = \left(\frac{4\,m^4 + \bar{r}^2\bar{t}^2}{2\,\bar{r}^2}\right)\frac{\partial}{\partial \bar{t}} - \bar{r}\bar{t}\frac{\partial}{\partial \bar{r}} - \frac{2\,m^2}{\bar{r}}\frac{\partial}{\partial \bar{\phi}}$$

To summarize, we have

In [24]:
```
all([xi1.bracket(xi2) == xi1,
     xi1.bracket(xi3) == xi2,
     xi2.bracket(xi3) == xi3])
```

Out[24]: True

To recognize a standard Lie algebra, let us perform a slight change of basis:

In [25]:
```
vE = -sqrt(2)*xi3
vF = sqrt(2)*xi1
vH = 2*xi2
```

We have then the following commutation relations:

In [26]:
```
all([vE.bracket(vF) == vH,
     vH.bracket(vE) == 2*vE,
     vH.bracket(vF) == -2*vF])
```

Out[26]: True

We recognize the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. Indeed, we have

In [27]:
```
sl2 = lie_algebras.sl(RR, 2)
E,F,H = sl2.gens()
all([E.bracket(F) == H,
     H.bracket(E) == 2*E,
     H.bracket(F) == -2*F])
```

Out[27]: True

Hence, we have

$$\mathrm{Lie}(G_3) = \mathfrak{sl}(2,\mathbb{R}).$$

At this stage, G_3 could be $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{PSL}(2,\mathbb{R})$ or $\overline{\mathrm{SL}(2,\mathbb{R})}$ (the universal covering group of $\mathrm{SL}(2,\mathbb{R})$). One can show that actually $G_3 = \mathrm{SL}(2,\mathbb{R})$. We conclude that the full isometry group of the near-horizon geometry is

$$G = \mathrm{U}(1) \times \mathrm{SL}(2,\mathbb{R}).$$

The full notebook is available at
https://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Notebooks/SM_extremal_Kerr_near_horizon.ipynb

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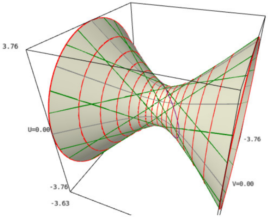
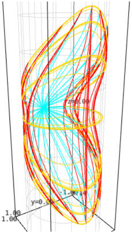
Conclusions

Many explicit computations on spacetimes of interest for the gauge/gravity duality can be performed with SageMath.
More examples at https://sagemanifolds.obspm.fr/examples.html
in particular the **AdS spacetime** example, exploring various coordinate systems, with many 3D plots:
https://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Notebooks/SM_anti_de_Sitter.ipynb





Want to join the SageManifolds project or simply to stay tuned?

visit https://sagemanifolds.obspm.fr/
(download, documentation, example notebooks, mailing list)