Extrinsic curvature in quasiequilibrium binary black hole spacetimes

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Plan

1. Evolution of binary black holes

2. The effective potential + Bowen-York approach

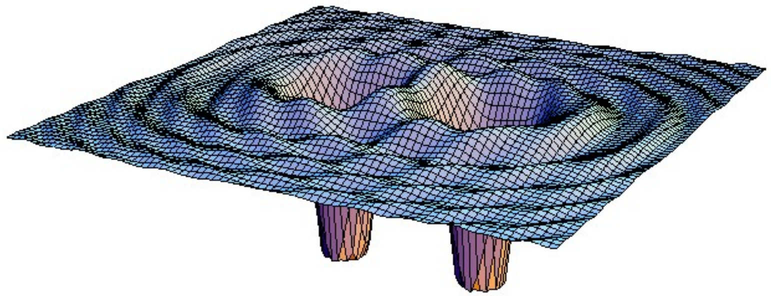
3. The helical Killing vector approach

4. Analysis of the extrinsic curvature

1

Evolution of binary black holes

Binary black holes inspiral and coalescence



From the GW detection point of view:

- the most promising source [Lipunov, Postnov & Prokhorov, New Astron. **2**, 43 (1997)]

From the theoretical point of view:

- Binary BH = the two body problem in General Relativity
- Extreme GR $\Longrightarrow$ probes GR in the strong field regime

From the astrophysical point of view:

- Rate of binary black hole coalescences $\Longrightarrow$ massive star evolution
- Inspiral GW signal $\Longrightarrow$ precise measure of Hubble constant H_0
- GW observations of supermassive BH at high z $\Longrightarrow$ formation of large structures

Three types of binary BH coalescence

(1) Coalescence of stellar BH: $3M_\odot \lesssim M \lesssim 50M_\odot$: from evolution of massive stars

event rate: • up to $\sim 20/\mathrm{Myr}$ per galaxy [Belczynski, Kalogera, Bulik, ApJ **572**, 407 (2002)]

• $1.6 \times 10^{-7}\ \mathrm{yr}^{-1}\mathrm{Mpc}^{-3}$ from binary BH formation in globular clusters [Portegies Zwart & McMillan, ApJ **528**, L17 (2000)]

(2) Coalescence of intermediate mass BH: $M \sim 10^3 - 10^4\, M_\odot$: in globular cluster centers (M15, Mayall 2)

event rate : ??

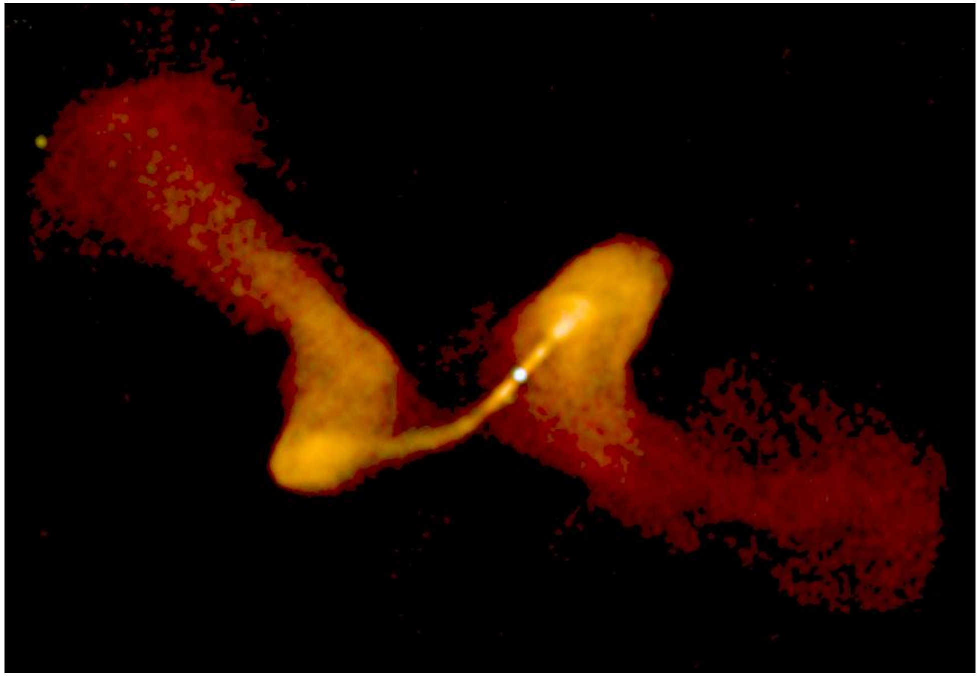
(3) Coalescence of massive BH: $M \sim 10^6 - 10^9\, M_\odot$: from galaxy encounters

event rate : possibly large

NB: Same physics (scaling with M)

Observational evidences for binary black holes

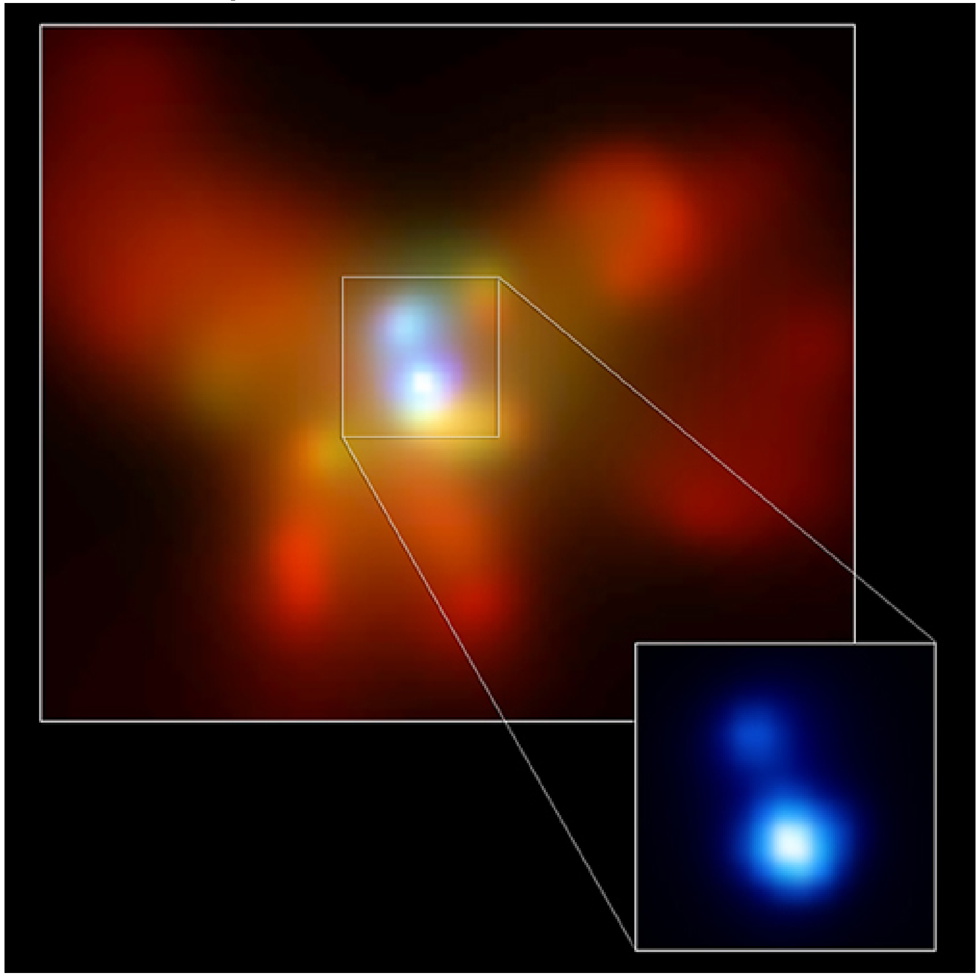
... in the past



Change of direction of NGC 326 jet

[Merrit & Eckers, Science **297**, 1310 (2002)]

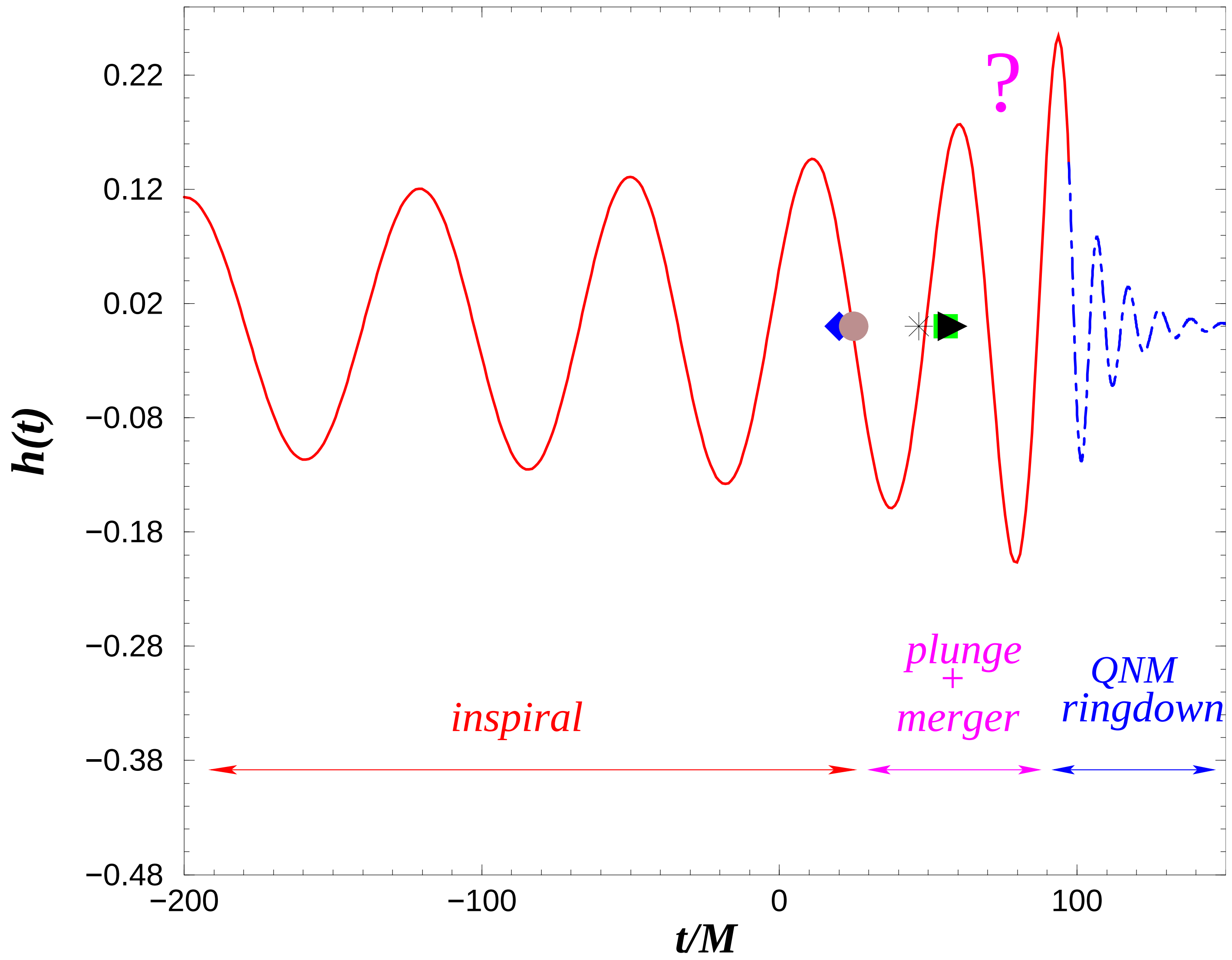
... in the present



X-ray view of double nucleus
of galaxy NGC 6240 (Chandra satellite)

[Hasinger et al., ApJL in press]

Gravitational waveform



[from Buonanno & Damour, PRD **62**, 064015 (2000)]

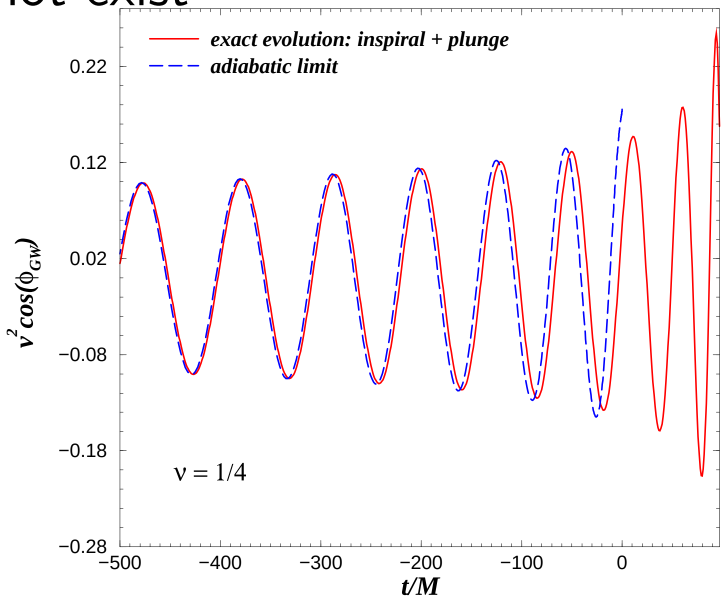
End of inspiral: the last stable orbit

cf. Luc Blanchet's talk

Very small mass ratio (Schwarzschild spacetime) : there exists an *innermost stable circular orbit (ISCO)* :

$$R_{\rm ISCO}^{\rm Schw} = 6M \qquad\qquad \Omega_{\rm ISCO}^{\rm Schw} = 6^{-3/2} M^{-1} \simeq 0.068\, M^{-1}$$

Equal mass ratio : gravitational radiation dissipation $\Longrightarrow$ strictly circular orbits do not exist



The ISCO is then defined in terms of the conservative part in the equation of motions, which give rise to circular orbits (adiabatic approximation). Consider a sequence of circular orbits of smaller and smaller radius, mimicking the inspiral. The ISCO is defined as the *turning point* in the *binding energy* of this sequence.

$\longleftarrow$ [Buonanno & Damour, PRD **62**, 064015 (2000)]

Computing quasiequilibrium configurations of close binary black holes

Last orbits of the inspiral

- Initial motivation: provide initial data for numerical computation of the plunge and merger

- But these configurations have interest from their own: they can lead to the (adiabatic) ISCO, which may be observed in gravitational waveformes

Remember from *Luc Blanchet's talk*: gravitational radiation reaction makes the orbital eccentricity to vanish $\Rightarrow$ one must deal only with circular orbits

2

The Effective Potential + Bowen-York (EPBY) approach

The Effective Potential approach (Cook 1994)

Procedure to get a quasiequilibrium configuration of binary black hole in circular orbit:

- Solve only for the 4 constraint equations of GR in vacuum on a spacelike **3-dimensional** surface with a non-trivial topology (for instance the Misner-Lindquist topology or the Brill-Lindquist topology, *cf. Luc Blanchet's talk*)
- Define the binding energy by $E = M_{\rm ADM} - M_1 - M_2$
- Define a circular orbit as an extremum of E with respect to proper separation l at fixed angular momentum and BH individual mass:

$$\left.\frac{\partial E}{\partial l}\right|_{M_1, M_2, J} = 0$$

- Compute the orbital angular velocity as $\Omega = \left.\dfrac{\partial E}{\partial J}\right|_{M_1, M_2, l}$

Ambiguity of the effective potential approach

Contrary to the ADM mass, the individual masses M_1 and M_2 are ill-defined quantities in GR.

Cook ansatz [PRD **50**, 5025 (1994)] : define the individual mass M_i from the apparent horizon area $\mathcal{A}_i$ and individual spin and via the Christodoulou formula:

$$M_i^2 := \frac{\mathcal{A}_i}{16\pi} + \frac{4\pi S_i^2}{\mathcal{A}_i}$$

Caveat 1: Christodoulou formula only established for a single stationary black hole (Kerr spacetime)

Caveat 2: moreover with $\mathcal{A}_i$ the area of the event horizon, not the apparent one

Caveat 3: The individual spin S_i suffers from the same lack of unambiguous definition as the individual mass.

Solving for the constraint equations

Spacelike hypersurface: induced metric γ_{ij}, extrinsic curvature K_{ij}

Hamiltonian constraint and momentum constraint :

$R + K^2 - K_{ij}K^{ij} = 0 \qquad\qquad D_j K^{ij} - D^i K = 0$

York-Lichnerowicz conformal decomposition:

Split K_{ij} into a traceless part A_{ij} and a trace part : $K_{ij} = A_{ij} + \frac{K}{3}\gamma_{ij}$

Introduce a conformal metric : $\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij}$

and note the identity $D_j A^{ij} = \Psi^{-10}\tilde{D}_j(\Psi^{10}A^{ij})$

$\Rightarrow$ introduce a conformal traceless extrinsic curvature: $A^{ij} = \Psi^{-10}\tilde{A}^{ij}$

Split $\tilde{A}^{ij}$ into a longitudinal and transverse part: $\tilde{A}^{ij} = (\tilde{L}X)^{ij} + \tilde{A}^{ij}_{\rm TT}$

with $(\tilde{L}X)^{ij} := \tilde{D}^j X^i + \tilde{D}^i X^j - \frac{2}{3}\tilde{D}_k X^k \,\tilde{\gamma}^{ij}$ and $\tilde{D}_j \tilde{A}^{ij}_{\rm TT} = 0$

Finally: $\boxed{K^{ij} = \Psi^{-10}\left[(\tilde{L}X)^{ij} + \tilde{A}^{ij}_{\rm TT}\right] + \frac{K}{3}\gamma^{ij}}$

The constraint equations in the conformal decomposition

Hamiltonian constraint : $$\boxed{8\tilde{D}_k\tilde{D}^k\Psi - \tilde{R}\Psi - \frac{2}{3}K^2\Psi^5 + \tilde{A}_{ij}\tilde{A}^{ij}\,\Psi^7 = 0}$$

Momentum constraint : $$\boxed{\tilde{D}_k\tilde{D}^k X^i + \frac{1}{3}\tilde{D}^i\tilde{D}_k X^k + \tilde{R}^i{}_j X^j = \frac{2}{3}\Psi^6\tilde{D}^i K}$$

Note: the momentum constraint involves only the longitudinal part (X^i) of $\tilde{A}^{ij}$, not the TT part ($\tilde{A}^{ij}_{\rm TT}$), which can thus be freely chosen.

The Bowen-York solution

Choose:
- Maximal slicing : $K = 0$ $\Rightarrow$ momentum constraint independent of Ψ
- Conformally flat 3-metric : $\tilde{\gamma}_{ij} = f_{ij}$
- Vanishing TT part of $\tilde{A}^{ij}$: $\tilde{A}^{ij}_{\rm TT} = 0$

Hamiltonian constraint $\Rightarrow$
$$\underline{\Delta}\Psi = -\frac{\Psi^7}{8}\tilde{A}_{ij}\tilde{A}^{ij} \qquad (1)$$

Momentum constraint $\Rightarrow$
$$\underline{\Delta}X^i + \frac{1}{3}\bar{\nabla}^i\bar{\nabla}_k X^k = 0 \qquad (2)$$

Bowen-York analytical solution of (2) [Bowen & York, PRD **21**, 2047 (1980)] :

For a single black hole : $X^i_{\rm BY_0} = -\frac{1}{4r}\left(7P^i + P_j\frac{x^j x^i}{r^2}\right) - \frac{1}{r^3}\epsilon^i{}_{jk}S^j x^k$

with $x^i = (x, y, z)$, $r^2 := x^2 + y^2 + z^2$

Two constant vector parameters : $\begin{cases} P^i & : \ \sim \text{linear momentum} \\ S^i & : \ \sim \text{angular momentum} \end{cases}$

The Bowen-York solution (con't)

Example: choose S^i perpendicular to P^i and choose Cartesian coordinate system (x, y, z) such that $P^i = (0, P, 0)$ and $S^i = (0, 0, S)$. Then

$$
\begin{aligned}
X^x_{\mathrm{BY}_0} &= -\frac{P}{4}\frac{xy}{r^3} + S\frac{y}{r^3} \\
X^y_{\mathrm{BY}_0} &= -\frac{P}{4r}\left(7 + \frac{y^2}{r^2}\right) - S\frac{x}{r^3} \\
X^z_{\mathrm{BY}_0} &= -\frac{P}{4}\frac{xz}{r^3}
\end{aligned}
$$

Bowen-Tork extrinsic curvature: $\tilde{A}^{ij}_{\mathrm{BY}_0} = (\bar{L}X_{\mathrm{BY}_0})^{ij}$

$$
\boxed{\tilde{A}^{ij}_{\mathrm{BY}_0} = \frac{3}{2r^3}\left[P^i x^j + P^j x^i - \left(\delta^{ij} - \frac{x^i x^j}{r^2}\right) P^k x_k\right] + \frac{3}{r^5}\left(\epsilon^i{}_{kl} S^k x^l x^j + \epsilon^j{}_{kl} S^k x^l x^i\right)}
$$

Using Bowen-York in the two black hole problem

Since Eq. (2) is linear

$$X^i = X^i_{\rm BY} := X^i_{{\rm BY}_0}(\boldsymbol{P}_1, \boldsymbol{S}_1, x^i \to x^i_1) + X^i_{{\rm BY}_0}(\boldsymbol{P}_2, \boldsymbol{S}_2, x^i \to x^i_2)$$

Accordingly

$$K^{ij} = \Psi^{-10}\left[\tilde{A}^{ij}_{{\rm BY}_0}(\boldsymbol{P}_1, \boldsymbol{S}_1, x^i \to x^i_1) + \tilde{A}^{ij}_{{\rm BY}_0}(\boldsymbol{P}_2, \boldsymbol{S}_2, x^i \to x^i_2)\right]$$

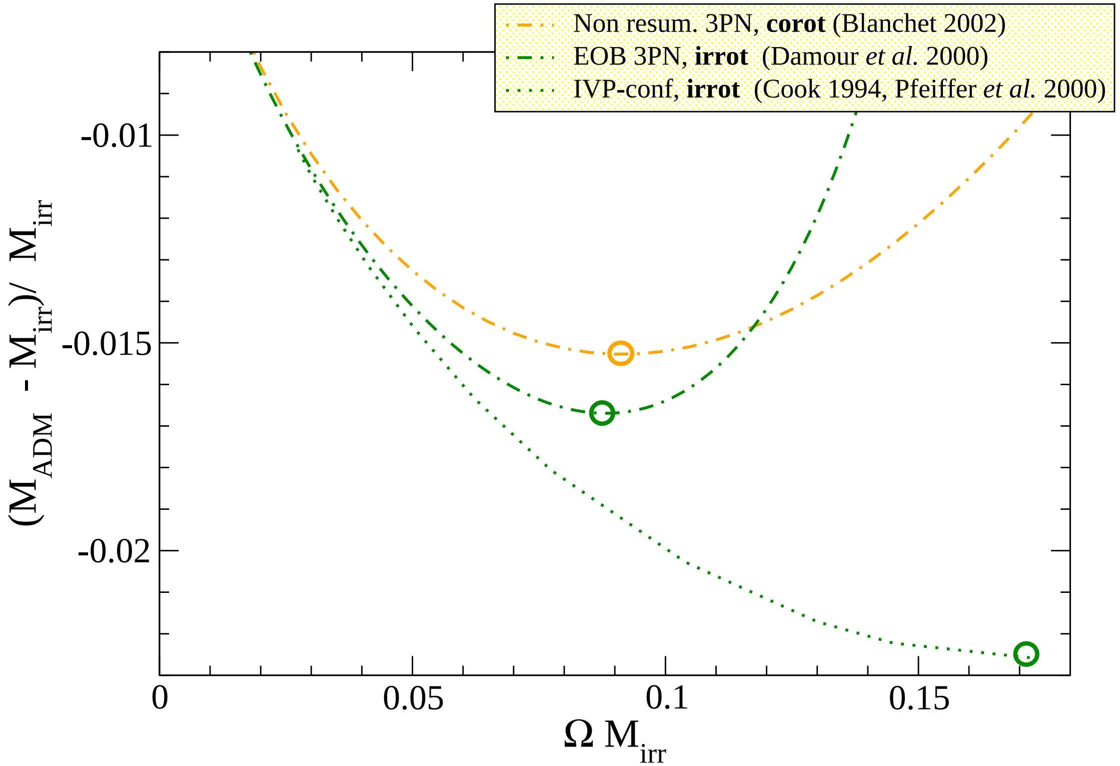
There remains to solve (numerically) the non-linear elliptic equation (1) to get Ψ.

Numerical implementations:

- Cook 1994 [PRD **50**, 5025 (1994)] : *Misner-Lindquist topology*
- Pfeiffer, Teukolsky & Cook 2000 [PRD **62**, 104018 (2000)] : *idem*
- Baumgarte 2000 [PRD **62**, 024018 (2000)] : *Brill-Lindquist topology*

Discrepancy between EPBY and post-Newtonian results

Binding energy along an evolutionary sequence of equal-mass binary black holes:



Post-Newtonian computations : at the 3-PN level:

- Damour, Jaranowski & Schäfer 2000 [PRD **62**, 084011 (2000)] : Effective One Body approach (EOB)
- Blanchet 2002 [PRD **65**, 124009 (2002)] : Non-resummed Taylor expansion

3

The Helical Killing Vector (HKV) approach

Basics

Problem treated:
Binary black holes in the pre-coalescence stage
$\Rightarrow$ the notion of **orbit** has still some meaning

Basic idea:
Construct an approximate, but full spacetime (i.e. **4-dimensional**) representing 2 orbiting black holes
Previous numerical treatments (IVP) : 3-dimensional (initial value problem on a spacelike 3-surface)
4-dimensional approach $\Rightarrow$ rigorous definition of orbital angular velocity

[Gourgoulhon, Grandclément & Bonazzola, PRD **65**, 044020 (2002)]

[Grandclément, Gourgoulhon & Bonazzola, PRD **65**, 044021 (2002)]

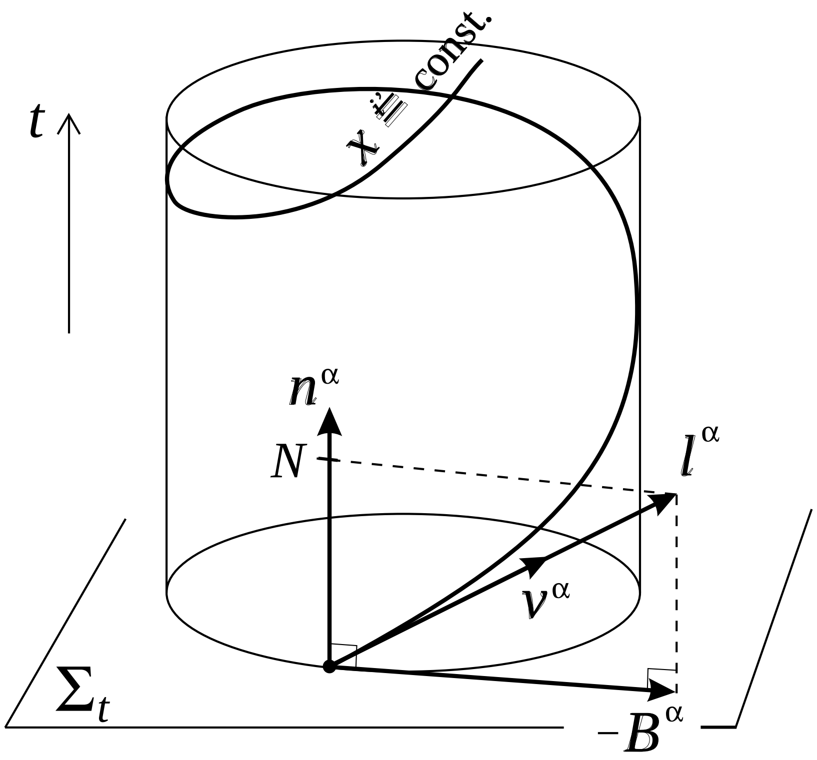
Helical symmetry

Physical assumption: when the two holes are sufficiently far apart, the radiation reaction can be neglected $\Rightarrow$ closed orbits
Gravitational radiation reaction circularizes the orbits $\Rightarrow$ circular orbits

Geometrical translation: there exists a Killing vector field $\boldsymbol{\ell}$ such that:

far from the system (asymptotically inertial coordinates $(t_0, r_0, \theta_0, \varphi_0)$),

$$\boldsymbol{\ell} \to \frac{\partial}{\partial t_0} + \Omega \frac{\partial}{\partial \varphi_0}$$



Helical symmetry: discussion

Helical symmetry is exact

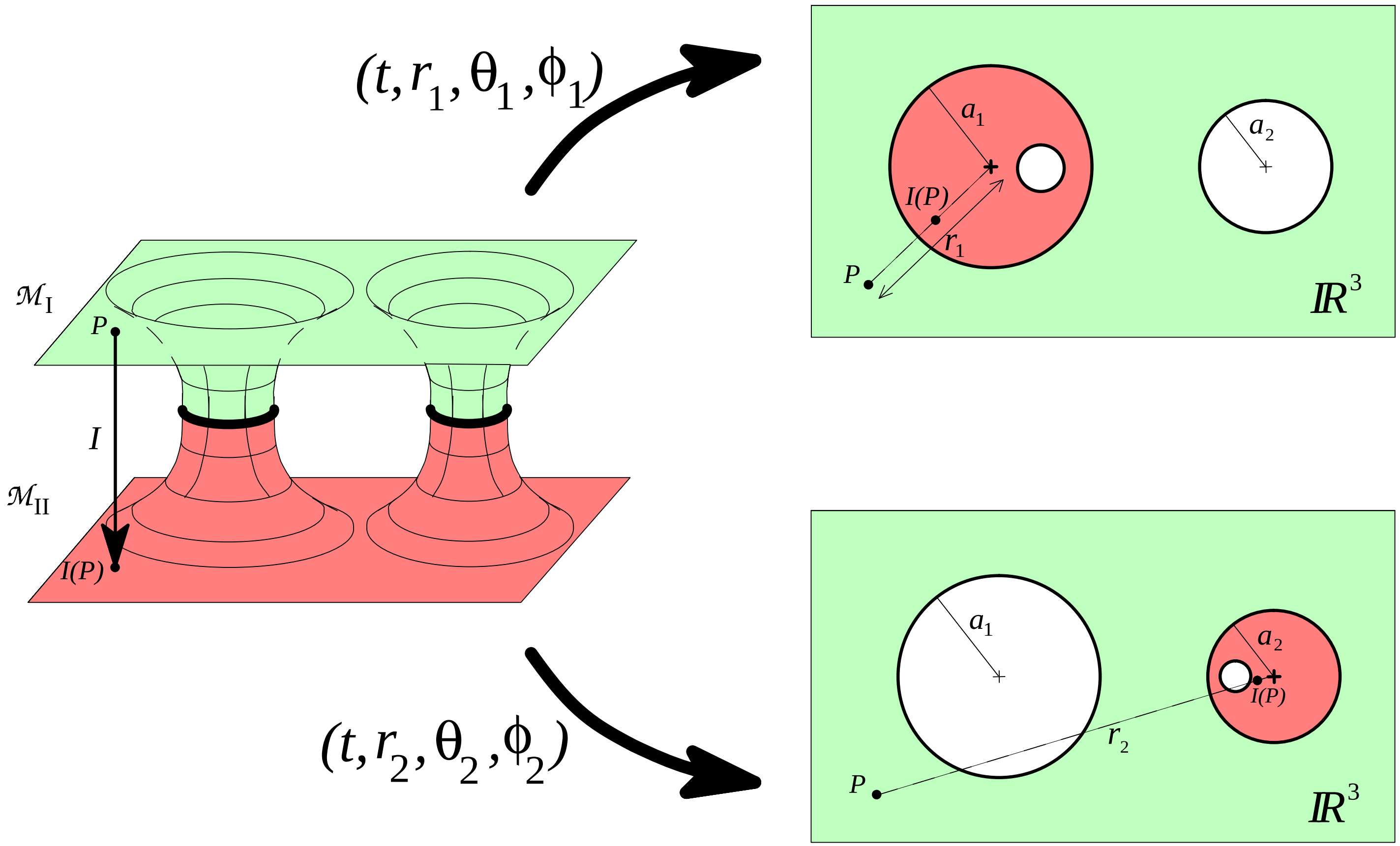
- in **Newtonian gravity** and in **2nd order Post-Newtonian gravity**
- in general relativity for a non-axisymmetric system (binary) only with **standing gravitational waves**

But a spacetime with a helical Killing vector and standing gravitational waves cannot be asymptotically flat in full GR [Gibbons & Stewart 1983].

We have used a truncated version of GR (the **Isenberg-Wilson-Mathews** approximation, which will be described below) which (i) admits the helical Killing vector and (ii) is asymptotically flat.

Spacetime manifold

Topology : $\mathbb{R} \times$ Misner-Lindquist



Canonical mapping: $I: \quad (t, r_1, \theta_1, \varphi_1) \mapsto \left(t, \frac{a_1^2}{r_1}, \theta_1, \varphi_1\right)$

Isometry between the two sheets

Assumption: the canonical mapping I is an isometry: $I_*\mathbf{g} = \mathbf{g}$

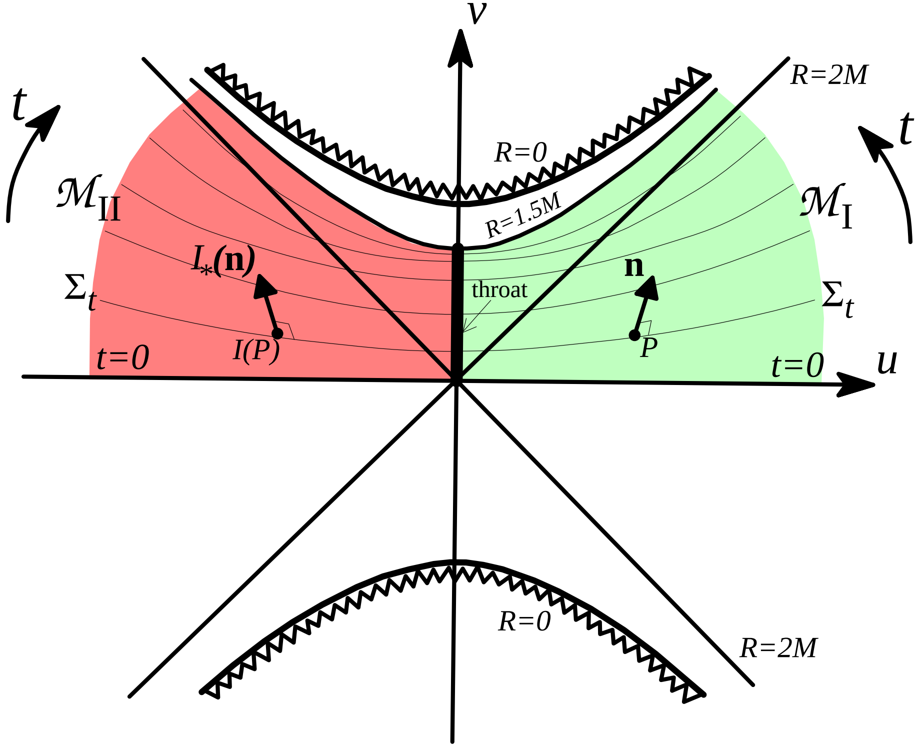
Consequences:

- $I_* t = t$ and $I_* \nabla t = \nabla t$
- $I_* \mathbf{n} = \pm \mathbf{n}$
- $I_* N = \pm N$ (same sign as $\mathbf{n}$)
- $I_* \boldsymbol{\beta} = \boldsymbol{\beta}$
- $I_* \boldsymbol{\gamma} = \boldsymbol{\gamma}$
- $I_* \mathbf{K} = \pm \mathbf{K}$ (same sign as $\mathbf{n}$)

Choice of the minus sign

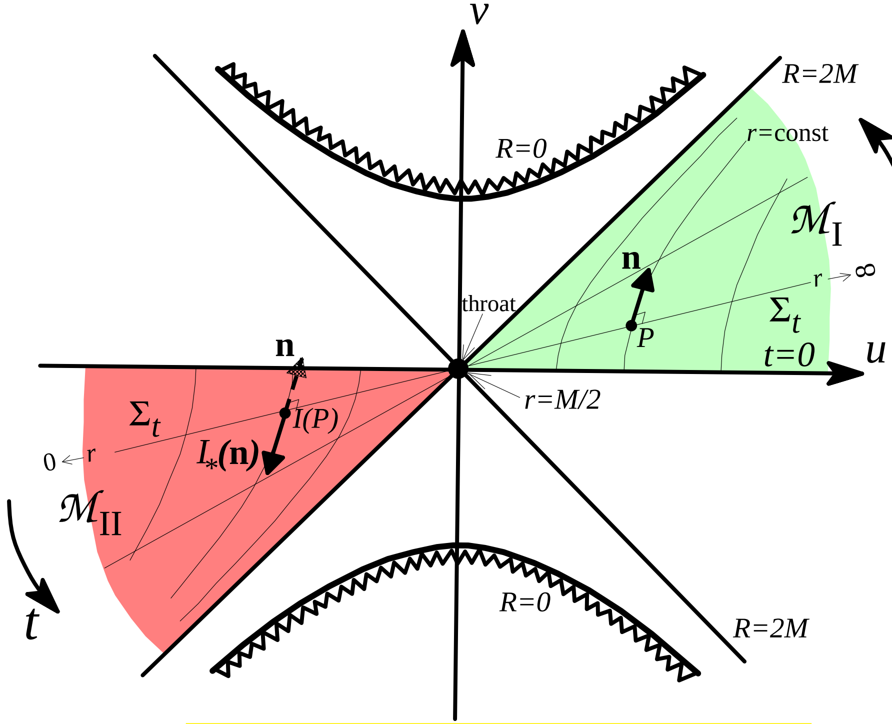
Two families of maximal slicing of the Schwarzschild spacetime:

$+$ sign (symmetric lapse)



time evolving

$-$ sign (antisymmetric lapse)



preserves the stationarity

Rotation state of each black hole

Choice: rotation synchronized with the orbital motion (corotating system)

Justifications:

- the only rotation state fully compatible with the helical symmetry [Friedman, Uryu & Shibata, PRD **65**, 064035 (2002)]
- for close systems, black hole "effective viscosity" might be very efficient in synchronizing the spins with the orbital motion [e.g. Price & Whelan, PRL **87**, 231101 (2001)]

Geometrical translation: the two horizons are Killing horizons *(cf. Koga's talk)* associated with $\boldsymbol{\ell}$:

$$\boldsymbol{\ell} \cdot \boldsymbol{\ell}|_{\mathcal{H}_1} = 0 \qquad \text{and} \qquad \boldsymbol{\ell} \cdot \boldsymbol{\ell}|_{\mathcal{H}_2} = 0 \; .$$

[cf. the rigidity theorem for a Kerr black hole]

Consequence on the shift vector: $\boldsymbol{\ell} \cdot \boldsymbol{\ell} = -N^2 + \boldsymbol{\beta} \cdot \boldsymbol{\beta}$

$\Rightarrow$ boundary conditions on the horizons: $\boldsymbol{\beta}|_{\mathcal{H}_1} = 0 \qquad \text{and} \qquad \boldsymbol{\beta}|_{\mathcal{H}_2} = 0$.

Einstein equations

Framework: 3+1 formalism with maximal slicing: $K = 0$

Isenberg-Wilson-Mathews approximation: conformally flat spatial metric: $\boldsymbol{\gamma} = \Psi^4 \boldsymbol{f}$

$\Rightarrow$ spacetime metric : $ds^2 = -N^2 dt^2 + \Psi^4 f_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$

Amounts to solve 5 of the 10 Einstein equations (one more than IVP !) :

$$\underline{\Delta}\Psi = -\frac{\Psi^5}{8}\hat{A}_{ij}\hat{A}^{ij} \qquad \textit{(Hamiltonian constraint)}$$

$$\underline{\Delta}\beta^i + \frac{1}{3}\bar{\nabla}^i\bar{\nabla}_j\beta^j = 2\hat{A}^{ij}\left(\bar{\nabla}_j N - 6N\bar{\nabla}_j \ln\Psi\right) \qquad \textit{(momentum constraint)}$$

$$\Delta N = N\Psi^4\hat{A}_{ij}\hat{A}^{ij} - 2\bar{\nabla}_j \ln\Psi\,\bar{\nabla}^j N \qquad \textit{(trace of } \frac{\partial K_{ij}}{\partial t} = \cdots\textit{)}$$

with $\hat{A}_{ij} := \Psi^{-4}K_{ij}$ and $\hat{A}^{ij} := \Psi^4 K^{ij}$

Extrinsic curvature : helical symmetry $\Rightarrow K_{ij} = D_i\beta_j + D_j\beta_i$

$$\hat{A}^{ij} = \frac{1}{2N}(\bar{L}\beta)^{ij} \text{ with } (\bar{L}\beta)^{ij} := \bar{\nabla}^i\beta^j + \bar{\nabla}^j\beta^i - \frac{2}{3}\bar{\nabla}_k\beta^k f^{ij} \qquad \textit{(traceless part)}$$

$$\bar{\nabla}_i\beta^i = -6\beta^i\bar{\nabla}_i \ln\Psi \qquad \textit{(trace part)}$$

Boundary conditions

isometry condition on γ_{rr}: $\left.\left(\frac{\partial\Psi}{\partial r_1}+\frac{\Psi}{2r_1}\right)\right\rvert_{\mathcal{S}_1}=0$	$\left.\left(\frac{\partial\Psi}{\partial r_2}+\frac{\Psi}{2r_2}\right)\right\rvert_{\mathcal{S}_2}=0$	asymptotic flatness: $\Psi\to 1$ when $r\to\infty$
corotating black holes: $\boldsymbol{\beta}\rvert_{\mathcal{S}_1}=0$	$\boldsymbol{\beta}\rvert_{\mathcal{S}_2}=0$	definition of $\boldsymbol{\ell}$: $\boldsymbol{\beta}\to\Omega\frac{\partial}{\partial\varphi_0}$ when $r\to\infty$
isometry condition on N: $N\rvert_{\mathcal{S}_1}=0$	$N\rvert_{\mathcal{S}_2}=0$	asymptotic flatness: $N\to 1$ when $r\to\infty$

Regularity on the horizons

Position of the problem:

$$\left.\begin{array}{l} K^{ij} = \dfrac{(L\beta)^{ij}}{2\Psi^4 N} \\ N|_{\mathcal{S}} = 0 \end{array}\right\} \Longrightarrow \text{one must have } L\boldsymbol{\beta}|_{\mathcal{S}} = 0 \text{ for } \mathbf{K} \text{ to be regular}$$

$$\text{One has} \quad \left.\begin{array}{lll} (1) & \boldsymbol{\beta}|_{\mathcal{S}} = 0 & \text{(rigid rotation)} \\ (2) & I_*\boldsymbol{\beta} = \boldsymbol{\beta} & \text{(isometry)} \\ (3) & \bar{D}_i\beta^i = -6\beta^i\bar{D}_i \ln\Psi & (K=0) \end{array}\right\} \Longrightarrow L\boldsymbol{\beta}|_{\mathcal{S}} = 0$$

However, only (1) and the part of (2) implied by (1) are really imposed when solving the vector Poisson equation for $\boldsymbol{\beta}$.

Adopted solution:

Set $\boldsymbol{\beta}_{\rm new} = \boldsymbol{\beta}_{\rm old} + \boldsymbol{\beta}_{\rm cor}$ with $\boldsymbol{\beta}_{\rm cor}$ chosen so that (2) and (3) are fulfilled on the throats.

At the end of the computation, $\boldsymbol{\beta}_{\rm cor}$ must be zero (to get an exact solution) or small (to get an approximate solution).

Determination of Ω

Virial assumption: $O(r^{-1})$ part of the metric ($r \to \infty$) same as Schwarzschild

[The only quantity "felt" at the $O(r^{-1})$ level by a distant observer is the total mass of the system.]

A priori

$$\Psi \sim 1 + \frac{M_{\rm ADM}}{2r} \qquad \text{and} \qquad N \sim 1 - \frac{M_{\rm K}}{r}$$

Hence

$$\text{(virial assumption)} \iff M_{\rm ADM} = M_{\rm K}$$

Note

$$\text{(virial assumption)} \iff \Psi^2 N \sim 1 + \frac{\alpha}{r^2}$$

Link with the classical virial theorem

Einstein equations $\Rightarrow$

$$\underline{\Delta} \ln(\Psi^2 N) \;=\; \Psi^4 \left[4\pi S_i{}^i + \frac{3}{4}\hat{A}_{ij}\hat{A}^{ij}\right] - \frac{1}{2}\left[\bar{\nabla}_i \ln N \bar{\nabla}^i \ln N + \bar{\nabla}_i \ln(\Psi^2 N)\bar{\nabla}^i \ln(\Psi^2 N)\right]$$

No monopolar $1/r$ term in $\Psi^2 N \iff$

$$\int_{\Sigma_t} \left\{4\pi S_i{}^i + \frac{3}{4}\hat{A}_{ij}\hat{A}^{ij} - \frac{\Psi^{-4}}{2}\left[\bar{\nabla}_i \ln N \bar{\nabla}^i \ln N + \bar{\nabla}_i \ln(\Psi^2 N)\bar{\nabla}^i \ln(\Psi^2 N)\right]\right\} \Psi^4 \sqrt{f}\, d^3x \;=\; 0$$

Newtonian limit is the classical virial theorem:

$$2E_{\rm kin} + 3P + E_{\rm grav} = 0$$

Defining an evolutionary sequence

An evolutionary sequence is defined by:

$$\left.\frac{dM_{\rm ADM}}{dJ}\right|_{\rm sequence} = \Omega$$

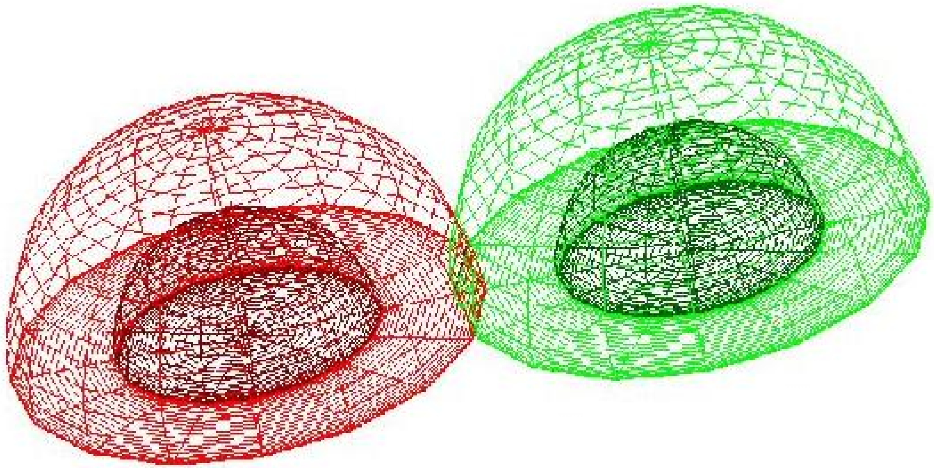
This is equivalent to requiring the constancy of the horizon area of each black hole, by virtue of the First law of thermodynamics for binary black holes :

$$dM_{\rm ADM} = \Omega\, dJ + \frac{1}{8\pi}\left(\kappa_1\, dA_1 + \kappa_2\, dA_2\right)$$

recently established by Friedman, Uryu & Shibata [PRD **65**, 064035 (2002)].

Note: Within the helical symmetry framework, a minimum in $M_{\rm ADM}$ along a sequence at fixed horizon area locates a change of orbital stability (ISCO) [Friedman, Uryu & Shibata, PRD **65**, 064035 (2002)].

Numerical integration



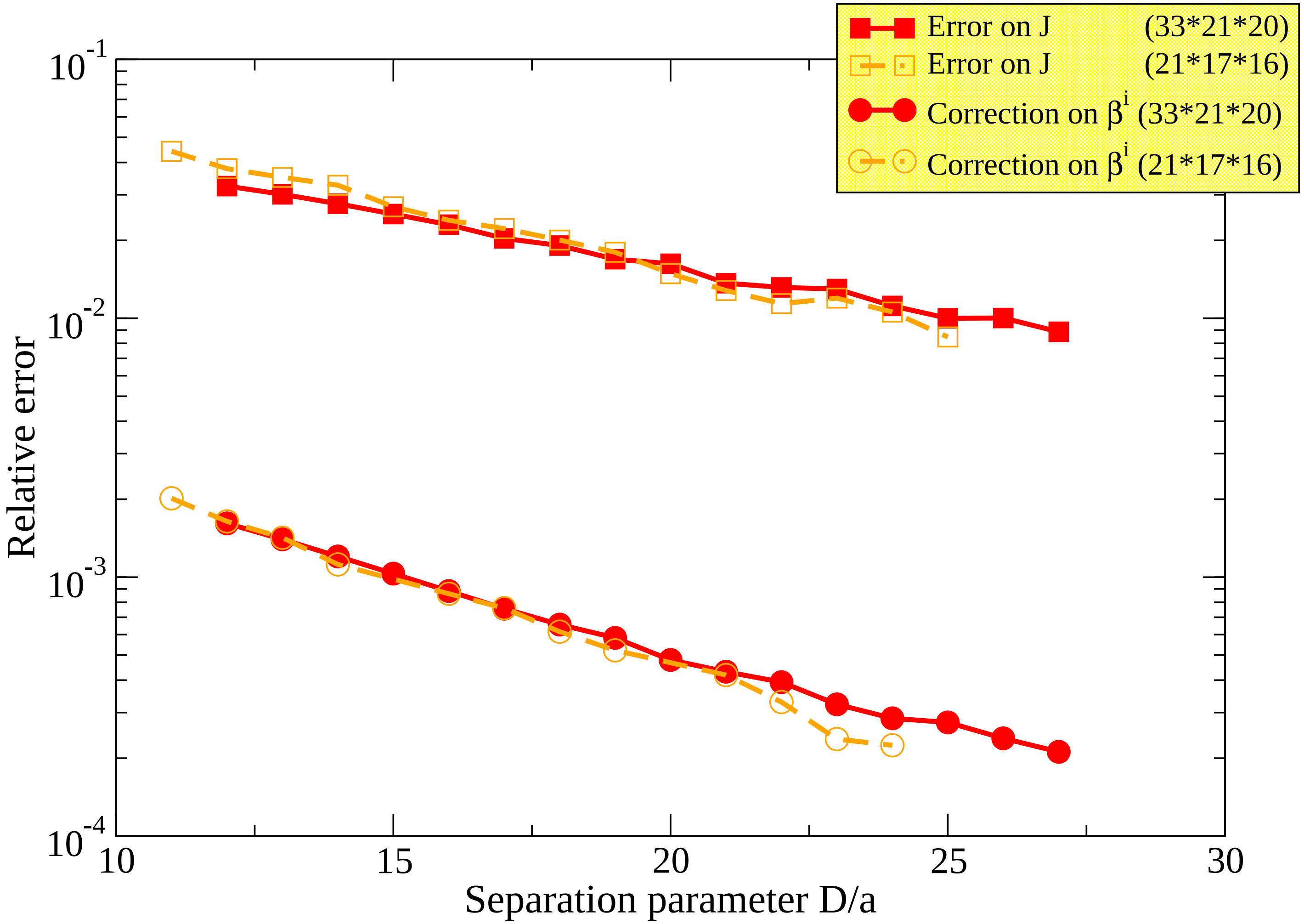
Numerical code based on a multidomain spectral method

Written with the LORENE C++ library

Many tests passed by the code

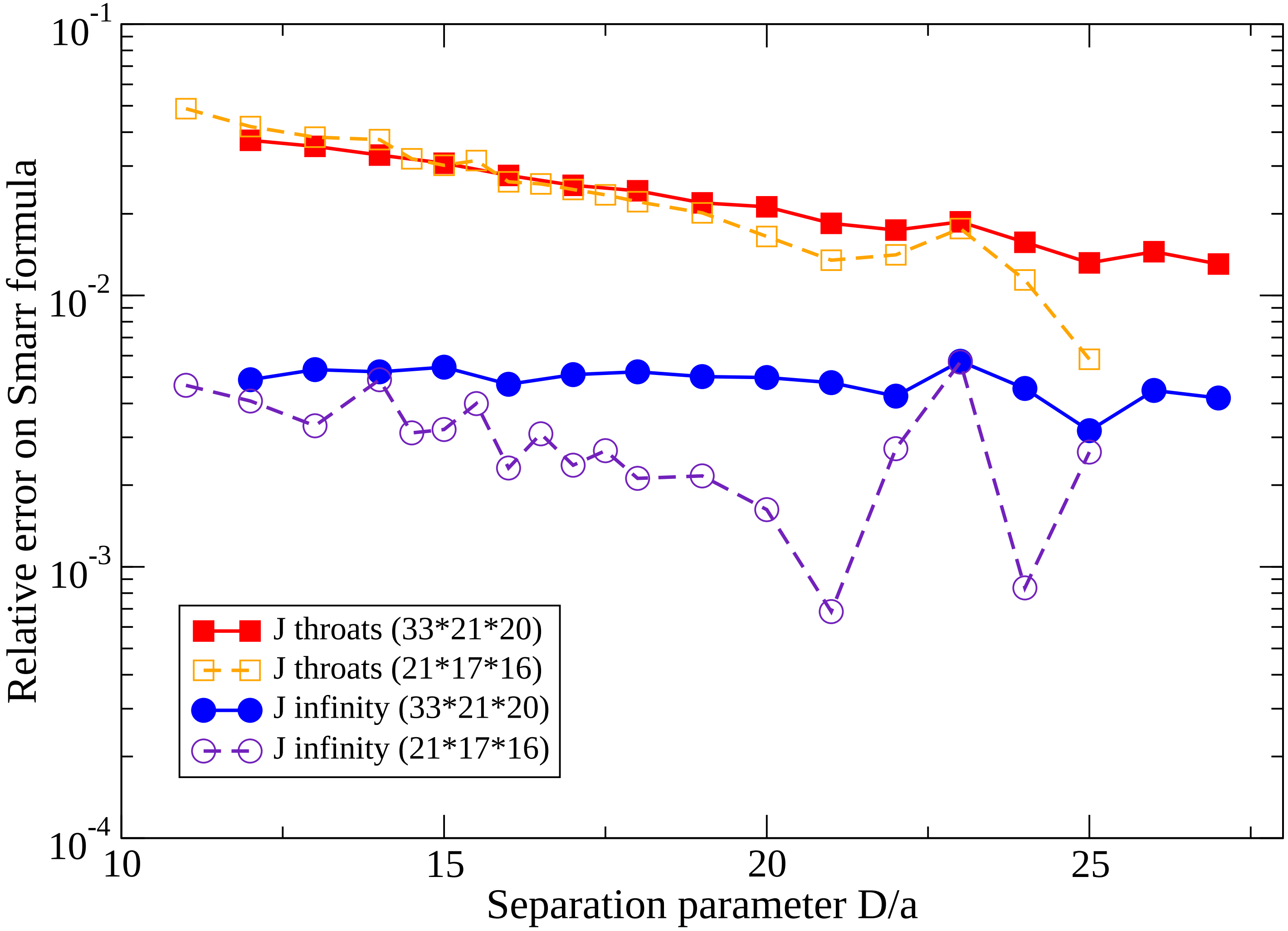
[Grandclément, Gourgoulhon & Bonazzola, PRD **65**, 044021 (2002)]

Test 1 : smallness of the correction function on the shift vector



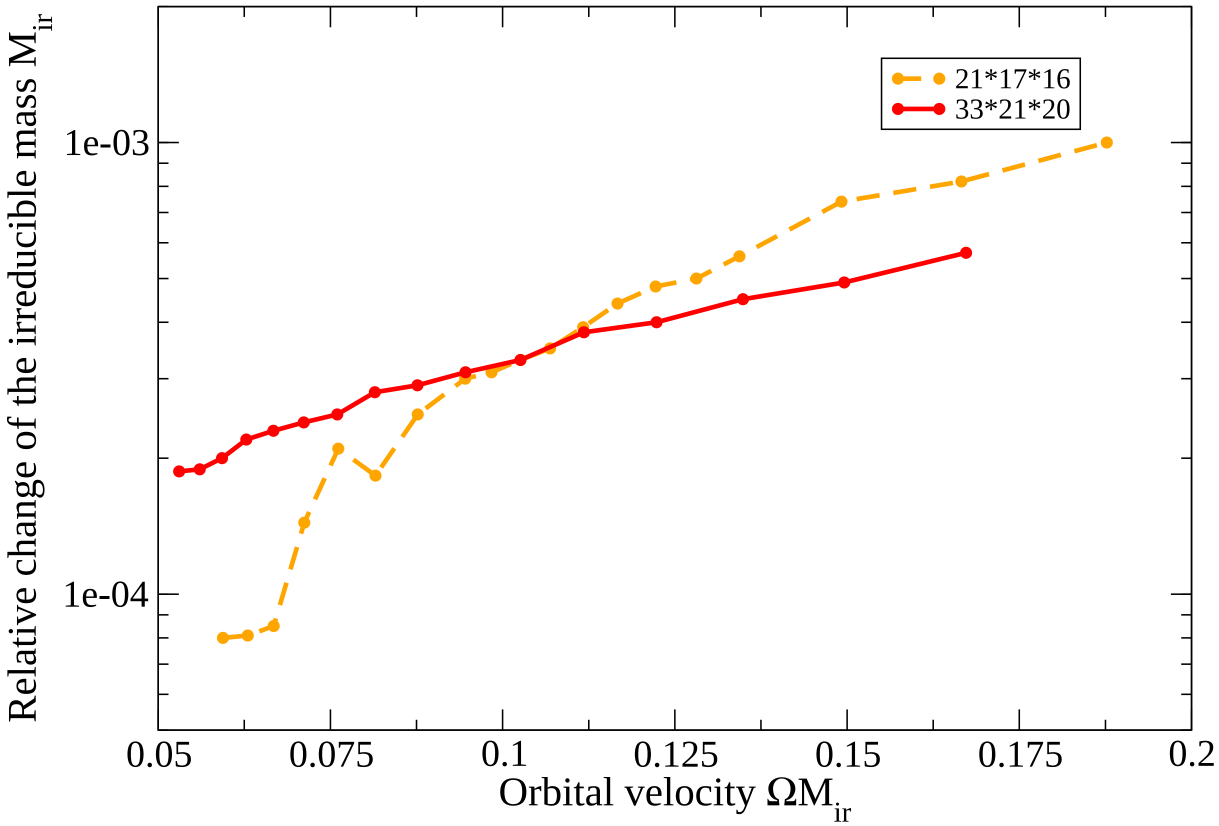
Relative amplitude of the correction function imposed on β and relative difference between the angular momentum J computed at spatial infinity and at the throats

Test 2 : error on Smarr formula



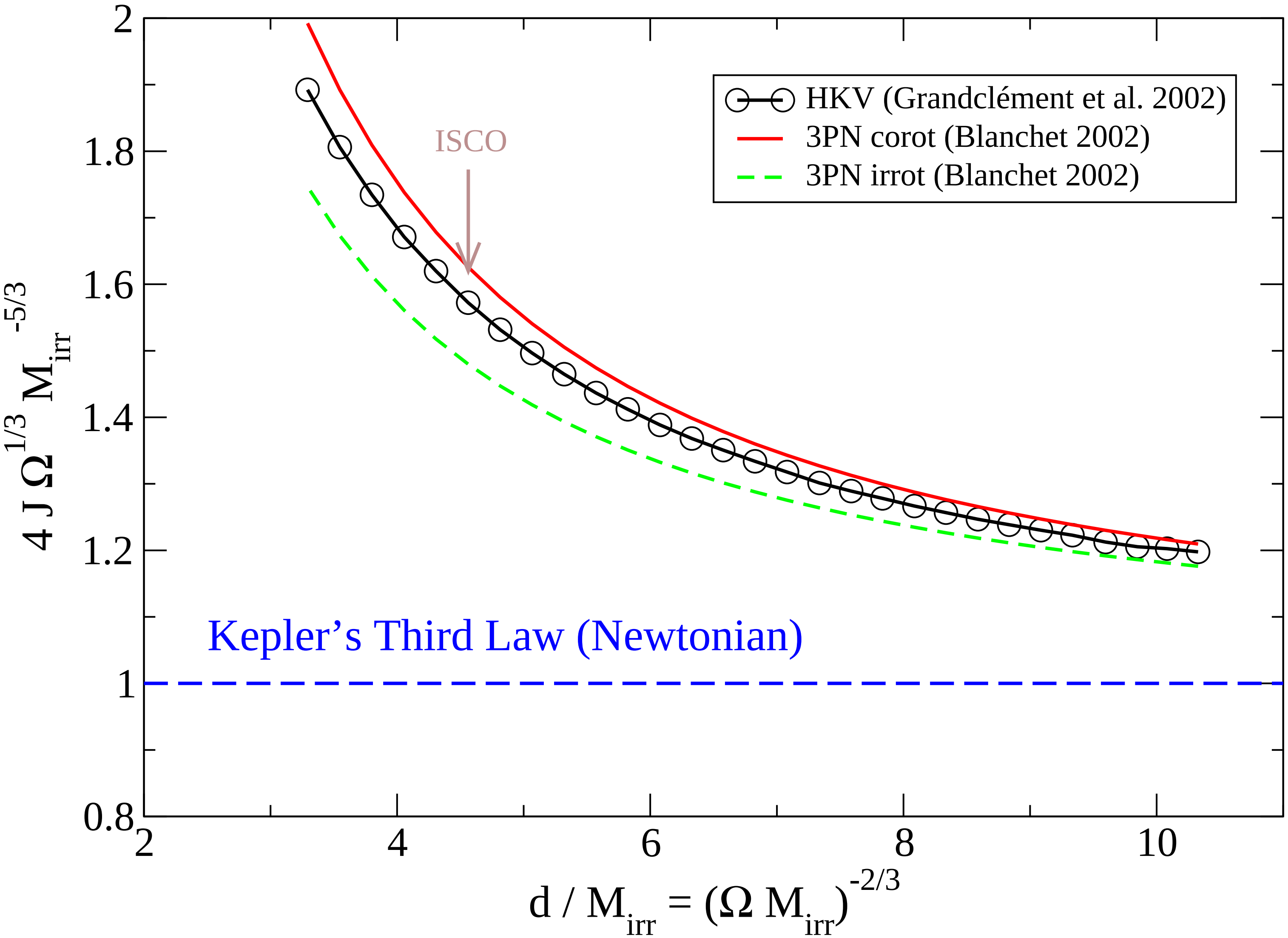
Relative error on Smarr formula $M - 2\Omega J = -1/4\pi \sum_i \oint_{S_i} \Psi^2 \bar{\nabla}_i N \, dS^i$

Test 3 : conservation of the horizon area along a sequence



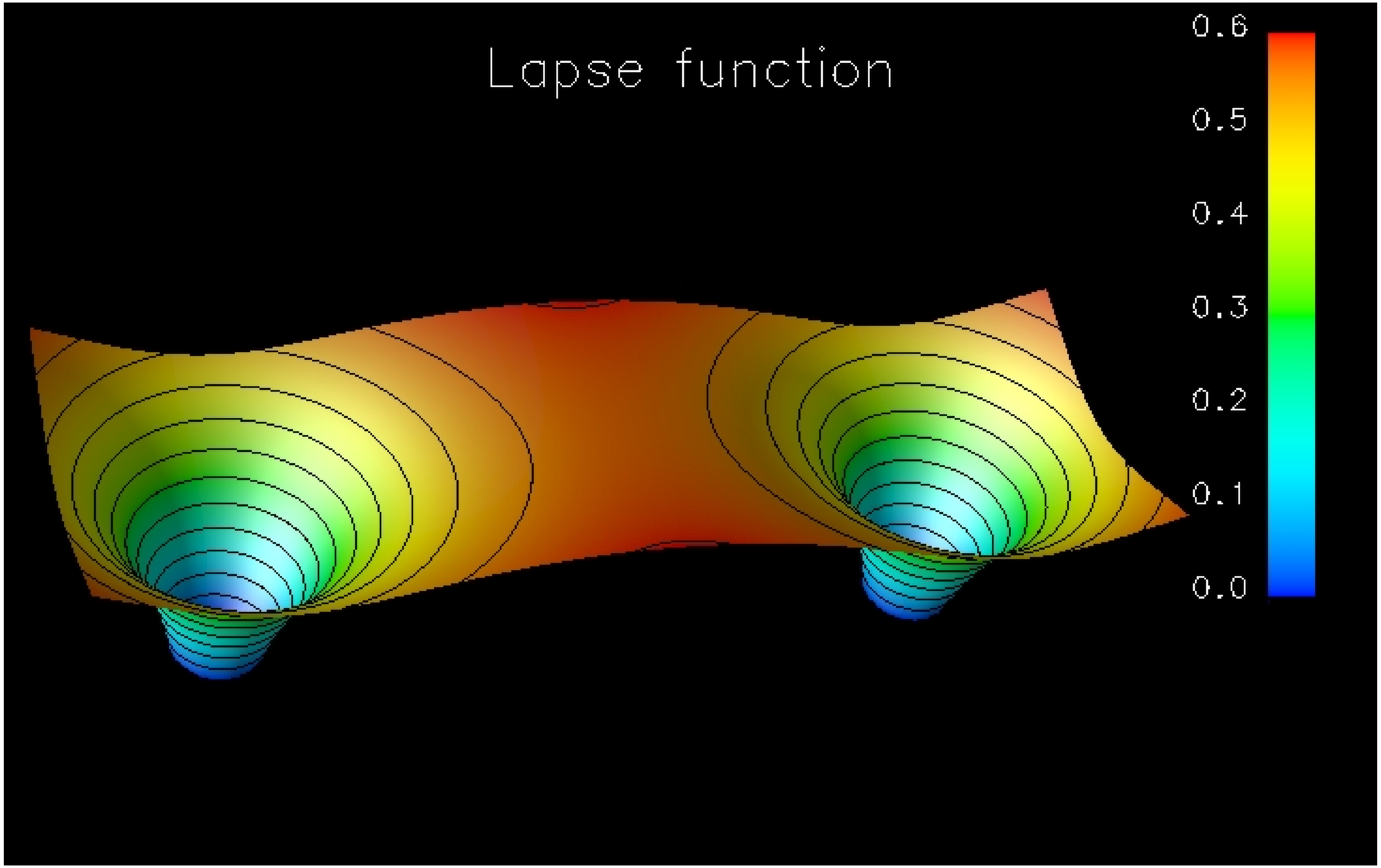
Relative change of the horizon area along an evolutionary sequence

Test 4 : getting Kepler's third law at large separation



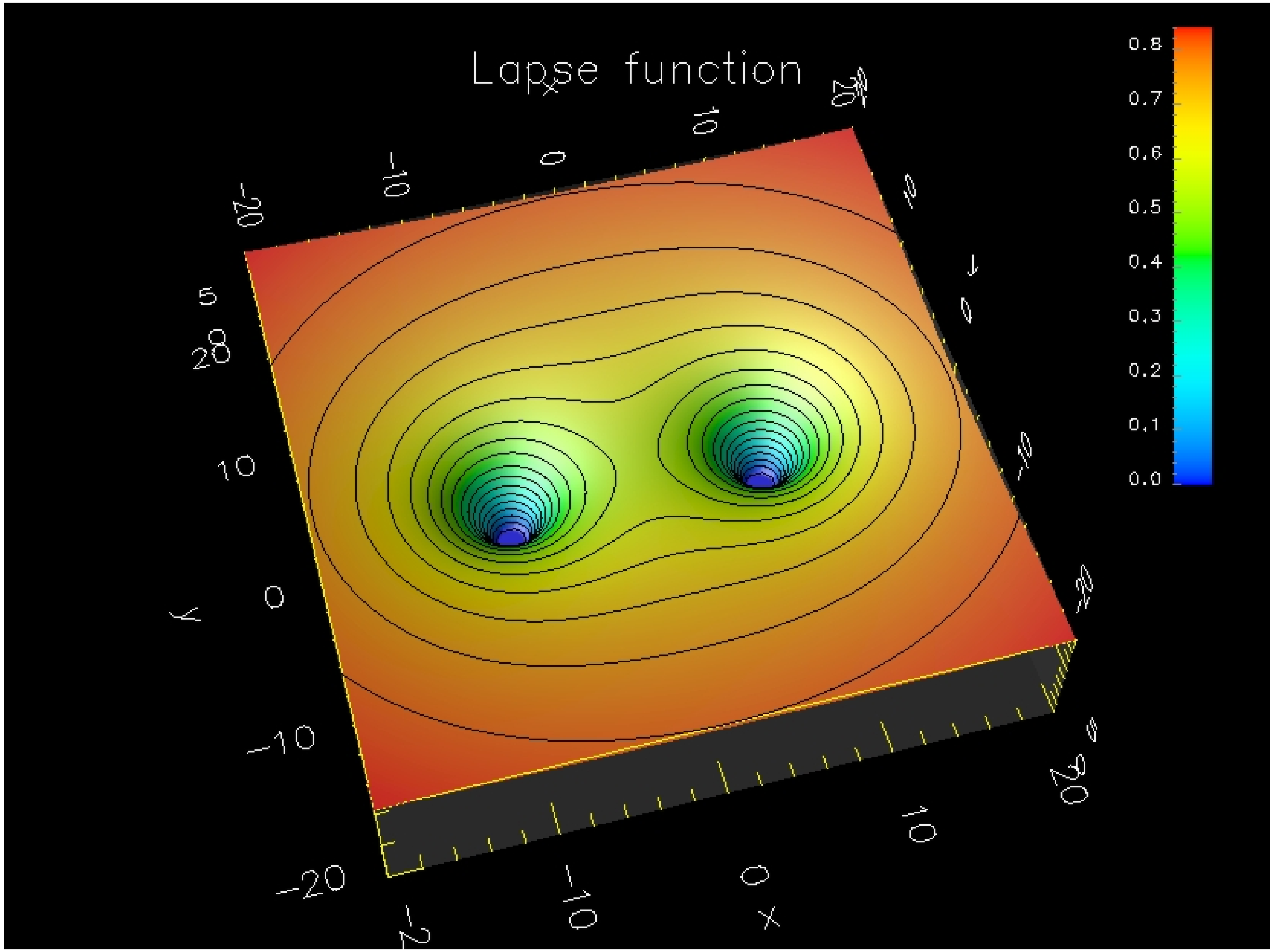
Check of the determination of Ω at large separation.

ISCO configuration



[Grandclément, Gourgoulhon, Bonazzola, PRD **65**, 044021 (2002)]

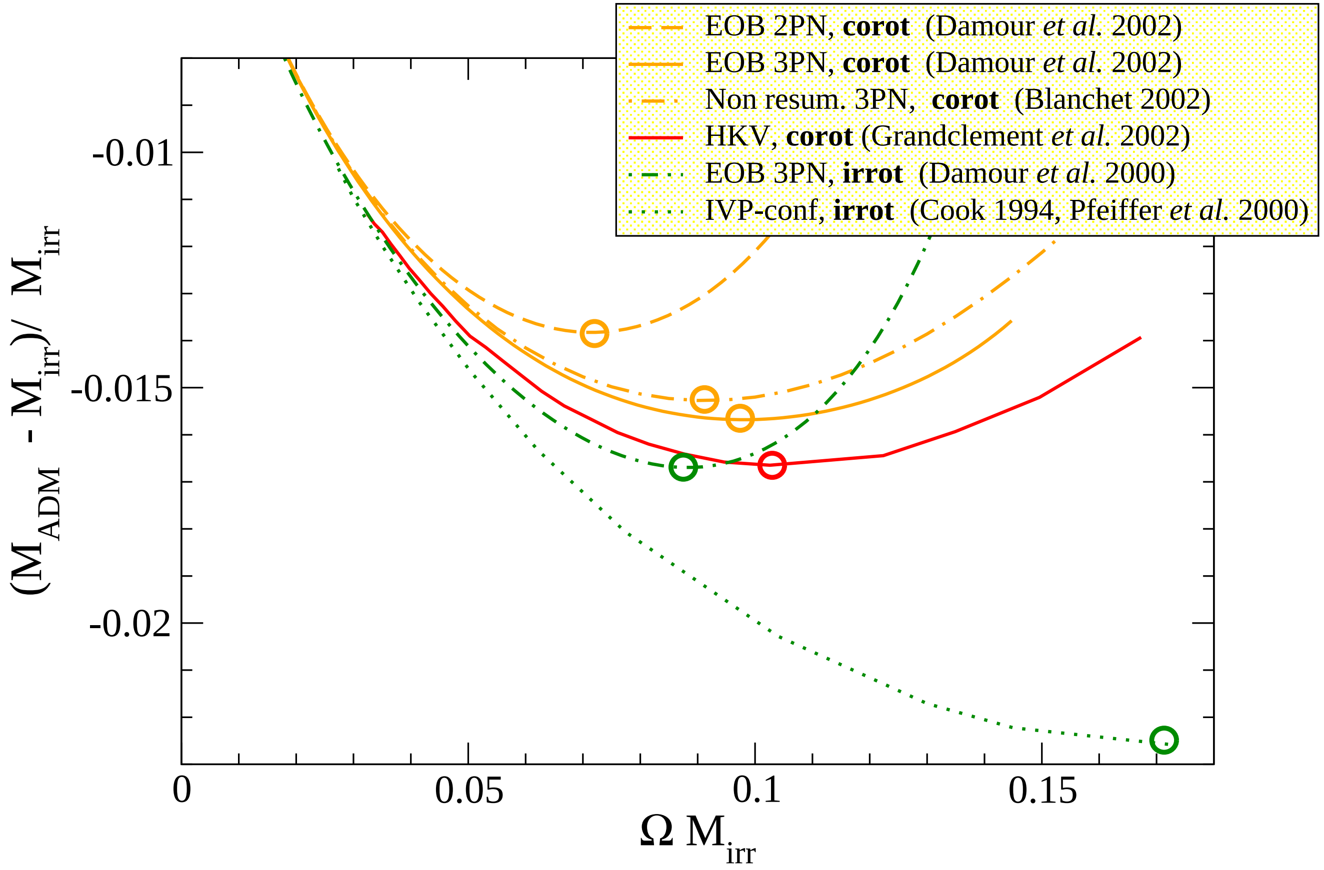
ISCO configuration



[Grandclément, Gourgoulhon, Bonazzola, PRD **65**, 044021 (2002)]

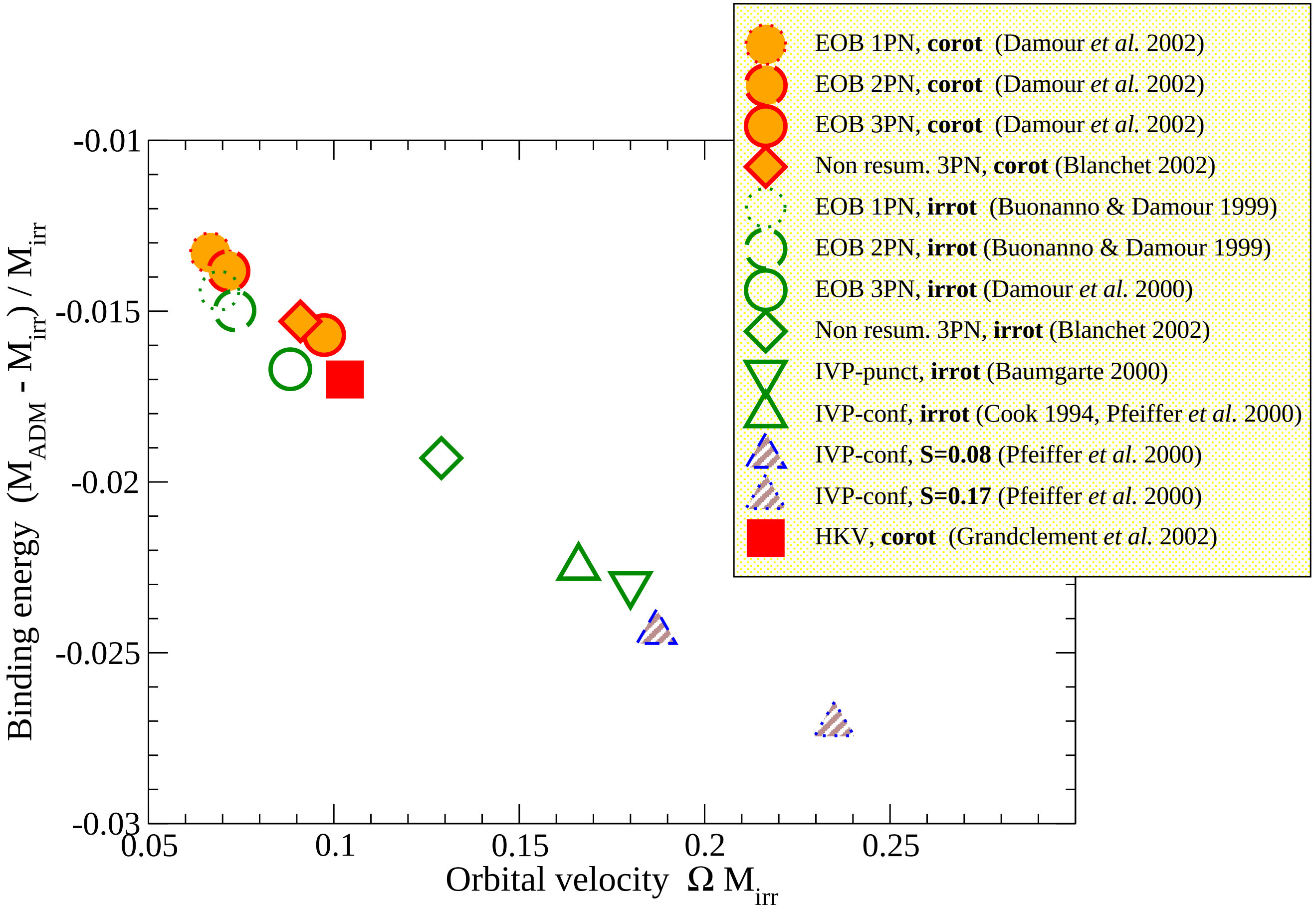
Comparison with Post-Newtonian computations

Binding energy along an evolutionary sequence of equal-mass binary black holes



[from Damour, Gourgoulhon, Grandclément, PRD **66**, 024007 (2002)]

Location of the ISCO



Gravitational wave frequency:

$$f = 320 \, \frac{\Omega M_{\rm ir}}{0.1} \, \frac{20 \, M_\odot}{M_{\rm ir}} \ \mathrm{Hz}$$

[from Damour, Gourgoulhon, Grandclément, PRD **66**, 024007 (2002)]

Source of the discrepancy between the two numerical approaches

Recall : both EPBY and HKV method employ a conformally flat 3-metric, so this cannot be the reason why EPBY was far from post-Newtonian results.

Two main differences between *effective potential + Bowen-York (EPBY)* and *helical Killing vector (HKV)* approaches:

- Criterion for a circular orbit and determination of the orbital angular velocity Ω
- Extrinsic curvature of the $t = \mathrm{const}$ hypersurface

Is the discrepancy due to the determination of circular orbits ?

EPBY definition of circular orbit and Ω lacks of rigor, due to the ad hoc definition of the binding energy. This is unavoidable, due to the intrinsic **3-dimensional** character of EPBY :

no time in EPBY $\Rightarrow$ no well-defined velocity !

On the contrary HKV is intrinsically **4-dimensional**, and its definition of Ω is unambiguous.

However, despite these differences, it turns out that the two ways of determining Ω for circular orbits yield the same result

- for irrotational black holes with the Bowen-York extrinsic curvature (Shibata 2002).
- for a simple analytical model of a spherical shell of collisionless particles (Skoge & Baumgarte 2002 [PRD **66**, 107501 (2002)])

$\Rightarrow$ Main source of discrepancy: the extrinsic curvature

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Analysis of the extrinsic curvature

Comparison between EPBY and HKV extrinsic curvatures

- For effective potential + Bowen-York :

$$K^{ij}_{\rm EPBY} = \Psi^{-10} (\tilde{L} X_{\rm BY})^{ij}$$

- For helical Killing vector :

$$K^{ij}_{\rm HKV} = \frac{\Psi^{-4}}{2N} (\tilde{L}\beta)^{ij}$$

where $(\tilde{L}V)^{ij} := \tilde{D}^j V^i + \tilde{D}^i V^j - \frac{2}{3} \tilde{D}_k V^k \, \tilde{\gamma}^{ij}$, with for both cases $\tilde{\gamma}^{ij} = f^{ij}$

Is $K^{ij}_{\rm HKV}$ of Bowen-York type ?

$$K^{ij}_{\rm HKV} = K^{ij}_{\rm EPBY} \iff (\tilde{L}\beta)^{ij} = \frac{2N}{\Psi^6} (\tilde{L} X_{\rm BY})^{ij}$$

This could be possible if $N = \Psi^{-6}$ and $\beta^i = 2 X^i_{\rm BY}$

But β^i is determined so that the spatial coordinate lines follow the integral lines of the Killing vector; it is not freely choosable.

Comparison between EPBY and HKV extrinsic curvatures (con't)

Thus one can write

$$K^{ij}_{\mathrm{EPBY}} = K^{ij}_{\mathrm{HKV}} + B^{ij}_{\mathrm{TT}}$$

where B^{ij}_{TT} is a transverse-traceless quantity which can be seen as the amount of spurious gravitational waves in the EPBY data.

Analysis in the conformal thin-sandwich framework

Conformal thin-sandwich formulation of the initial value problem [York, PRL **82**, 1350 (1999)]: an alternative to the classical York-Lichnerowicz conformal decomposition:

$$\begin{aligned} K^{ij}_{\rm CTS} &= \frac{1}{2N}\left[(L\beta)^{ij} - u^{ij}\right] + \frac{1}{3}K\,\gamma^{ij} \\ &= \frac{\Psi^{-4}}{2N}\left[(\tilde{L}\beta)^{ij} - \tilde{u}^{ij}\right] + \frac{1}{3}K\,\gamma^{ij} \end{aligned}$$

where u^{ij} and $\tilde{u}^{ij}$ represent the evolution of the conformal 3-metric between two neighbouring slices: $u_{ij} := \gamma^{1/3}\partial_t(\gamma^{-1/3}\gamma_{ij}) = \partial_t\gamma_{ij} - \frac{1}{3}\gamma^{kl}\partial_t\gamma_{kl}\,\gamma_{ij}$ and $\tilde{u}_{ij} := \partial_t\tilde{\gamma}_{ij} = \Psi^{-4}u_{ij}$

$\tilde{u}^{ij}$ represents the freely specifiable (i.e. not determined by the constraint equations) part of K^{ij}, instead of $\tilde{A}^{ij}_{\rm TT}$ in the classical formulation. $\tilde{u}^{ij}$ has much more physical meaning than $\tilde{A}^{ij}_{\rm TT}$.
In particular, for a stationary spacetime : $\tilde{u}^{ij} = 0$ and for maximal slices $(K = 0)$:

$$K^{ij}_{\rm CTS} = K^{ij}_{\rm HKV}$$

Conclusion:

HKV extrinsic curvature is of the conformal thin-sandwich type, and not of the Bowen-York type

Asymptotic behavior of the extrinsic curvature

When $r \to \infty$, $N \sim 1 - \frac{M}{r}$, $\Psi \sim 1 + \frac{M}{2r}$, $K^{ij} = O\left(r^{-3}\right)$,
so that the HKV momentum constraint equation reduces to

$$\underline{\Delta}\beta^i + \frac{1}{3}\bar{\nabla}^i\bar{\nabla}_j\beta^j = O\left(r^{-5}\right)$$

Compare with the EPBY momentum constraint and conclude that

$$\beta^i \sim \beta^i_{\rm kin} + 2X^i_{\rm BY} \qquad \text{when} \quad r \to \infty$$

with the pure kinematical shift (corotating coordinates) $\beta^i_{\rm kin} := (-\Omega y, \Omega x, 0)$

Hence

$$\boxed{K^{ij}_{\rm HKV} \sim K^{ij}_{\rm EPBY} \qquad \text{when} \quad r \to \infty}$$

Asymptotic behavior of the shift vector

β^i can be split in three parts [Gourgoulhon, Grandclément & Bonazzola, PRD **65**, 044020 (2002)] :

$$\beta^i = \beta^i_{\rm kin} + \beta^i_{\rm angu} + \beta^i_{\rm quad}$$

involving 3 constant parameters : Ω, J and $\dot{I}_{xy}$:

- $\beta^i_{\rm kin} := (-\Omega y,\ \Omega x,\ 0) \ = \ O(r)$
- $\beta^i_{\rm angu} := \left(2J\frac{y}{r^3},\ -2J\frac{x}{r^3},\ 0\right) \ = \ O(r^{-2})$
- $\beta^i_{\rm quad} := \left(-\frac{3}{2}\dot{I}_{xy}\,\frac{y}{r^3}\left[1+\frac{x^2}{r^2}\right],\ -\frac{3}{2}\dot{I}_{xy}\,\frac{x}{r^3}\left[1+\frac{y^2}{r^2}\right],\ -\frac{3}{2}\dot{I}_{xy}\,\frac{xyz}{r^5}\right) \ = \ O(r^{-2})$

J to be identified to the total angular momentum and $\dot{I}_{xy}$ represents, at the Newtonian limit, the only non vanishing time derivative of the mass-quadrupole moment.

Corresponding decomposition of the extrinsic curvature

Since $K^{ij} \sim \frac{1}{2}(\bar{L}\beta)^{ij}$ when $r \to \infty$, the above decomposition of the shift vector induces

$$K^{ij} = K^{ij}_{\rm angu} + K^{ij}_{\rm quad}$$

with $K^{ij}_{\rm angu} := \frac{1}{2}(\bar{L}\beta_{\rm angu})^{ij}$, $K^{ij}_{\rm quad} := \frac{1}{2}(\bar{L}\beta_{\rm quad})^{ij}$ (notice $(\bar{L}\beta_{\rm kin})^{ij} = 0$)

$$K^{ij}_{\rm angu} = \frac{3J}{r^5} \begin{pmatrix} -2xy & x^2 - y^2 & -yz \\ & 2xy & xz \\ \text{sym.} & & 0 \end{pmatrix} = O(r^{-3})$$

$$K^{ij}_{\rm quad} = \frac{3\dot{I}_{xy}}{2r^5} \begin{pmatrix} xy(5x^2/r^2 - 1) & 5x^2y^2/r^2 - z^2 & yz(5x^2/r^2 + 1) \\ & xy(5y^2/r^2 - 1) & xz(5y^2/r^2 + 1) \\ \text{sym.} & & xy(5z^2/r^2 - 3) \end{pmatrix} = O(r^{-3})$$

Total angular momentum

York's definition as a surface integral at spatial infinity:

$$\mathcal{J}[K^{ij}] = \frac{1}{8\pi} \oint_\infty \left(K^i{}_j - K^k{}_k\, f^i{}_j\right) \left(\frac{\partial}{\partial\varphi}\right)^j dS_i$$

Plugin $K^{ij}_{\rm angu}$ and $K^{ij}_{\rm quad}$ results in

$$\boxed{\mathcal{J}[K^{ij}_{\rm angu}] = J} \quad \text{and} \quad \boxed{\mathcal{J}[K^{ij}_{\rm quad}] = 0}$$

Helical symmetry and asymptotic flatness

Dynamical Einstein equations: $\dfrac{\partial K^{ij}}{\partial t} - \pounds_{\boldsymbol{\beta}} K^{ij} = \cdots$

A direct computation shows that

$$\pounds_{\boldsymbol{\beta}} K^{ij}_{\rm angu} = O(r^{-4})$$

$$\pounds_{\boldsymbol{\beta}} K^{ij}_{\rm quad} = O(r^{-3})$$

- For axisymmetric stationary spacetimes, $K^{ij}_{\rm quad} = 0$ and we recover that $\pounds_{\boldsymbol{\beta}} K^{ij} = O(r^{-4})$

- For binary systems, $\pounds_{\boldsymbol{\beta}} K^{ij} = O(r^{-3})$ only and this is the source of the non-flatness problem

Conclusions and future prospects

- The classical Bowen-York extrinsic curvature does not represent well binary black holes in quasiequilibrium orbital motion

- This explains why the effective potential method employed by Cook and subsequent authors fails: it amounts to minimize the binding energy among a set of non-steady solutions

- The conformal thin sandwich formalism recently introduced by York seems a much better physical treatment of the initial value problem

- The helical Killing vector approach can be described in this formalism and results in very good agreement with post-Newtonian computations

- Next computational step: relaxing the conformal flatness hypothesis, while keeping the helical symmetry