# Evolution of 3+1 Einstein equations via a constrained scheme

Eric Gourgoulhon

Laboratoire de l'Univers et de ses Théories (LUTH)
CNRS / Observatoire de Paris
F-92195 Meudon, France

eric.gourgoulhon@obspm.fr

*based on collaboration with*
Silvano Bonazzola, Philippe Grandclément, José Luis Jaramillo,
François Limousin, Lap-Ming Lin, Jérôme Novak & Motoyuki Saijo

Department of Earth Science and Astronomy, University of Tokyo
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# Outline

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- Ó Murchadha & York (1974): Conformal transverse-traceless (CTT) method for solving the constraint equations
- Smarr & York (1978): Radiation gauge for numerical relativity: elliptic-hyperbolic system with asymptotic TT behavior
- York (1999): Conformal thin-sandwich (CTS) method for solving the constraint equations

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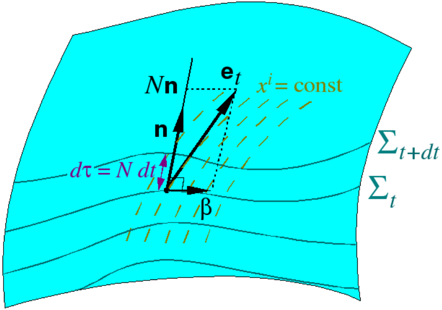
- Shibata & Nakamura (1995), Baumgarte & Shapiro (1999): *BSSN formulation:* conformal decomposition of the 3+1 equations and promotion of some connection function as an independent variable
- Shibata (2000): 3-D full computation of binary neutron star merger: *first full GR 3-D solution of the Cauchy problem of astrophysical interest*

# Outline

# 3+1 decomposition of spacetime

*Foliation of spacetime* by a family of spacelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$ ; on each hypersurface, pick a coordinate system $(x^i)_{i\in\{1,2,3\}} \Longrightarrow$
$(x^\mu)_{\mu\in\{0,1,2,3\}} = (t, x^1, x^2, x^3)$ = coordinate system on spacetime




$\boldsymbol{n}$ : future directed unit normal to $\Sigma_t$ :
$\boldsymbol{n} = -N\,\mathbf{d}t$, $N$ : lapse function
$\boldsymbol{e}_t = \partial/\partial t$ : time vector of the natural basis associated with the coordinates $(x^\mu)$

$N$ : lapse function
$\boldsymbol{\beta}$ : shift vector
$\Big\}\ \boldsymbol{e}_t = N\boldsymbol{n} + \boldsymbol{\beta}$

**Geometry of the hypersurfaces $\Sigma_t$:**

– induced metric $\boldsymbol{\gamma} = \boldsymbol{g} + \boldsymbol{n}\otimes\boldsymbol{n}$

– extrinsic curvature : $\boldsymbol{K} = -\frac{1}{2}\mathcal{L}_{\boldsymbol{n}}\boldsymbol{\gamma}$

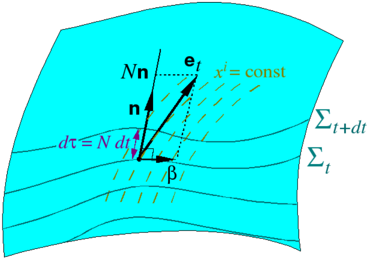
$$g_{\mu\nu}\,dx^\mu\,dx^\nu = -N^2\,dt^2 + \gamma_{ij}\,(dx^i + \beta^i dt)\,(dx^j + \beta^j\,dt)$$

# Choice of coordinates within the 3+1 formalism

$$(x^\mu) = (t, x^i) = (t, x^1, x^2, x^3)$$

Choice of the lapse function $N$ $\iff$ choice of the slicing $(\Sigma_t)$

Choice of the shift vector $\boldsymbol{\beta}$ $\iff$ choice of the spatial coordinates $(x^i)$ on each hypersurface $\Sigma_t$




A well-spread choice of slicing: *maximal slicing*: $K := \mathrm{tr}\,\boldsymbol{K} = 0$

[Lichnerowicz 1944]

# 3+1 decomposition of Einstein equation

**Orthogonal projection of Einstein equation onto $\Sigma_t$ and along the normal to $\Sigma_t$ :**

- Hamiltonian constraint: $R + K^2 - K_{ij}K^{ij} = 16\pi E$
- Momentum constraint : $D_j K^{ij} - D^i K = 8\pi J^i$
- Dynamical equations :
$$\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}} K_{ij} = -D_i D_j N + N \left[ R_{ij} - 2K_{ik}K^k_{\ j} + K K_{ij} + 4\pi((S-E)\gamma_{ij} - 2S_{ij}) \right]$$

$E := \boldsymbol{T}(\boldsymbol{n},\boldsymbol{n}) = T_{\mu\nu}\, n^\mu n^\nu$, $J_i := -\gamma_i^{\ \mu}\, T_{\mu\nu}\, n^\nu$, $S_{ij} := \gamma_i^{\ \mu}\, \gamma_j^{\ \nu}\, T_{\mu\nu}$, $S := S_i^{\ i}$

$D_i$ : covariant derivative associated with $\boldsymbol{\gamma}$, $R_{ij}$ : Ricci tensor of $D_i$, $R := R_i^{\ i}$

Kinematical relation between $\boldsymbol{\gamma}$ and $\boldsymbol{K}$: $\frac{\partial \gamma^{ij}}{\partial t} + D^i \beta^j + D^j \beta^i = 2N K^{ij}$

Resolution of Einstein equation $\equiv$ Cauchy problem

# Free vs. constrained evolution in 3+1 numerical relativity

Einstein equations split into

$$\begin{cases} \text{dynamical equations} & \dfrac{\partial}{\partial t} K_{ij} = \ldots \\ \text{Hamiltonian constraint} & R + K^2 - K_{ij} K^{ij} = 16\pi E \\ \text{momentum constraint} & D_j K_i^{\ j} - D_i K = 8\pi J_i \end{cases}$$

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# Free vs. constrained evolution in 3+1 numerical relativity

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  - fully constrained schemes: Evans (1989), Shapiro & Teukolsky (1992), Abrahams et al. (1994)
- **3-D computations (from mid 90's):** Almost all based on free evolution schemes: BSSN, symmetric hyperbolic formulations, etc...

  $\Longrightarrow$ **problem:** exponential growth of **constraint violating modes**

  [e.g. Frauendiener & Vogel, CQG **22**, 1769 (2005)]

# Attempts to suppress the constraint violating modes

- Constraint projection
  [Holst, Lindblom, Owen, Pfeiffer, Scheel & Kidder, PRD **70**, 084017 (2004)]
- Constraint-preserving boundary conditions
  [Kidder, Lindblom, Scheel, Buchman & Pfeiffer, PRD **71**, 064020 (2005)]
- Constraints as evolution equations
  [Gentle, George, Kheyfets & Miller, CQG **21**, 83 (2004)]
- Hamiltonian constraint as a parabolic equation ("Hamiltonian relaxation")
  [Marronetti, CQG **22**, 2433 (2005)]
- ...

... but the easiest way to get rid of the constraint violating modes would be to use a constrained scheme

# Why not using a constrained scheme ?

"Standard issue" **1** :

The constraints usually involve elliptic equations and 3-D elliptic solvers are CPU-time expensive !

# Cartesian vs. spherical coordinates in 3+1 numerical relativity

- **1-D and 2-D computations:** massive usage of spherical coordinates $(r, \theta, \varphi)$
- **3-D computations:** almost all based on Cartesian coordinates $(x, y, z)$, although spherical coordinates are better suited to study objects with spherical topology (black holes, neutron stars). Two exceptions:
  – Nakamura et al. (1987): evolution of pure gravitational wave spacetimes in spherical coordinates (but with Cartesian components of tensor fields)
  – Stark (1989): attempt to compute 3D stellar collapse in spherical coordinates

**"Standard issue" 2 :**

Spherical coordinates are singular at $r = 0$ and $\theta = 0$ or $\pi$ !

# "Standard issues" 1 and 2 can be overcome

## "Standard issues" 1 and 2 are neither *mathematical* nor *physical*

they are *technical* ones
$\Longrightarrow$ they can be overcome with appropriate techniques

## **Spectral methods** allow for

- an automatic treatment of the singularities of spherical coordinates (issue 2)
- **fast** 3-D elliptic solvers in spherical coordinates: 3-D Poisson equation reduced to a system of 1-D algebraic equations with banded matrices [Grandclément, Bonazzola, Gourgoulhon & Marck, J. Comp. Phys. **170**, 231 (2001)] (issue 1)

# Outline

# A new scheme for 3+1 numerical relativity

Constrained scheme built upon **maximal slicing** and **Dirac gauge**

[Bonazzola, Gourgoulhon, Grandclément & Novak, PRD **70**, 104007 (2004)]

# Conformal metric and dynamics of the gravitational field

## Dynamical degrees of freedom of the gravitational field:

York (1972) : they are carried by the conformal "metric"

$$\hat{\gamma}_{ij} := \gamma^{-1/3}\,\gamma_{ij} \qquad \text{with } \gamma := \det \gamma_{ij}$$

$\hat{\gamma}_{ij}$ = *tensor density* of weight $-2/3$

To work with *tensor fields* only, introduce an *extra structure* on $\Sigma_t$: a *flat metric* $\boldsymbol{f}$ such that $\frac{\partial f_{ij}}{\partial t} = 0$ and $\gamma_{ij} \sim f_{ij}$ at spatial infinity *(asymptotic flatness)*

Define $\tilde{\gamma}_{ij} := \Psi^{-4}\,\gamma_{ij}$ or $\gamma_{ij} =: \Psi^4\,\tilde{\gamma}_{ij}$ with $\Psi := \left(\frac{\gamma}{f}\right)^{1/12}$, $f := \det f_{ij}$

$\tilde{\gamma}_{ij}$ is invariant under any conformal transformation of $\gamma_{ij}$ and verifies $\det \tilde{\gamma}_{ij} = f$

*Notations:*
- $\tilde{\gamma}^{ij}$: inverse conformal metric : $\tilde{\gamma}_{ik}\,\tilde{\gamma}^{kj} = \delta_i^{\ j}$
- $\tilde{D}_i$ : covariant derivative associated with $\tilde{\gamma}_{ij}$, $\tilde{D}^i := \tilde{\gamma}^{ij}\,\tilde{D}_j$
- $\mathcal{D}_i$ : covariant derivative associated with $f_{ij}$, $\mathcal{D}^i := f^{ij}\,\mathcal{D}_j$

# Dirac gauge: definition

**Conformal decomposition** of the metric $\gamma_{ij}$ of the spacelike hypersurfaces $\Sigma_t$:

$$\gamma_{ij} =: \Psi^4 \, \tilde\gamma_{ij} \qquad \text{with} \qquad \tilde\gamma^{ij} =: f^{ij} + h^{ij}$$

where $f_{ij}$ is a flat metric on $\Sigma_t$, $h^{ij}$ a symmetric tensor and $\Psi$ a scalar field defined by $\Psi := \left(\frac{\det \gamma_{ij}}{\det f_{ij}}\right)^{1/12}$

Dirac gauge (Dirac, 1959) = *divergence-free* condition on $\tilde\gamma^{ij}$:

$$\mathcal{D}_j \tilde\gamma^{ij} = \mathcal{D}_j h^{ij} = 0$$

where $\mathcal{D}_j$ denotes the covariant derivative with respect to the flat metric $f_{ij}$.
Compare

- minimal distortion (Smarr & York 1978) : $D_j \left(\partial\tilde\gamma^{ij}/\partial t\right) = 0$
- pseudo-minimal distortion (Nakamura 1994) : $\mathcal{D}^j \left(\partial\tilde\gamma^{ij}/\partial t\right) = 0$

*Notice:* Dirac gauge $\Longleftrightarrow$ BSSN connection functions vanish: $\tilde\Gamma^i = 0$

# Dirac gauge: motivation

Expressing the Ricci tensor of conformal metric as a second order operator:
In terms of the covariant derivative $\mathcal{D}_i$ associated with the flat metric $\boldsymbol{f}$:

$$\tilde{\gamma}^{ik}\tilde{\gamma}^{jl}\tilde{R}_{kl} = \frac{1}{2}\left(\tilde{\gamma}^{kl}\mathcal{D}_k\mathcal{D}_l h^{ij} - \tilde{\gamma}^{ik}\mathcal{D}_k H^j - \tilde{\gamma}^{jk}\mathcal{D}_k H^i\right) + \mathcal{Q}(\tilde{\gamma}, \boldsymbol{\mathcal{D}}\tilde{\gamma})$$

with $H^i := \mathcal{D}_j h^{ij} = \mathcal{D}_j \tilde{\gamma}^{ij} = -\tilde{\gamma}^{kl}\Delta^i{}_{kl} = -\tilde{\gamma}^{kl}(\tilde{\Gamma}^i{}_{kl} - \bar{\Gamma}^i{}_{kl})$

and $\mathcal{Q}(\tilde{\gamma}, \boldsymbol{\mathcal{D}}\tilde{\gamma})$ is quadratic in first order derivatives $\boldsymbol{\mathcal{D}h}$

Dirac gauge: $H^i = 0 \Longrightarrow$ Ricci tensor becomes an elliptic operator for $h^{ij}$
Similar property as harmonic coordinates for the 4-dimensional Ricci tensor:

$${}^4R_{\alpha\beta} = -\frac{1}{2} g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} g_{\alpha\beta} + \text{quadratic terms}$$

# Dirac gauge: motivation (con't)

- spatial harmonic coordinates: $\mathcal{D}_j \left[ \left( \frac{\gamma}{f} \right)^{1/2} \gamma^{ij} \right] = 0$

  $\Longrightarrow$ makes the Ricci tensor $R_{ij}$ (associated with the **physical** 3-metric $\gamma_{ij}$) an elliptic operator for $\gamma^{ij}$ [Andersson & Moncrief, Ann. Henri Poincaré **4**, 1 (2003)]

- Dirac gauge: $\mathcal{D}_j \left[ \left( \frac{\gamma}{f} \right)^{1/3} \gamma^{ij} \right] = 0$

  $\Longrightarrow$ makes the Ricci tensor $\tilde{R}_{ij}$ (associated with the **conformal** 3-metric $\tilde{\gamma}_{ij}$) an elliptic operator for $\tilde{\gamma}^{ij}$

# Dirac gauge: discussion

- introduced by Dirac (1959) in order to fix the coordinates in some *Hamiltonian formulation* of general relativity; originally defined for Cartesian coordinates only: $\frac{\partial}{\partial x^j}\left(\gamma^{1/3}\,\gamma^{ij}\right) = 0$

  but trivially extended by us to more general type of coordinates (e.g. spherical) thanks to the introduction of the flat metric $f_{ij}$:
  $\mathcal{D}_j\left((\gamma/f)^{1/3}\gamma^{ij}\right) = 0$
- first discussed in the context of numerical relativity by Smarr & York (1978), as a candidate for a radiation gauge, but disregarded for not being covariant under coordinate transformation $(x^i) \mapsto (x^{i'})$ in the hypersurface $\Sigma_t$, contrary to the *minimal distortion gauge* proposed by them
- fully specifies (up to some boundary conditions) the coordinates in each hypersurface $\Sigma_t$, including the initial one $\Rightarrow$ allows for the search for *stationary solutions*
- Shibata, Uryu & Friedman [PRD **70**, 044044 (2004)] propose to use Dirac gauge to compute quasiequilibrium configurations of binary neutron stars beyond the IWM approximation

# Dirac gauge: discussion (con't)

Dirac gauge

- leads asymptotically to transverse-traceless (TT) coordinates (same as minimal distortion gauge). Both gauges are analogous to *Coulomb gauge* in electrodynamics
- turns the Ricci tensor of conformal metric $\tilde{\gamma}_{ij}$ into an elliptic operator for $h^{ij}$ $\Longrightarrow$ the dynamical Einstein equations become a *wave equation* for $h^{ij}$
- insures that the Ricci scalar $\tilde{R}$ (arising in the Hamiltonian constraint) does not contain any second order derivative of $h^{ij}$
- results in a *vector elliptic equation* for the shift: vector $\beta^i$
- is fulfilled by conformally flat initial data : $\tilde{\gamma}_{ij} = f_{ij} \Longrightarrow h^{ij} = 0$: this allows for the direct use of many currently available initial data sets

# Maximal slicing + Dirac gauge

Our choice of coordinates to solve numerically the Cauchy problem:

- choice of $\Sigma_t$ foliation: maximal slicing: $K := \mathrm{tr}\, \boldsymbol{K} = 0$
- choice of $(x^i)$ coordinates within $\Sigma_t$: Dirac gauge: $\mathcal{D}_j h^{ij} = 0$

*Note:* the Cauchy problem has been shown to be locally strongly well posed for a similar coordinate system, namely *constant mean curvature* $(K = t)$ and *spatial harmonic coordinates* $\left(\mathcal{D}_j \left[ (\gamma/f)^{1/2} \gamma^{ij} \right] = 0\right)$
[Andersson & Moncrief, Ann. Henri Poincaré **4**, 1 (2003)]

# 3+1 Einstein equations in maximal slicing + Dirac gauge

[Bonazzola, Gourgoulhon, Grandclément & Novak, PRD **70**, 104007 (2004)]

- 5 elliptic equations (4 constraints + $K = 0$ condition) ($\Delta := \mathcal{D}_k \mathcal{D}^k$):

$$\Delta N = \Psi^4 N \left[4\pi(E+S) + \tilde{A}_{kl} A^{kl}\right] - h^{kl}\mathcal{D}_k\mathcal{D}_l N - 2\tilde{D}_k \ln\Psi \, \tilde{D}^k N$$

$$\begin{aligned} \Delta(\Psi^2 N) &= \Psi^6 N \left(4\pi S + \frac{3}{4}\tilde{A}_{kl}A^{kl}\right) - h^{kl}\mathcal{D}_k\mathcal{D}_l(\Psi^2 N) \\ &+\Psi^2 \Bigg[ N \left( \frac{1}{16}\tilde{\gamma}^{kl}\mathcal{D}_k h^{ij}\mathcal{D}_l\tilde{\gamma}_{ij} - \frac{1}{8}\tilde{\gamma}^{kl}\mathcal{D}_k h^{ij}\mathcal{D}_j\tilde{\gamma}_{il} \right. \\ &\left. + 2\tilde{D}_k \ln\Psi \, \tilde{D}^k \ln\Psi \right) + 2\tilde{D}_k \ln\Psi \, \tilde{D}^k N \Bigg]. \end{aligned}$$

$$\begin{aligned} \Delta\beta^i + \frac{1}{3}\mathcal{D}^i\left(\mathcal{D}_j\beta^j\right) &= 2A^{ij}\mathcal{D}_j N + 16\pi N\Psi^4 J^i - 12 N A^{ij}\mathcal{D}_j \ln\Psi \\ &- 2\Delta^i{}_{kl} N A^{kl} - h^{kl}\mathcal{D}_k\mathcal{D}_l\beta^i - \frac{1}{3}h^{ik}\mathcal{D}_k\mathcal{D}_l\beta^l \end{aligned}$$

# 3+1 equations in maximal slicing + Dirac gauge (cont'd)

- Evolution equations:

$$\frac{\partial^2 h^{ij}}{\partial t^2} - \frac{N^2}{\psi^4}\Delta h^{ij} - 2\pounds_{\boldsymbol{\beta}}\frac{\partial h^{ij}}{\partial t} + \pounds_{\boldsymbol{\beta}}\pounds_{\boldsymbol{\beta}} h^{ij} = \mathcal{S}^{ij}$$

where $\mathcal{S}^{ij}$ is a complicated source which does not contain any second-order derivative of $h^{ij}$, except for the non-linear term $h^{kl}\mathcal{D}_k\mathcal{D}_l h^{ij}$.

These 6 equations, after taking into account the 3 Dirac conditions and the condition $\det\tilde{\gamma}_{ij} = \det f_{ij}$ are reduced to 2 scalar wave equations for two scalar potentials $\chi$ and $\mu$ :

$$-\frac{\partial^2 \chi}{\partial t^2} + \Delta\chi = S_\chi$$

$$-\frac{\partial^2 \mu}{\partial t^2} + \Delta\mu = S_\mu$$

# Reduction to 2 scalar wave equations

- **TT decomposition of $h^{ij}$:** $h^{ij} =: \bar{h}^{ij} + \frac{1}{2}\left(h\, f^{ij} - \mathcal{D}^i\mathcal{D}^j\phi\right)$

  where $h := f_{ij}h^{ij}$ and $\phi$ is solution of $\Delta\phi = h$
  $\bar{h}^{ij}$ is TT with respect to metric $f_{ij}$: $\mathcal{D}_j\bar{h}^{ij} = 0$ and $f_{ij}\bar{h}^{ij} = 0$
- **Expression of $\bar{h}^{ij}$ in terms of 2 potentials:** Components of $\bar{h}^{ij}$ with respect to a spherical $\boldsymbol{f}$-orthonormal frame:

  $$\bar{h}^{rr} = \frac{\chi}{r^2}, \quad \bar{h}^{r\theta} = \frac{1}{r}\left(\frac{\partial\eta}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial\mu}{\partial\phi}\right), \quad \bar{h}^{r\phi} = \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial\eta}{\partial\phi} + \frac{\partial\mu}{\partial\theta}\right), \text{ etc...}$$

  with $\Delta_{\theta\phi}\eta = -\partial\chi/\partial r - \chi/r$ (first Dirac gauge condition $\mathcal{D}_j\bar{h}^{rj} = 0$)
  The other two Dirac gauge conditions ($\mathcal{D}_j\bar{h}^{\theta j} = 0$ and $\mathcal{D}_j\bar{h}^{\phi j} = 0$) are used to compute $\bar{h}^{\theta\phi}$ and $\bar{h}^{\phi\phi}$.
  Finally the trace-free condition is used to get $\bar{h}^{\theta\theta}$.
- **Iterative computation of the trace $h$ to ensure** $\det\tilde{\gamma}_{ij} = \det f_{ij}$:

  $$\det\tilde{\gamma}_{ij} = \det f_{ij} \iff h = -h^{rr}h^{\theta\theta} - h^{rr}h^{\phi\phi} - h^{\theta\theta}h^{\phi\phi} + (h^{r\theta})^2 + (h^{r\phi})^2 + (h^{\theta\phi})^2 - h^{rr}h^{\theta\theta}h^{\phi\phi} - 2h^{r\theta}h^{r\phi}h^{\theta\phi} + h^{rr}(h^{\theta\phi})^2 + h^{\theta\theta}(h^{r\phi})^2 + h^{\phi\phi}(h^{r\theta})^2.$$

# The constrained scheme

Slice $\Sigma_t$ up to date

$\Downarrow$

Evolution of $\chi$ and $\mu$ to next time slice $\Sigma_{t+\delta t}$

$\Downarrow$

Deduce $\bar{h}^{ij}$ from $\chi$ and $\mu$ via the Dirac gauge and trace-free conditions

$\Downarrow$

Deduce the trace $h$ from $\det \tilde{\gamma}_{ij} = \det f_{ij}$
$\Rightarrow h^{ij}$ and $\tilde{\gamma}^{ij}$ on $\Sigma_{t+\delta t}$

$\Downarrow$

Solve iteratively the elliptic system for $N$, $\Psi^2 N$ and $\beta^i$ on $\Sigma_{t+\delta t}$

# Numerical implementation

Numerical code based on the C++ library LORENE (http://www.lorene.obspm.fr) with the following main features:

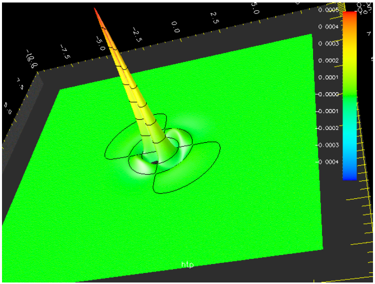
- multidomain spectral methods based on spherical coordinates $(r, \theta, \varphi)$, with compactified external domain ($\Longrightarrow$ spatial infinity included in the computational domain for elliptic equations)
- very efficient outgoing-wave boundary conditions, ensuring that all modes with spherical harmonics indices $\ell = 0$, $\ell = 1$ and $\ell = 2$ are perfectly outgoing
  [Novak & Bonazzola, J. Comp. Phys. **197**, 186 (2004)]
  (*recall:* Sommerfeld boundary condition works only for $\ell = 0$, which is too low for gravitational waves)

# Results on a pure gravitational wave spacetime

**Initial data:** similar to [Baumgarte & Shapiro, PRD **59**, 024007 (1998)], namely a momentarily static ($\partial\tilde{\gamma}^{ij}/\partial t = 0$) Teukolsky wave $\ell = 2$, $m = 2$:

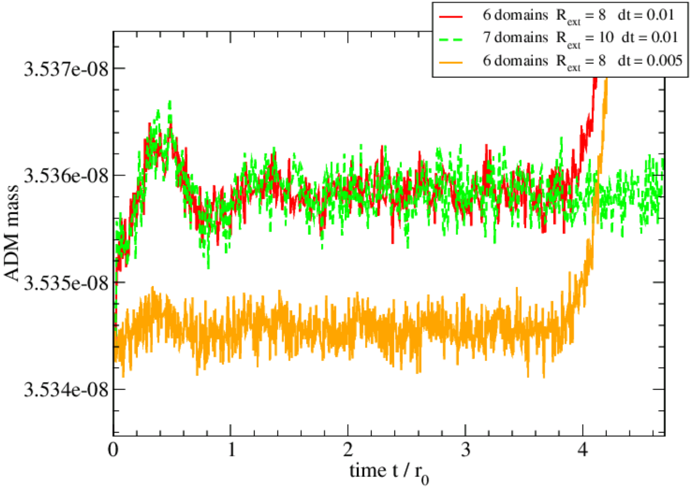
$$\begin{cases} \chi(t=0) = \dfrac{\chi_0}{2}\, r^2 \exp\left(-\dfrac{r^2}{r_0^2}\right) \sin^2\theta \, \sin 2\varphi \\ \mu(t=0) = 0 \end{cases} \quad \text{with} \quad \chi_0 = 10^{-3}$$

Preparation of the initial data by means of the *conformal thin sandwich* procedure




Evolution of $h^{\phi\phi}$ in the plane $\theta = \frac{\pi}{2}$

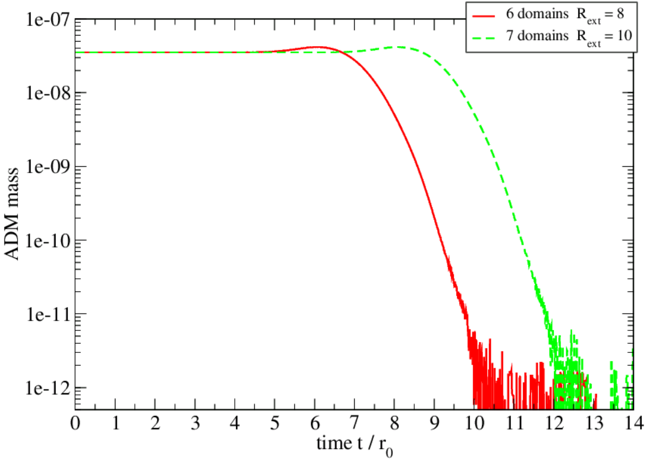
# Test: conservation of the ADM mass




Number of coefficients in each domain: $N_r = 17$, $N_\theta = 9$, $N_\varphi = 8$

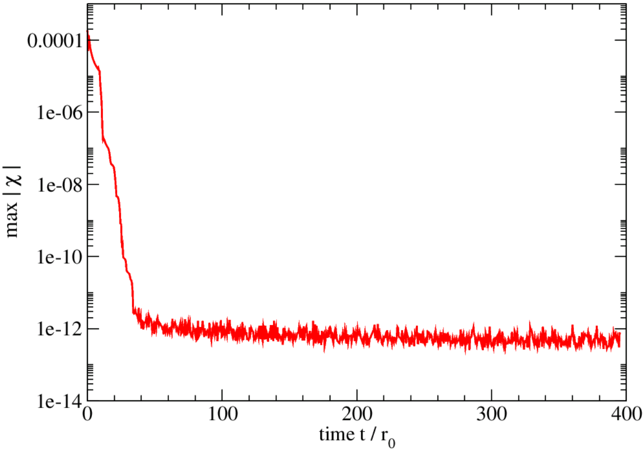
For $dt = 5\,10^{-3} r_0$, the ADM mass is conserved within a relative error lower than $10^{-4}$

# Late time evolution of the ADM mass




At $t > 10\, r_0$, the wave has completely left the computation domain
$\Longrightarrow$ Minkowski spacetime

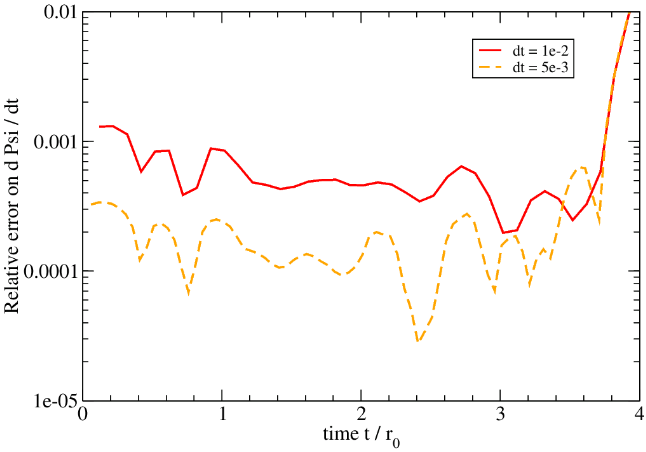
# Long term stability




Nothing happens until the run is switched off at $t = 400\, r_0$ !

# Another test: check of the $\frac{\partial \Psi}{\partial t}$ relation

The relation $\frac{\partial}{\partial t} \ln \Psi - \beta^k \mathcal{D}_k \ln \Psi = \frac{1}{6} \mathcal{D}_k \beta^k$ (trace of the definition of the extrinsic curvature as the time derivative of the spatial metric) is not enforced in our scheme $\Longrightarrow$ this provides an additional test:



# Outline

# Rigidly rotating neutron stars

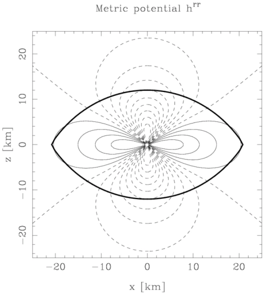
## Initial data for time evolution

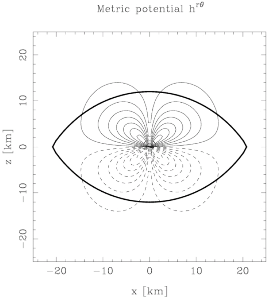
Stationary axisymmetric configurations within Dirac gauge and maximal slicing:
Equations are the same as in the dynamical case, with $\frac{\partial}{\partial t} \longrightarrow 0$

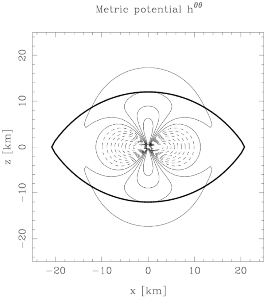
### Model considered here:

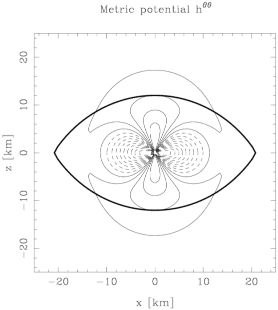
- EOS: polytropic: $\gamma = 2$
- central density: $\rho_c = 2.9\rho_{nuc}$
- maximum rotation rate (mass shedding limit)
- gravitational mass (ADM mass) : $M = 1.51\, M_\odot$
- baryon mass: $M_B = 1.60\, M_\odot$

# Rigidly rotating neutron stars




Conformal metric potentials $h^{ij}$

$h^{r\varphi} = h^{\theta\varphi} = 0$

for the other comp., $\max h^{ij} \sim 0.005$

Test: virial identities:

$\mathrm{GRV2} = 1.5\ 10^{-4}$, $\mathrm{GRV3} = 2.1\ 10^{-4}$

[Lin et al., in preparation]

# Rigidly rotating neutron stars

## Comparison with the quasi-isotropic gauge

Quasi-isotropic gauge:

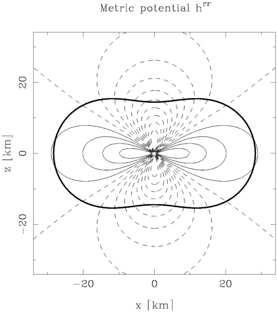
$$ds^2 = -N^2 dt^2 + A^2(dr^2 + r^2 d\theta^2) + B^2 r^2 \sin^2\theta (d\varphi + \beta^\varphi dt)^2$$

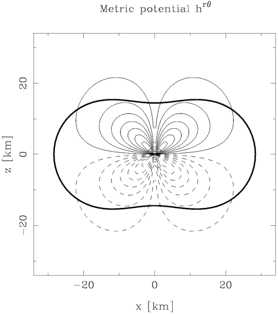
used in all rotating neutron stars studies,
see e.g. [Nozawa, Stergioulas, Gourgoulhon & Eriguchi, A&A Suppl. **132**, 431 (1998)]

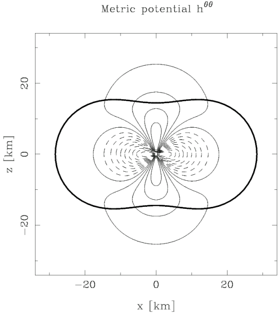
### Relative difference on global quantities

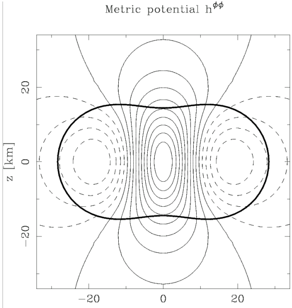
| | |
|---|---|
| $N(r=0)$ | $10^{-5}$ |
| $M$ | $10^{-4}$ |
| $M_{\rm B}$ | $10^{-4}$ |
| $R_{\rm circ}$ | $4\ 10^{-4}$ |
| $J$ | $3\ 10^{-4}$ |

# Differentially rotating neutron stars




Conformal metric potentials $h^{ij}$

Central lapse: $N_c = 0.59$

Test: virial identities:

$\mathrm{GRV2} = 6.0\ 10^{-5}$, $\mathrm{GRV3} = 1.7\ 10^{-4}$

[Saijo et al., in preparation]

# Outline

# Summary

- Dirac gauge + maximal slicing reduces the Einstein equations into a system of
  – two scalar elliptic equations (including the Hamiltonian constraint)
  – one vector elliptic equations (the momentum constraint)
  – two scalar wave equations (evolving the two dynamical degrees of freedom of the gravitational field)
- The usage of spherical coordinates and spherical components of tensor fields is crucial in reducing the dynamical Einstein equations to two scalar wave equations
- The unimodular character of the conformal metric ($\det \tilde{\gamma}_{ij} = \det f_{ij}$) is ensured in our scheme
- Easy extraction of gravitational radiation (asymp. TT)
- First numerical results show that Dirac gauge + maximal slicing seems a promising choice for stable evolutions of 3+1 Einstein equations and gravitational wave extraction

# Future prospects

- Quasiequilibrium configurations of binary neutron stars in Dirac gauge (K. Uryu & F. Limousin)
- Stellar core collapse ("Mariage des maillages" project) (J. Novak, H. Dimmelmeier & L.M. Lin)
- Evolving neutron star spacetimes
  - slow evolution (cf. [Schäfer & Gopakumar, PRD **69**, 021501(R) (2004)])
  - dynamical evolution
- Evolving black hole spacetimes