SageManifolds

Éric Gourgoulhon

Laboratoire Univers et Théories (LUTH)
CNRS / Observatoire de Paris / Université Paris Diderot
92190 Meudon, France

http://luth.obspm.fr/~luthier/gourgoulhon/

based on a collaboration with
Michał Bejger, Marco Mancini, Travis Scrimshaw

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Outline

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Differential geometry in Sage

Sage is well developed in many domains of mathematics but not too much in the area of differential geometry:

Already in Sage

- differential forms on an open subset of Euclidean space *(with a fixed set of coordinates)* (J. Vankerschaver)
- parametrized 2-surfaces in 3-dim. Euclidean space (M. Malakhaltsev, J. Vankerschaver, V. Delecroix)

On Trac

- 2-D hyperbolic geometry (V. Delecroix, M. Raum, G. Laun, trac ticket #9439)

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The SageManifolds project

http://sagemanifolds.obspm.fr/

Aim

Implement real smooth manifolds of arbitrary dimension in Sage and tensor calculus on them, in a coordinate/frame-independent manner

In particular:

- one should be able to introduce an arbitrary number of coordinate charts on a given manifold, with the relevant transition maps
- tensor fields must be manipulated as such and not through their components with respect to a specific (possibly coordinate) vector frame

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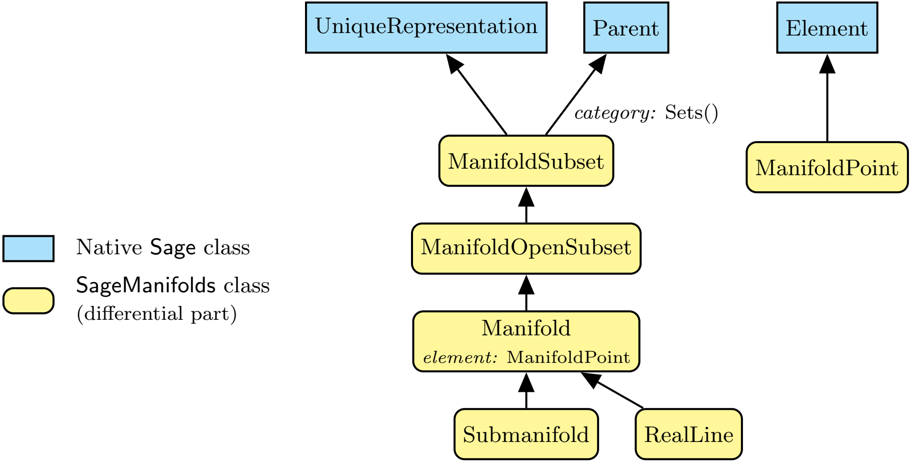
Concretely, the project amounts to creating new Python classes, such as `Manifold`, `Chart`, `TensorField` or `Metric`, within Sage's **Parent/Element framework**.

Implementating manifolds and their subsets



Implementing coordinate charts

Given a (topological) manifold M of dimension $n \geq 1$, a coordinate chart $\varphi : U \to \mathbb{R}^n$ on an open subset $U \subset M$ is implemented in SageManifolds via the class `Chart`, whose main data is U and a n-tuple of *Sage symbolic variables* `x`, `y`, ..., each of them representing a coordinate

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In general, more than one (regular) chart is required to cover the entire manifold:

Examples:

- at least 2 charts are necessary to cover the n-dimensional sphere $\mathbb{S}^n$ $(n \geq 1)$ and the torus $\mathbb{T}^2$
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In SageManifolds, an arbitrary number of charts can be introduced

To fully specify the manifold, one shall also provide the *transition maps* on overlapping chart domains (SageManifolds class `CoordChange`)

Implementing scalar fields

A **scalar field** on manifold M is a smooth mapping

$$\begin{array}{llcl} f: & U \subset M & \longrightarrow & \mathbb{R} \\ & p & \longmapsto & f(p) \end{array}$$

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The various coordinate representations F, $\hat{F}$, ... of f are stored as a *Python dictionary* whose keys are the charts C, $\hat{C}$, ...:

$$f.\_\mathtt{express} = \left\{ C : F,\ \hat{C} : \hat{F}, \ldots \right\}$$

with $f(\underbrace{p}_{\text{point}}) = F(\underbrace{x^1, \ldots, x^n}_{\substack{\text{coord. of } p \\ \text{in chart } C}}) = \hat{F}(\underbrace{\hat{x}^1, \ldots, \hat{x}^n}_{\substack{\text{coord. of } p \\ \text{in chart } \hat{C}}}) = \ldots$

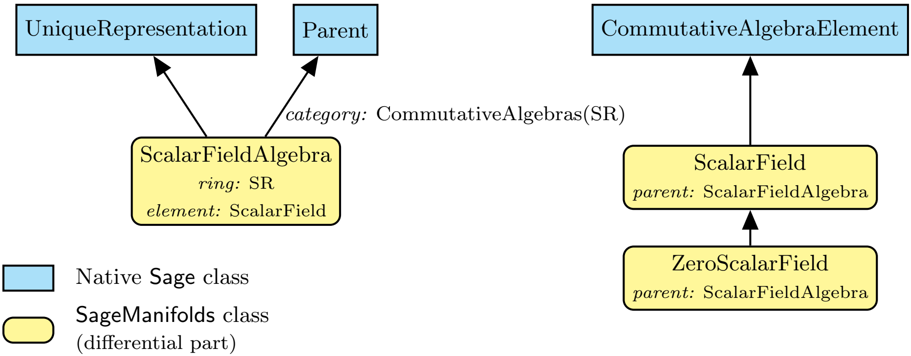
The scalar field algebra

Given an open subset $U \subset M$, the set $C^\infty(U)$ of scalar fields defined on U has naturally the structure of a **commutative algebra over** $\mathbb{R}$: it is clearly a vector space over $\mathbb{R}$ and it is endowed with a commutative ring structure by pointwise multiplication:

$$\forall f, g \in C^\infty(U), \quad \forall p \in U, \quad (f.g)(p) := f(p)g(p)$$

The algebra $C^\infty(U)$ is implemented in SageManifolds via the class `ScalarFieldAlgebra`.

Classes for scalar fields



Vector field modules

Given an open subset $U \subset M$, the set $\mathcal{X}(U)$ of smooth vector fields defined on U has naturally the structure of a **module over the scalar field algebra** $C^\infty(U)$.

$\mathcal{X}(U)$ is a free module $\iff$ U admits a global vector frame $(\boldsymbol{e}_a)_{1\leq a\leq n}$:

$$\forall \boldsymbol{v} \in \mathcal{X}(U), \quad \boldsymbol{v} = v^a \boldsymbol{e}_a, \quad \text{with } v^a \in C^\infty(U)$$

At a point $p \in U$, the above translates into an identity in the *tangent vector space* T_pM:

$$\boldsymbol{v}(p) = v^a(p)\, \boldsymbol{e}_a(p), \quad \text{with } v^a(p) \in \mathbb{R}$$

Example:

If U is the domain of a coordinate chart $(x^a)_{1\leq a\leq n}$, $\mathcal{X}(U)$ is a free module of rank n over $C^\infty(U)$, a basis of it being the coordinate frame $(\partial/\partial x^a)_{1\leq a\leq n}$.

Parallelizable manifolds

M is a **parallelizable manifold** $\iff$ M admits a global vector frame
$\iff$ $\mathcal{X}(M)$ is a free module
$\iff$ M's tangent bundle is trivial: $TM \simeq M \times \mathbb{R}^n$

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Examples of parallelizable manifolds

- $\mathbb{R}^n$ (global coordinate charts $\Rightarrow$ global vector frames)
- the circle $\mathbb{S}^1$ (NB: no global coordinate chart)
- the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$
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- the 7-sphere $\mathbb{S}^7$
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Examples of non-parallelizable manifolds

- the sphere $\mathbb{S}^2$ (hairy ball theorem!) and any n-sphere $\mathbb{S}^n$ with $n \notin \{1, 3, 7\}$
- the real projective plane $\mathbb{RP}^2$

Implementing vector fields

Ultimately, in SageManifolds, vector fields are to be described by their components w.r.t. various vector frames.

If the manifold M is not parallelizable, we assume that it can be covered by a finite number N of parallelizable open subsets U_i $(1 \leq i \leq N)$ (OK for M compact). We then consider **restrictions** of vector fields to these domains:

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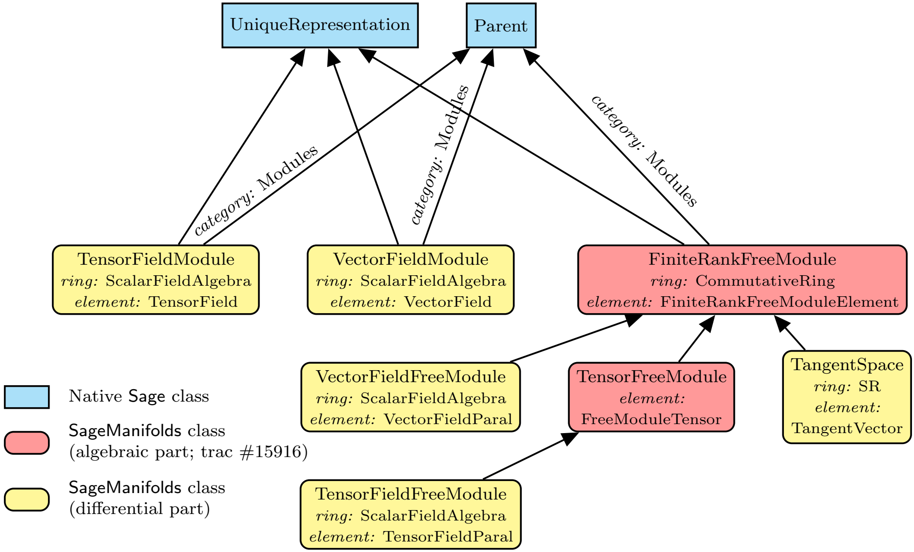
For each i, $\mathcal{X}(U_i)$ is a free module of rank $n = \dim M$ and is implemented in SageManifolds as an instance of `VectorFieldFreeModule`, which is a subclass of `FiniteRankFreeModule`.

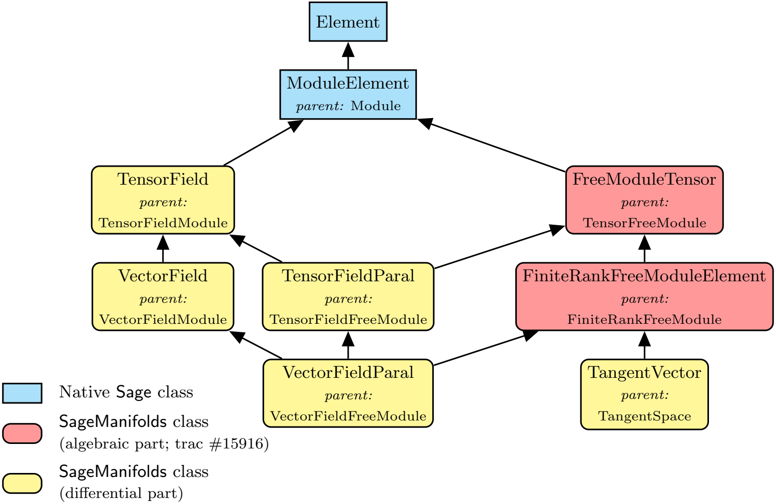
Each vector field $\boldsymbol{v} \in \mathcal{X}(U_i)$ has different set of components $(v^a)_{1\leq a\leq n}$ in different vector frames $(\boldsymbol{e}_a)_{1\leq a\leq n}$ introduced on U_i. They are stored as a *Python dictionary* whose keys are the vector frames:

$$\boldsymbol{v}.\texttt{\_components} = \{(\boldsymbol{e}) : (v^a),\ (\hat{\boldsymbol{e}}) : (\hat{v}^a), \ldots\}$$

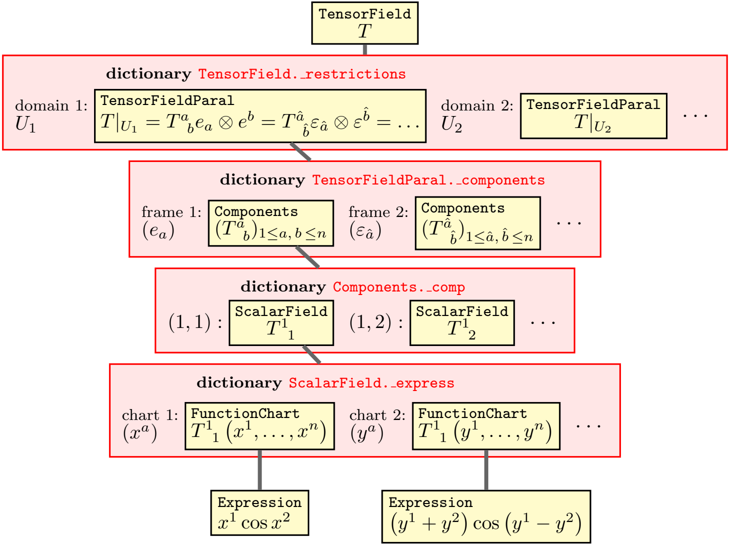
Module classes in SageManifolds



Tensor field classes



Tensor field storage



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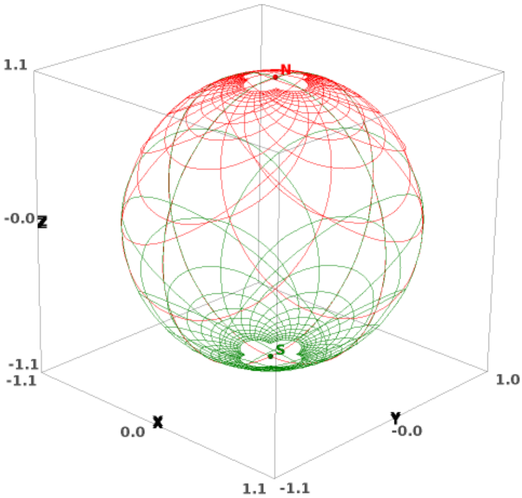
The 2-sphere example



Function `Chart.plot()`

Stereographic coordinates on the 2-sphere

Two charts:

- X_1: $\mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$
- X_2: $\mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^2$

See the worksheet at
http://sagemanifolds.obspm.fr/examples/html/SM_sphere_S2_days64.html

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Conclusion and perspectives

- **SageManifolds** is a work in progress
  $\sim$ 47,000 lines of Python code up to now (including comments and doctests)
- A preliminary version (v0.7) is freely available (GPL) at
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Example: installing SageManifolds 0.7 in a branch of a Sage 6.5 install

```
cd <your Sage root directory>
git remote add sm-github https://github.com/sagemanifolds/sage.git
git fetch -t sm-github sm-v0.7
git checkout -b sagemanifolds
git merge FETCH_HEAD
make
```

More details at http://sagemanifolds.obspm.fr/download.html

Current status

Already present (v0.7):

- maps between manifolds, pullback operator
- submanifolds, pushforward operator
- curves in manifolds
- standard tensor calculus (tensor product, contraction, symmetrization, etc.), even on non-parallelizable manifolds
- all monoterm tensor symmetries
- exterior calculus (wedge product, exterior derivative, Hodge duality)
- Lie derivatives of tensor fields
- affine connections, curvature, torsion
- pseudo-Riemannian metrics, Weyl tensor
- some plotting capabilities (charts, points, curves)

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  - computation of geodesics (numerical integration via Sage/GSL or `Gyoto`)
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  - add more graphical outputs
  - add more functionalities: symplectic forms, fibre bundles, spinors, variational calculus, etc.
  - connection with numerical relativity: using Sage to explore numerically generated spacetimes

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- *Developments within Sage* (to be discussed in this workshop):
  - introduce the category of topological spaces
  - introduce Lie groups

Integration into Sage

- The algebraic part (tensors on free modules of finite ranks) is submitted to Sage Trac as ticket #15916 and is under review (thank you Travis!)
- The ticket devoted to the differential part (#14865) must be reorganized (split in smaller tickets)

Acknowledgements: the SageManifolds project has benefited from many discussions with Sage developers around the world, and especially in Paris area.