General relativity calculus with SageManifolds

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Outline

1. An overview of Sage
2. The SageManifolds project
3. Conclusion and perspectives

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Sage in a few words

- Sage is a **free open-source** mathematics software
- it is based on the Python programming language
- it makes use of **many pre-existing open-sources packages**, among which
  - Maxima (symbolic calculations, since 1967 !)
  - GAP (group theory)
  - PARI/GP (number theory)
  - Singular (polynomial computations)
  - matplotlib (high quality figures)

  and provides a uniform interface to them
- William Stein (Univ. of Washington) created Sage in 2005; since then, ∼**150 developers** have joined the Sage team

The mission

Create a viable free open source alternative to Magma, Maple, Mathematica and Matlab.

Advantages of Sage

Sage is free

Freedom means

1. everybody can use it, by downloading the software from `http://sagemath.org`
2. everybody can examine the source code and improve it

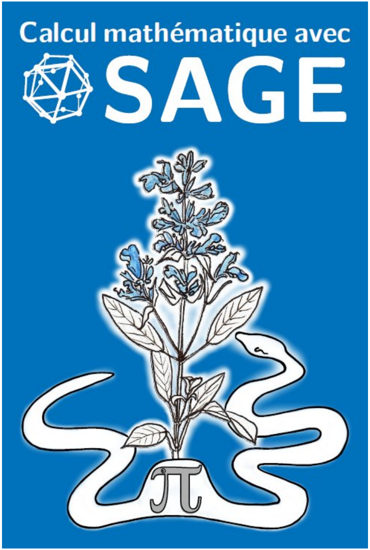
Sage is based on Python

- no need to learn a specific syntax to use it
- easy access for students
- Python is a very powerful object oriented language, with a neat syntax

Sage is developing and spreading fast

...sustained by an important community of developers

The Sage book



by Paul Zimmermann et al. (2013)

Released under *Creative Commons* license:

- freely downloadable from http://sagebook.gforge.inria.fr/
- printed copies can be ordered at moderate price (10 €)

English translation in progress...

A quick overview of Sage

It's time for a demo...

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Existing softwares for differential geometry

Packages for proprietary softwares:

- xAct free package for Mathematica [J.-M. Martin-Garcia]
- Ricci free package for Mathematica [J. L. Lee]
- MathTensor package for Mathematica [S. M. Christensen & L. Parker]
- DifferentialGeometry included in Maple [I. M. Anderson & E. S. Cheb-Terrab]
- Atlas 2 for Maple and Mathematica
- ...

Standalone softwares:

- SHEEP, Classi, STensor, based on Lisp, developed in 1970's and 1980's (free) [R. d'Inverno, I. Frick, J. Åman, J. Skea, et al.]
- Cadabra field theory (free) [K. Peeters]
- SnapPy topology and geometry of 3-manifolds, based on Python (free) [M. Culler, N. M. Dunfield & J. R. Weeks]
- ...

cf. pretty complete list on `http://www.xact.es/links.html`

Existing softwares for differential geometry

Two types of **tensor computations**:

Abstract computations

- xAct/xTensor
- MathTensor
- Ricci
- Cadabra

Component computations

- xAct/xCoba
- Atlas 2
- DifferentialGeometry
- SageManifolds

The situation in Sage

Sage is well developed in many domains of mathematics: number theory, group theory, linear algebra, etc.

...but not too much in the area of differential geometry:

Already in Sage

- differential forms on an open subset of Euclidean space *(with a fixed set of coordinates)* (J. Vankerschaver)

Proposed extensions (Trac)

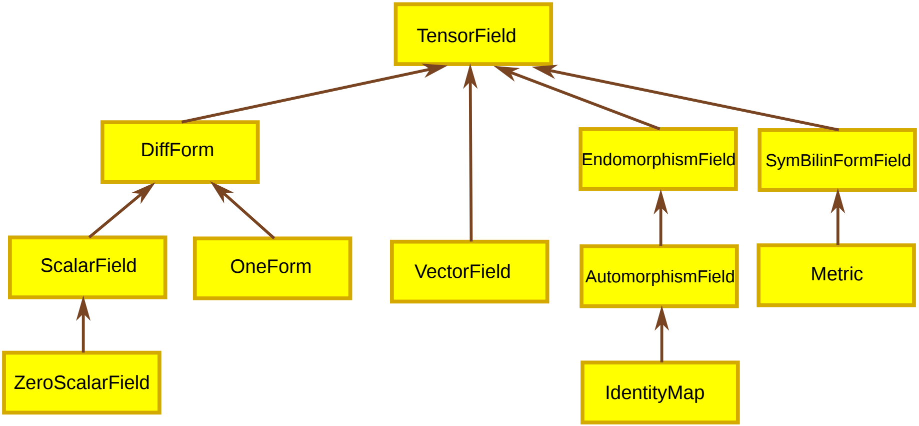
- parametrized 2-surfaces in 3-dim. Euclidean space (M. Malakhaltsev, J. Vankerschaver, V. Delecroix, trac #10132)
- 2-D hyperbolic geometry (V. Delecroix, M. Raum, trac #9439)

SageManifolds

A new set of *Python classes* implementing differential geometry in Sage:

- `Manifold`: differentiable manifolds over $\mathbb{R}$, of arbitrary dimension
- `SubManifold`: submanifolds
- `Point`: points on a manifold
- `Chart`: charts
- `DiffMapping`, `Diffeomorphism`: differential mappings between manifolds
- `ScalarField`: differential mappings to $\mathbb{R}$
- `TensorField`, `VectorField`, `SymBilinFormField`, etc.: tensor fields on a manifold
- `DiffForm`, `OneForm`: p-forms
- `VectorFrame`, `CoordBasis`: vector frames on a manifold, including tetrads and coordinate bases
- `Components`, `CompWithSym`, etc.: components of a tensor field in a given vector frame
- `AffConnection`, `LeviCivitaConnection`: affine connections
- `Metric`, `RiemannMetric`, `LorentzMetric`: pseudo-Riemannian metrics

Inheritance diagram of the tensor field classes



Basic SageManifolds objects are coordinate-free

1. The scalar field case

Scalar field on the manifold $\mathcal{M}$: (differentiable) mapping $f : U \subset \mathcal{M} \to \mathbb{R}$

A scalar field maps *points*, not *coordinates*, to real numbers
$\Longrightarrow$ an object f in the `ScalarField` class has different *coordinate representations* in different charts defined on $\mathcal{M}$.

The various coordinate representations are stored as a *Python dictionary* whose keys are the charts:

$$f.\mathrm{express} = \left\{ C : F,\ \hat{C} : \hat{F}, \ldots \right\}$$

with $$f(\underbrace{\,p\,}_{\text{point}}) = F(\underbrace{x^1, \ldots, x^n}_{\substack{\text{coord. of } p \\ \text{in chart } C}}) = \hat{F}(\underbrace{\hat{x}^1, \ldots, \hat{x}^n}_{\substack{\text{coord. of } p \\ \text{in chart } \hat{C}}}) = \ldots$$

Basic SageManifolds objects are coordinate-free

2. The tensor field case

Given a vector frame $(\boldsymbol{e}_i)$ with dual coframe $(\boldsymbol{e}^i)$, the components of a tensor field $\boldsymbol{T}$ in this frame are *scalar fields*, since

$$T^{i\ldots}_{\ \ j\ldots} = \boldsymbol{T}(\boldsymbol{e}^i, \ldots, \boldsymbol{e}_j, \ldots)$$

$\Longrightarrow$ an object $\boldsymbol{T}$ in the `TensorField` class has different sets of components $T^{i\ldots}_{\ \ j\ldots}$ in different vector frames, each component in a given set being an object of the `ScalarField` class

The various sets of components are stored as a *Python dictionary* whose keys are the vector frames:

$$\boldsymbol{T}.\mathrm{components} = \left\{ (\boldsymbol{e}) : (T^{i\ldots}_{\ \ j\ldots}),\ (\hat{\boldsymbol{e}}) : (\hat{T}^{i\ldots}_{\ \ j\ldots}), \ldots \right\}$$

Tensor field implementation

`TensorField`

$$T$$

$$T = T^a{}_b \, e_a \otimes e^b = T^{\hat{a}}{}_{\hat{b}} \, \varepsilon_{\hat{a}} \otimes \varepsilon^{\hat{b}} = \cdots$$

Dictionary `TensorField.components`

frame 1: (e_a) — `Components` $\left(T^a{}_b\right)_{1\le a,b\le n}$

frame 2: (ε_a) — `Components` $\left(T^{\hat{a}}{}_{\hat{b}}\right)_{1\le a,b\le n}$

...

Dictionary `Components._comp`

(1,1) : `ScalarField` $T^1{}_1$

(1,2) : `ScalarField` $T^1{}_2$

...

Dictionary `ScalarField.express`

chart 1: (x^a) — `FunctionChart` $T^1{}_1(x^1, \ldots, x^n)$

chart 2: (y^a) — `FunctionChart` $T^1{}_1(y^1, \ldots, y^n)$

...

`Expression` $x^1 \cos x^2$

`Expression` $(y^1 + y^2)\cos(y^1 - y^2)$

SageManifolds at work: the Kerr-Newman example

1. Checking Maxwell equations

SM_Kerr_Newman -- Sage - Mozilla Firefox

SageManifolds: examples | SM_Kerr_Newman – Sage

localhost:8080/home/admin/80/

```
dX = M.coframe('BL_b') ; dX
```

$$(dt, dr, d\theta, d\phi)$$

The electromagnetic field tensor F is formed as [cf. e.g. Eq. (33.5) of Misner, Thorne & Wheeler (1973)]

```
F = q/rho2^2 * (r^2-a^2*cos(th)^2)* dX(1).wedge( dX(0) - a*sin(th)^2* dX(3) ) + \
    2*q/rho2^2 * a*r*cos(th)*sin(th)* dX(2).wedge( (r^2+a^2)* dX(3) - a* dX(0) )
```

```
F.set_name('F') ; F.show()
```

$$F = \left(\frac{a^2 q\cos(\theta)^2 - qr^2}{a^4\cos(\theta)^4 + 2a^2r^2\cos(\theta)^2 + r^4}\right) dt\wedge dr + \left(\frac{2a^2qr\sin(\theta)\cos(\theta)}{a^4\cos(\theta)^4 + 2a^2r^2\cos(\theta)^2 + r^4}\right) dt\wedge d\theta + \left(\frac{\left(a^3q\cos(\theta)^2 - aqr^2\right)\sin(\theta)^2}{a^4\cos(\theta)^4 + 2a^2r^2\cos(\theta)^2 + r^4}\right) dr\wedge d\phi + \left(\frac{2\left(a^3qr+aqr\right.}{a^4\cos(\theta)^4+2}\right.$$

The Hodge dual of F:

```
star_F = F.hodge_star(g) ; star_F.show()
```

$$\star F = \left(\frac{2aqr\cos(\theta)}{a^4\cos(\theta)^4 + 2a^2r^2\cos(\theta)^2 + r^4}\right) dt\wedge dr + \left(-\frac{\left(a^3q\cos(\theta)^2 - aqr^2\right)\sin(\theta)}{a^4\cos(\theta)^4 + 2a^2r^2\cos(\theta)^2 + r^4}\right) dt\wedge d\theta + \left(-\frac{2\left(a^4qr\sin(\theta)^4\cos(\theta) - \left(a^4qr + a^2qr^3\right)\sin(\theta)^2\cos(\theta)\right)}{a^6\cos(\theta)^6 + 3a^4r^2\cos(\theta)^4 + 3a^2r^4\cos(\theta)^2 + r^6}\right)$$

Maxwell equations

Let us check that F obeys the two (source-free) Maxwell equations:

```
xder(F).show()
```

$$dF = 0$$

```
xder(star_F).show()
```

$$d\star F = 0$$

SageManifolds at work: the Kerr-Newman example

2. Checking Einstein equations

Copy of SM_Kerr_Newman -- Sage - Mozilla Firefox

SageManifolds: examples | Copy of SM_Kerr_Newman -...

localhost:8080/home/admin/86/

The Einstein tensor is

```
G = g.ricci() - 1/2*g.ricci_scalar()*g ; print G
```

```
field of symmetric bilinear forms '+Ric(g)' on the 4-dimensional
manifold 'M'
```

Since the Ricci scalar is zero, the Einstein tensor reduces to the Ricci tensor:

```
G == g.ricci()
```

True

The invariant $F_{\mu\nu}F^{\mu\nu}$ of the electromagnetic field:

```
F2 = F.contract(0, F.up(g), 0).self_contract(0, 1) ; print F2
```

```
scalar field on the 4-dimensional manifold 'M'
```

```
F2.show()
```

$$(t, r, \theta, \phi) \mapsto -\frac{2\left(a^4 q^2 \cos(\theta)^4 - 6\,a^2 q^2 r^2 \cos(\theta)^2 + q^2 r^4\right)}{a^8 \cos(\theta)^8 + 4\,a^6 r^2 \cos(\theta)^6 + 6\,a^4 r^4 \cos(\theta)^4 + 4\,a^2 r^6 \cos(\theta)^2 + r^8}$$

The energy-momentum tensor of the electromagnetic field:

```
T = 1/(4*pi)*( F.contract(0, F.up(g, 0), 0) - 1/4*F2 * g ); print T
```

```
tensor field of type (0,2) on the 4-dimensional manifold 'M'
```

Check of the Einstein equation:

```
G == 8*pi*T
```

True

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Conclusion and perspectives

- **SageManifolds** is a work in progress
  $\sim$ 20,000 lines of Python code up to now (including comments and doctests)
- A preliminary version (v0.3) is freely available (GPL) as a Sage package at `http://sagemanifolds.obspm.fr/`
- The current development version is available on the Git repository `https://gitroc.obspm.fr/gitweb/SageManifolds.git`
- *Already present:*
  - maps between manifolds, pullback operator
  - submanifolds, pushforward operator
  - standard tensor calculus (tensor product, contraction, symmetrization, etc.)
  - exterior calculus, Hodge duality
  - Lie derivatives
  - affine connections, curvature, torsion
  - pseudo-Riemannian metrics, Weyl tensor

Conclusion and perspectives

- *Not implemented yet (but should be soon):*
    - extrinsic geometry of pseudo-Riemannian submanifolds
    - computation of geodesics (numerical integration via Sage/GSL or `Gyoto`)
    - integrals on submanifolds
- *To do:*
    - improve the integration into Sage
    - merge other diff. geom. projects into SageManifolds, such as the parametrized 2-surfaces developed by M. Malakhaltsev, J. Vankerschaver & V. Delecroix (trac #10132)
    - graphical outputs
    - add more functionalities: symplectic forms, fibre bundles, spinors, variational calculus, etc.
    - connection with `Lorene`, `CoCoNuT`, ...

Want to join the project or simply to stay tuned?

visit `http://sagemanifolds.obspm.fr/`
(access to Git repository, documentation, example worksheets, mailing list)