Differential geometry with SageMath

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based on a collaboration with

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Outline

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- In 1965, J.G. Fletcher developed the `GEOM` program, to compute the Riemann tensor of a given metric
- In 1969, during his PhD under Pirani supervision, Ray d'Inverno wrote `ALAM (Atlas Lisp Algebraic Manipulator)` and used it to compute the Riemann tensor of Bondi metric. The original calculations took Bondi and his collaborators 6 months to go. The computation with ALAM took 4 minutes and yielded to the discovery of 6 errors in the original paper [J.E.F. Skea, *Applications of SHEEP* (1994)]

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- Since then, many software tools for tensor calculus have been developed... A rather exhaustive list: http://www.xact.es/links.html
  $\Longrightarrow$ cf. Maximilian Hasler's review talk on Friday.

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The mission

Create a viable free open source alternative to Magma, Maple, Mathematica and Matlab.

Some advantages of SageMath

SageMath is free

Freedom means

1. everybody can use it, by downloading the software from `http://sagemath.org`
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- no need to learn any specific syntax to use it
- easy access for students
- Python is a very powerful *object oriented language*, with a neat syntax

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SageMath is developing and spreading fast

...sustained by an enthusiastic community of developers

Object-oriented notation in Python

As an object-oriented language, Python (and hence SageMath) makes use of the following **postfix notation** (same in C++, Java, etc.):

$$\texttt{result} = \texttt{object.function(arguments)}$$

In a procedural language, this would be written as

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```

Examples

1. `riem = g.riemann()`
2. `lie_t_v = t.lie_der(v)`

NB: no argument in example 1

SageMath approach to computer mathematics

SageMath relies on a Parent / Element scheme: each object x on which some calculus is performed has a "parent", which is another SageMath object X representing the set to which x belongs.

The calculus rules on x are determined by the *algebraic structure* of X.

Conversion rules prior to an operation, e.g. $x + y$ with x and y having different parents, are defined at the level of the parents

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Example

```
sage: x = 4 ; x.parent()
Integer Ring
sage: y = 4/3 ; y.parent()
Rational Field
sage: s = x + y ; s.parent()
Rational Field
sage: y.parent().has_coerce_map_from(x.parent())
True
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This approach is similar to that of Magma and is different from that of Mathematica, in which everything is a tree of symbols

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The SageManifolds project

http://sagemanifolds.obspm.fr/

Aim

Implement smooth manifolds of arbitrary dimension in SageMath and tensor calculus on them

In particular:

- one should be able to introduce an arbitrary number of coordinate charts on a given manifold, with the relevant transition maps
- tensor fields must be manipulated as such and not through their components with respect to a specific (possibly coordinate) vector frame

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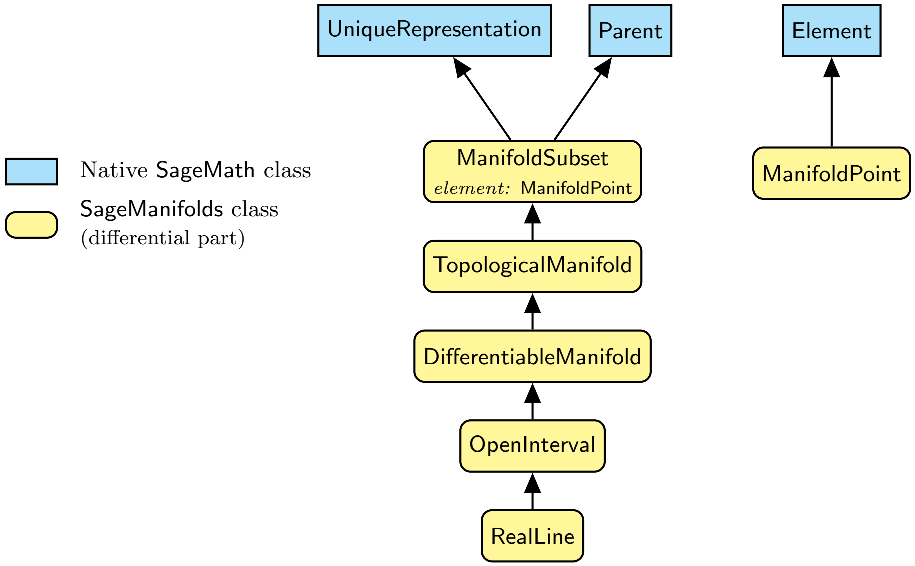
Concretely, the project amounts to creating new Python classes, such as `TopologicalManifold`, `DifferentiableManifold`, `Chart`, `TensorField` or `Metric`, within SageMath's **Parent/Element framework**.

Implementing manifolds and their subsets



Implementing coordinate charts

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In general, more than one chart is required to cover the entire manifold:

Examples:

- at least 2 charts are necessary to cover the n-dimensional sphere $\mathbb{S}^n$ $(n \geq 1)$ and the torus $\mathbb{T}^2$
- at least 3 charts are necessary to cover the real projective plane $\mathbb{RP}^2$

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In SageManifolds, an arbitrary number of charts can be introduced

To fully specify the manifold, one shall also provide the *transition maps* on overlapping chart domains (SageManifolds class `CoordChange`)

Implementing scalar fields

A **scalar field** on manifold M is a smooth mapping

$$\begin{array}{llcl} f: & U \subset M & \longrightarrow & \mathbb{R} \\ & p & \longmapsto & f(p) \end{array}$$

where U is an open subset of M.

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The various coordinate representations F, $\hat{F}$, ... of f are stored as a *Python dictionary* whose keys are the charts C, $\hat{C}$, ...:

$$f.\texttt{\_express} = \left\{ C : F, \ \hat{C} : \hat{F}, \ldots \right\}$$

with $\underbrace{f(\ p\ )}_{\text{point}} = F(\underbrace{x^1, \ldots, x^n}_{\substack{\text{coord. of } p \\ \text{in chart } C}}) = \hat{F}(\underbrace{\hat{x}^1, \ldots, \hat{x}^n}_{\substack{\text{coord. of } p \\ \text{in chart } \hat{C}}}) = \ldots$

The scalar field algebra

The parent of the scalar field $f : U \to \mathbb{R}$ is the set $C^\infty(U)$ of scalar fields defined on the open subset U.

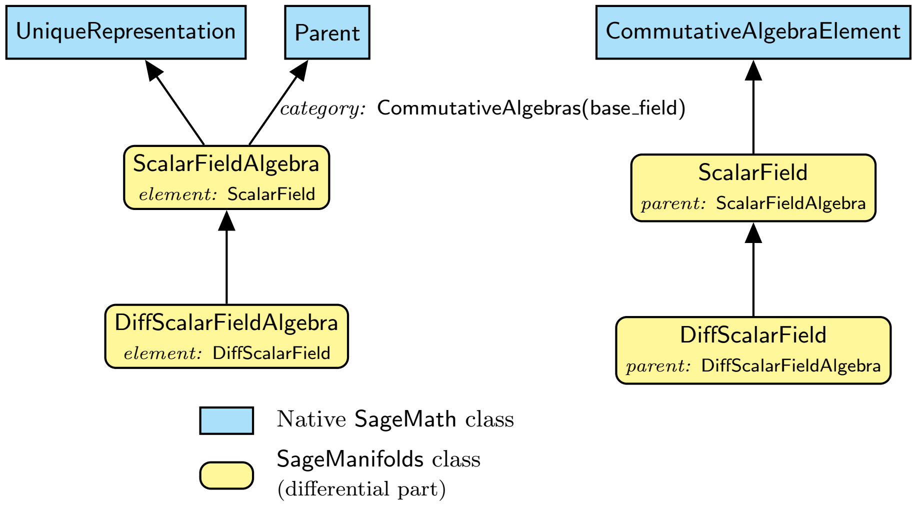
$C^\infty(U)$ has naturally the structure of a **commutative algebra over** $\mathbb{R}$:

1. it is clearly a vector space over $\mathbb{R}$
2. it is endowed with a commutative ring structure by pointwise multiplication:

$$\forall f, g \in C^\infty(U), \quad \forall p \in U, \quad (f.g)(p) := f(p)g(p)$$

The algebra $C^\infty(U)$ is implemented in SageManifolds via the class `ScalarFieldAlgebra`.

Classes for scalar fields



Vector field modules

Given an open subset $U \subset M$, the set $\mathcal{X}(U)$ of smooth vector fields defined on U has naturally the structure of a **module over the scalar field algebra** $C^\infty(U)$.

$\mathcal{X}(U)$ is a free module $\iff$ U admits a global vector frame $(\boldsymbol{e}_a)_{1\leq a\leq n}$:

$$\forall \boldsymbol{v} \in \mathcal{X}(U), \quad \boldsymbol{v} = v^a \boldsymbol{e}_a, \quad \text{with } v^a \in C^\infty(U)$$

At any point $p \in U$, the above translates into an identity in the *tangent vector space* T_pM:

$$\boldsymbol{v}(p) = v^a(p)\, \boldsymbol{e}_a(p), \quad \text{with } v^a(p) \in \mathbb{R}$$

Example:

If U is the domain of a coordinate chart $(x^a)_{1\leq a\leq n}$, $\mathcal{X}(U)$ is a free module of rank n over $C^\infty(U)$, a basis of it being the coordinate frame $(\partial/\partial x^a)_{1\leq a\leq n}$.

Parallelizable manifolds

M is a **parallelizable manifold** $\iff$ M admits a global vector frame
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$\iff$ M's tangent bundle is trivial: $TM \simeq M \times \mathbb{R}^n$

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Examples of parallelizable manifolds

- $\mathbb{R}^n$ (global coordinate charts $\Rightarrow$ global vector frames)
- the circle $\mathbb{S}^1$ (NB: no global coordinate chart)
- the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$
- the 3-sphere $\mathbb{S}^3 \simeq \mathrm{SU}(2)$, as any Lie group
- the 7-sphere $\mathbb{S}^7$
- any orientable 3-manifold (Steenrod theorem)

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Examples of non-parallelizable manifolds

- the sphere $\mathbb{S}^2$ (hairy ball theorem!) and any n-sphere $\mathbb{S}^n$ with $n \notin \{1, 3, 7\}$
- the real projective plane $\mathbb{RP}^2$

Implementing vector fields

Ultimately, in SageManifolds, vector fields are to be described by their components w.r.t. various vector frames.

If the manifold M is not parallelizable, we assume that it can be covered by a finite number N of parallelizable open subsets U_i $(1 \leq i \leq N)$ (OK for M compact). We then consider **restrictions** of vector fields to these domains:

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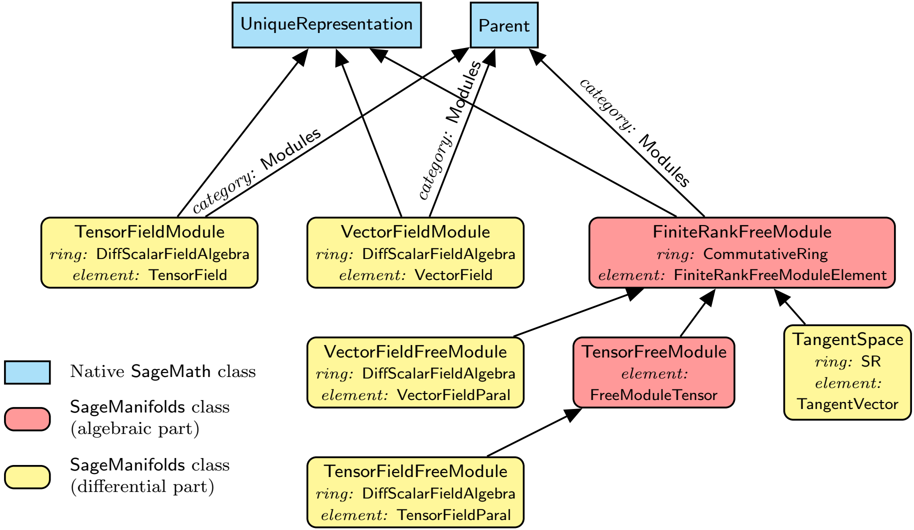
For each i, $\mathcal{X}(U_i)$ is a free module of rank $n = \dim M$ and is implemented in SageManifolds as an instance of `VectorFieldFreeModule`, which is a subclass of `FiniteRankFreeModule`.

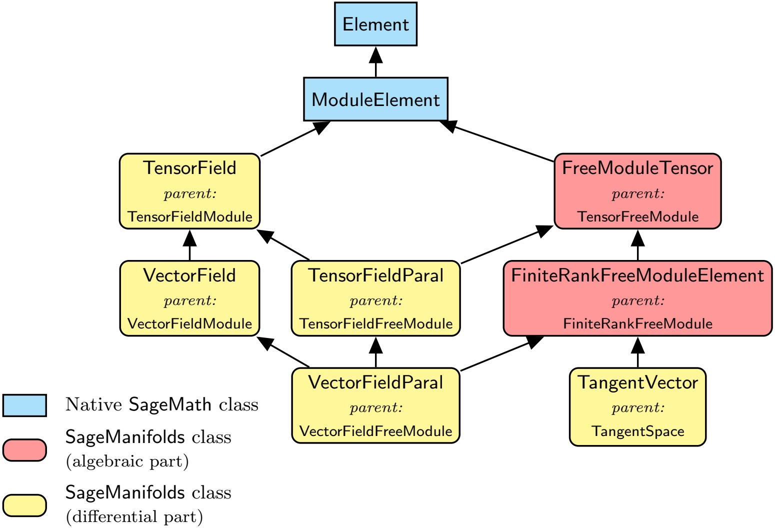
Each vector field $\boldsymbol{v} \in \mathcal{X}(U_i)$ has different set of components $(v^a)_{1 \leq a \leq n}$ in different vector frames $(\boldsymbol{e}_a)_{1 \leq a \leq n}$ introduced on U_i. They are stored as a *Python dictionary* whose keys are the vector frames:

$$\boldsymbol{v}.\texttt{\_components} = \{(\boldsymbol{e}) : (v^a),\ (\hat{\boldsymbol{e}}) : (\hat{v}^a), \ldots\}$$

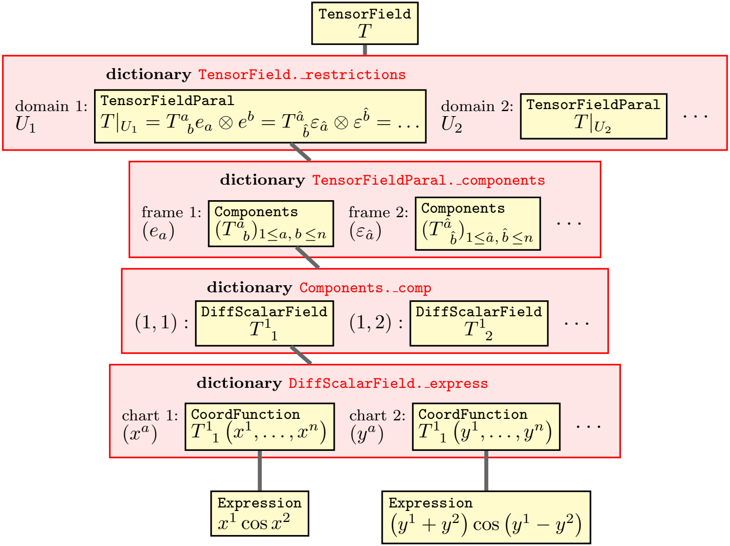
Module classes in SageManifolds



Tensor field classes

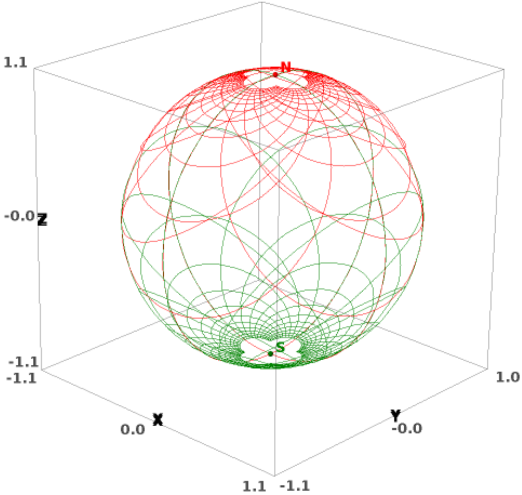


Tensor field storage



Outline

The 2-sphere example



Stereographic coordinates on the 2-sphere

Two charts:

- X_1: $\mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$
- X_2: $\mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^2$

← picture obtained via function `RealChart.plot()`

See the worksheet at `http://sagemanifolds.obspm.fr/examples.html`

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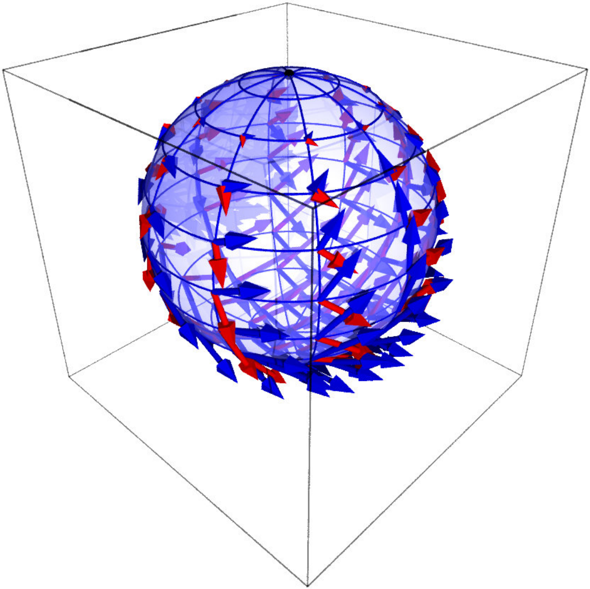

Vector frame associated with the stereographic coordinates (x, y) from the North pole

- $\frac{\partial}{\partial x}$
- $\frac{\partial}{\partial y}$

← picture obtained via the function
`VectorField.plot()`

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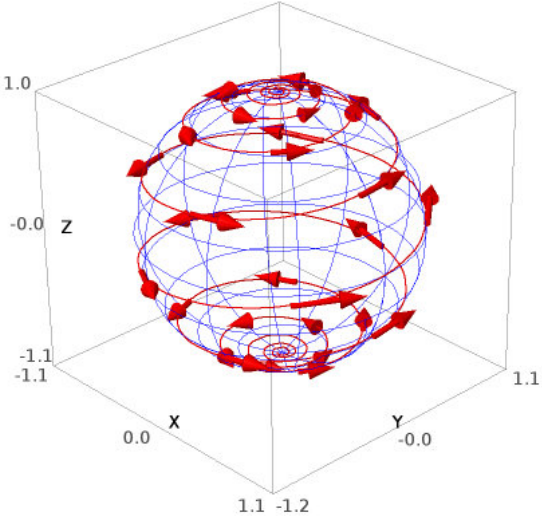
The 2-sphere example



A curve in $\mathbb{S}^2$: a loxodrome and its tangent vector field

← picture obtained via the functions `DifferentiableCurve.plot()` and `VectorField.plot()`

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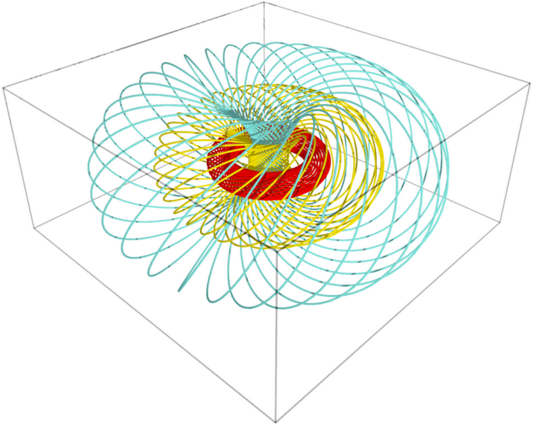
The 3-sphere example

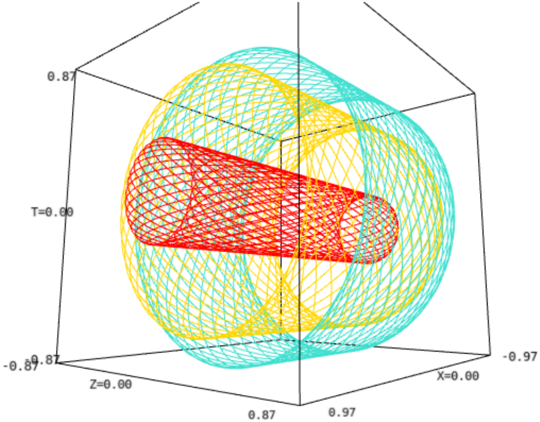


Some fibers of the **Hopf fibration** of $\mathbb{S}^3$ viewed in stereographic coordinates

← picture obtained via the function `DifferentiableCurve.plot()`

See the worksheet at http://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Worksheets/v1.0/SM_sphere_S3_Hopf.ipynb

The 3-sphere example



The same fibers but viewed in the Cartesian coordinates (T, X, Y) of $\mathbb{R}^4$ via the canonical embedding $\mathbb{S}^3 \to \mathbb{R}^4$

← picture obtained via the function DifferentiableCurve.plot()

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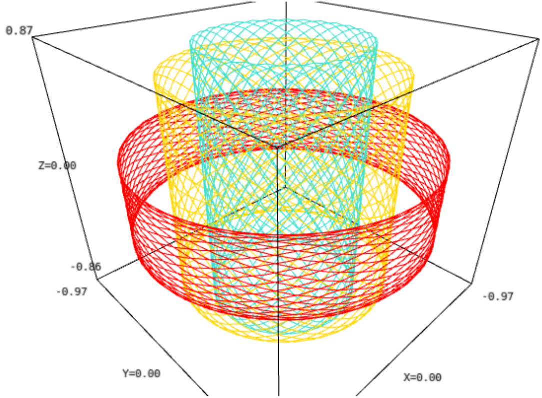

Again the same fibers but viewed in the Cartesian coordinates (X, Y, Z) of $\mathbb{R}^4$ via the canonical embedding $\mathbb{S}^3 \to \mathbb{R}^4$

$\leftarrow$ picture obtained via the function `DifferentiableCurve.plot()`

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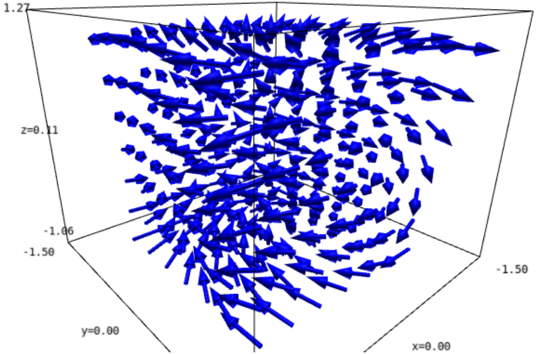
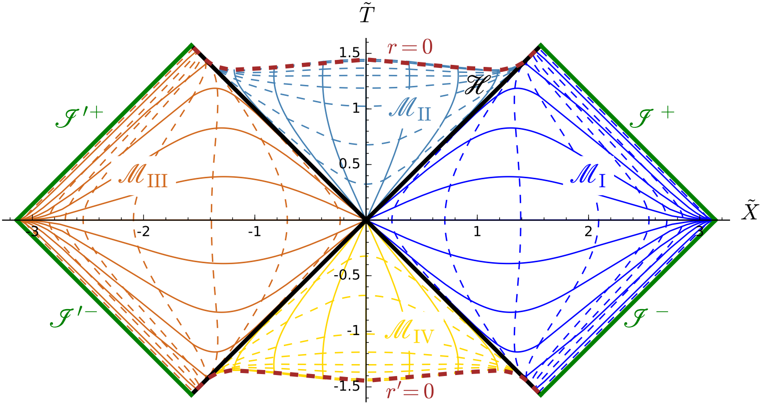


One of the vector fields of a left-invariant global vector frame of $\mathbb{S}^3$, viewed in stereographic coordinates

← picture obtained via the function `VectorField.plot()`

See the worksheet at http://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Worksheets/v1.0/SM_sphere_S3_vectors.ipynb

Charts on Schwarzschild spacetime
The Carter-Penrose diagram



Two charts of standard Schwarzschild-Droste coordinates (t, r, θ, φ) plotted in terms of Frolov-Novikov compactified coordinates $(\tilde{T}, \tilde{X}, \theta, \varphi)$; see the worksheet at http://luth.obspm.fr/~luthier/gourgoulhon/bh16/sage.html

Outline

Summary

SageManifolds: extends the modern computer algebra system SageMath towards differential geometry and tensor calculus

- http://sagemanifolds.obspm.fr/
- free software (GPL), as SageMath
- $\sim$ 65,000 lines of Python code (including comments and doctests)
- submitted to SageMath community as a sequence of 14 tickets
  $\rightarrow$ first ticket accepted in March 2015,
  the 14th one in Nov. 2016
- 5 developers, 3 reviewers

SageManifolds 1.0 released on 11 Jan. 2017 and fully included in SageMath 7.5

Current status

Already present (v1.0):

- topological manifolds: charts, open subsets, maps between manifolds, scalar fields
- differentiable manifolds: tangent spaces, vector frames, tensor fields, curves, pullback and pushforward operators
- standard tensor calculus (tensor product, contraction, symmetrization, etc.), even on non-parallelizable manifolds
- taking into account any monoterm tensor symmetry
- exterior calculus (wedge product, exterior derivative, Hodge duality)
- Lie derivatives of tensor fields
- affine connections (curvature, torsion)
- pseudo-Riemannian metrics
- some plotting capabilities (charts, points, curves, vector fields)
- parallelization (on tensor components) of CPU demanding computations, via the Python library `multiprocessing`

Current status

Future prospects:

- extrinsic geometry of pseudo-Riemannian submanifolds
- computation of geodesics (numerical integration via SageMath/GSL or `Gyoto`)
- integrals on submanifolds
- more graphical outputs
- more functionalities: symplectic forms, fibre bundles, spinors, variational calculus, etc.
- connection with numerical relativity: using SageMath to explore numerically-generated spacetimes

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Want to join the project or simply to stay tuned?

visit http://sagemanifolds.obspm.fr/
(download, documentation, example worksheets, mailing list)