Differential geometry with SageMath and applications to gravity

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NEB-19 Recent Developments in Gravity

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Basilis Xanthopoulos and Macsyma

cf. Kostas Kokkotas' talk this morning

Proc. R. Soc. Lond. A **423**, 387–400 (1989)
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Two black holes attached to strings

By Subrahmanyan Chandrasekhar[1], F.R.S.,
and B. C. Xanthopoulos[2]

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(*e*) *The absence of curvature singularities in the space external to the horizons*

The curvature invariant $R^{ab}R_{ab}$ was evaluated with the aid of the symbolic manipulation language MACSYMA with the result,

$$R^{ab}R_{ab} = \frac{64\alpha^{16}q^4(\eta^2-\mu^2)^6 K^2}{(p^2\Delta+q^2\delta)^8\,[(1+p\eta)^2-q^2\mu^2]^8}, \qquad (37)$$

`Macsyma` was initiated in 1968 at MIT and became GPL `Maxima` in 1998

List of software tools for tensor calculus: http://www.xact.es/links.html

Outline

1. A short presentation of SageMath
2. Example 1: Poincaré horizon in AdS spacetime
3. Example 2: Near-horizon geometry of the extremal Kerr black hole
4. Example 3: Rotating quark-gluon plasma in gauge/gravity duality
5. Other examples
6. Conclusions

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SageMath in a few words

SageMath (*nickname:* Sage) is a **free open-source** computer algebra system

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SageMath is free (GPL v2)

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SageMath is based on Python

- no need to learn any specific syntax to use it
- Python is a powerful *object oriented language*, with a neat syntax
- SageMath benefits from the Python ecosystem (e.g. Jupyter notebook, NumPy, Matplotlib)

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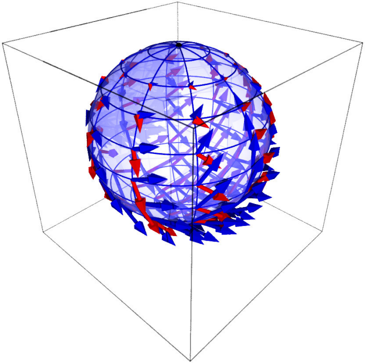
SageMath is developed by an enthusiastic community

- mostly composed of mathematicians
- welcoming newcomers

Differential geometry with SageMath

SageManifolds project: extends `SageMath` towards **differential geometry** and **tensor calculus**



Stereographic-coordinate frame on $\mathbb{S}^2$

- `https://sagemanifolds.obspm.fr`
- more than 110,000 lines of Python code
- fully included in `SageMath` (after review process)
- $\sim$ 25 contributors (developers and reviewers) cf. `https://sagemanifolds.obspm.fr/authors.html`
- dedicated mailing list
- help: `https://ask.sagemath.org`

Everybody is welcome to contribute
$\Longrightarrow$ visit `https://sagemanifolds.obspm.fr/contrib.html`

Current status

Already present (SageMath 9.4):

- differentiable manifolds: tangent spaces, vector frames, tensor fields, curves, pullback and pushforward operators, submanifolds
- vector bundles (tangent bundle, tensor bundles)
- standard tensor calculus (tensor product, contraction, symmetrization, etc.), even on non-parallelizable manifolds, and with all monoterm tensor symmetries taken into account
- Lie derivative along a vector field
- differential forms: exterior and interior products, exterior derivative, Hodge duality
- multivector fields: exterior and interior products, Schouten-Nijenhuis bracket
- affine connections (curvature, torsion)
- pseudo-Riemannian metrics
- computation of geodesics (numerical integration)

Current status

Already present *(cont'd)*:

- some plotting capabilities (charts, points, curves, vector fields)
- parallelization (on tensor components) of CPU demanding computations
- extrinsic geometry of pseudo-Riemannian submanifolds
- series expansions of tensor fields
- 2 symbolic backends: Pynac/Maxima (SageMath's default) and SymPy

Future prospects:

- more symbolic backends (Giac, FriCAS, ...)
- more graphical outputs
- symplectic forms, spinors, integrals on submanifolds, variational calculus, etc.
- connection with numerical relativity: use SageMath to explore numerically-generated spacetimes

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Poincaré horizon in anti-de Sitter spacetime

In AdS spacetime, the ***Poincaré horizon*** is the hypersurface bounding the patch of Poincaré coordinates.

Let us show that it is a degenerate Killing horizon by means of SageMath:

https://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Notebooks/SM_anti_de_Sitter_Poincare_hor.ipynb

(In the nbviewer menu, click on  to run an interactive version on a Binder server)

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Near-horizon geometry of the extremal Kerr black hole

Extremal Kerr black hole: $a = m \iff \kappa = 0$ (degenerate horizon)

Near-horizon geometry of extremal Kerr BH is similar to $\mathrm{AdS}_2 \times \mathbb{S}^2$
$\Longrightarrow$ extended isometry group: $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{U}(1)$

[Carter, Les Houches lecture (1973)] [Bardeen & Horowitz, PRD **60**, 104030 (1999)]

Near-horizon geometry of extremal Kerr black hole is at the basis of the **Kerr/CFT correspondence** (see [Compère, LRR **20**, 1 (2017)] for a review)

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Let us explore this geometry with a SageMath notebook:
https://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Notebooks/SM_extremal_Kerr_near_horizon.ipynb

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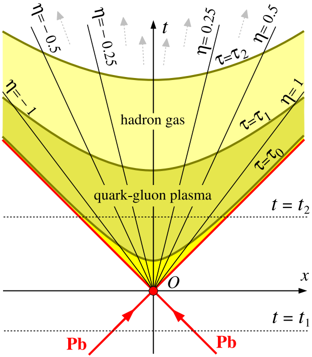
Quark-gluon plasma in gauge/gravity duality



Spacetime diagram of a heavy-ion collision (LHC)

$\tau_0 \simeq 0.2 \text{ fm}/c = 6\ 10^{-25}$ s

$\tau_1 \sim 10\tau_0$

Gauge/gravity duality ("holographic principle")

4d strongly-coupled gauge theory $\equiv$ 5d gravitation

Example: AdS/CFT correspondence

Rotating quark-gluon plasma (QGP) in non-central heavy-ion collisions at RHIC and LHC

Gauge theory: QCD (actually $\mathcal{N} = 4$ supersymmetric Yang-Mills)

Gravity: 5d Kerr-AdS spacetime

Quark-gluon plasma in gauge/gravity duality

An experimental probe of the QGP:
Energy loss of heavy quarks moving through the QGP observed via the phenomenon of jet-quenching

Holographic approach

Drag force of an open test string (Nambu-Goto action) with an endpoint (the heavy quark) attached to the boundary of 5d Kerr-AdS spacetime

Cf. the notebook
https://cocalc.com/share/6850015be0320058cadba1ce63e7a3eedf4eef89/Kerr-AdS-5D-string-b_na.ipynb
(computing the induced metric on the string's 2d worldsheet embedded in 5d Kerr-AdS spacetime; forming the Euler-Lagrange equations and solving them)

[Aref'eva, Golubtsova & Gourgoulhon, JHEP **04(2021)**, 169]

[Golubtsova, Gourgoulhon & Usova, arXiv:2107.11672]

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Other examples

- **Computation of geodesics in Kerr spacetime:**
  https://nbviewer.jupyter.org/github/BlackHolePerturbationToolkit/kerrgeodesic_gw/blob/master/Notebooks/Kerr_geodesics.ipynb
- **Gravitational waves from circular orbits around a Kerr black hole:**
  https://nbviewer.jupyter.org/github/BlackHolePerturbationToolkit/kerrgeodesic_gw/blob/master/Notebooks/grav_waves_circular.ipynb
  Application: Gravitational waves from bodies orbiting the Galactic Center black hole and their detectability by LISA
  [Gourgoulhon, Le Tiec, Vincent & Warburton, A&A **627**, A92 (2019)]

Image of an accretion disk surrounding a Schwarzschild BH

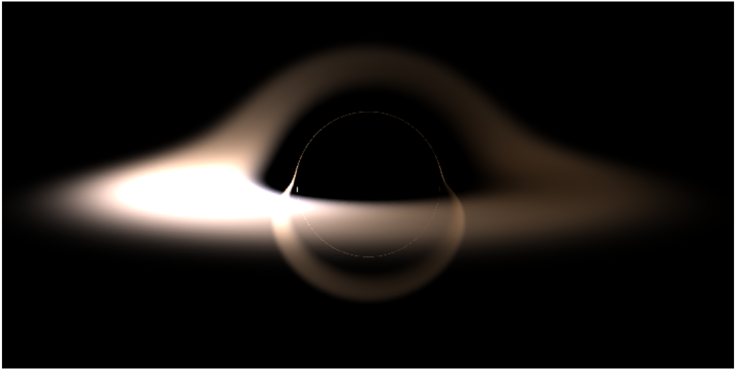

Image computed with SageMath by integrating null geodesics, cf. the notebook
https://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Notebooks/SM_black_hole_rendering.ipynb

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- runs on fully specified smooth manifolds (described by an atlas)

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Many examples available at
https://sagemanifolds.obspm.fr/examples.html

Want to join the SageManifolds project or to simply stay tuned?

visit https://sagemanifolds.obspm.fr/
(download, documentation, example notebooks, mailing list)