

# Black holes: from event horizons to trapping horizons

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**Laboratoire d'Astrophysique de Toulouse-Tarbes**

Toulouse, France  
10 December 2009

# Plan

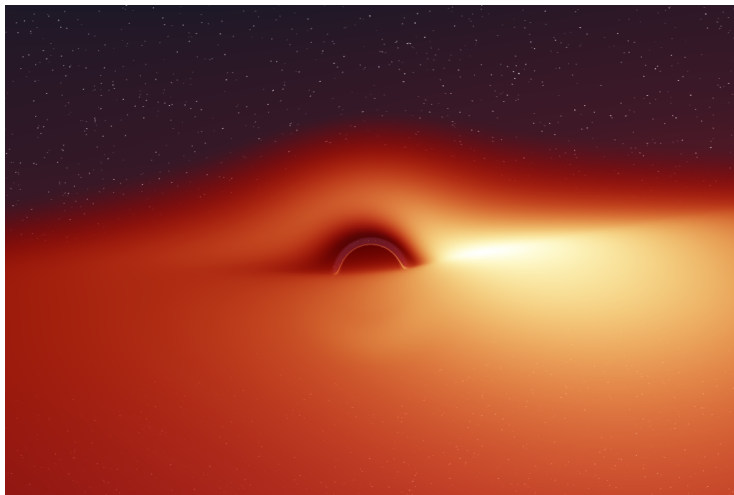
- 1 Concept of black hole and event horizon
- 2 Local approaches to black holes
- 3 Viscous fluid analogy
- 4 Angular momentum and area evolution laws
- 5 Applications to numerical relativity

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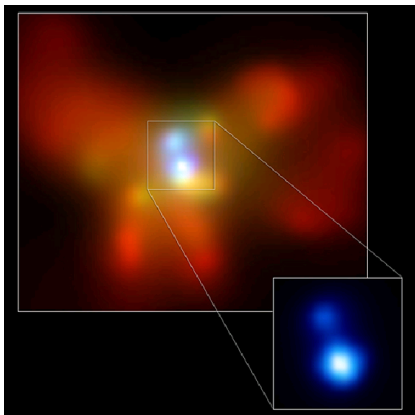
... for the astrophysicist: a very deep gravitational potential well



[J.A. Marck, CQG 13, 393 (1996)]

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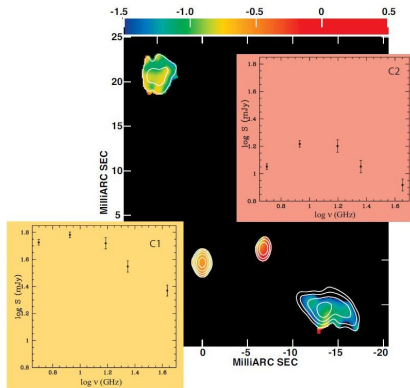
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Binary BH in galaxy NGC 6240

$d = 1.4$  kpc

[Komossa et al., ApJ 582, L15 (2003)]

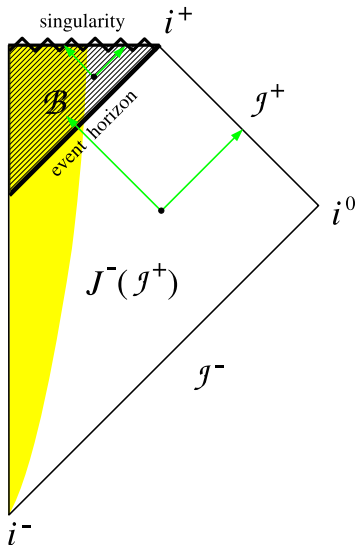


Binary BH in radio galaxy 0402+379

$d = 7.3$  pc

[Rodriguez et al., ApJ 646, 49 (2006)]

# What is a black hole ?



... for the mathematical physicist:

$$\mathcal{B} := \mathcal{M} - J^-(\mathcal{I}^+)$$

i.e. the region of spacetime where light rays cannot escape to infinity

- $(\mathcal{M}, g)$  = asymptotically flat manifold
- $\mathcal{I}^+$  = future null infinity
- $J^-(\mathcal{I}^+)$  = causal past of  $\mathcal{I}^+$

**event horizon:**  $\mathcal{H} := \partial J^-(\mathcal{I}^+)$   
(boundary of  $J^-(\mathcal{I}^+)$ )

$\mathcal{H}$  smooth  $\implies \mathcal{H}$  null hypersurface



# Drawbacks of the classical definition

- not applicable in **cosmology**, for in general  $(\mathcal{M}, g)$  is not asymptotically flat



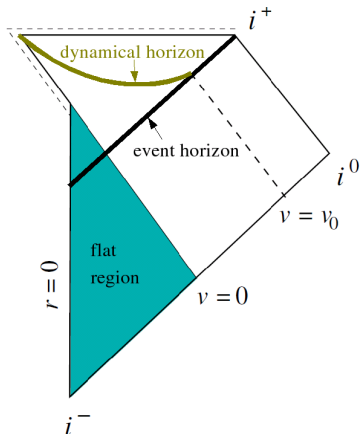
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Determination of  $J^-(\mathcal{I}^+)$  requires the knowledge of the entire future null infinity. Moreover this is *not locally linked with the notion of strong gravitational field*:



Example of event horizon in a **flat** region of spacetime:

Vaidya metric, describing incoming radiation from infinity:

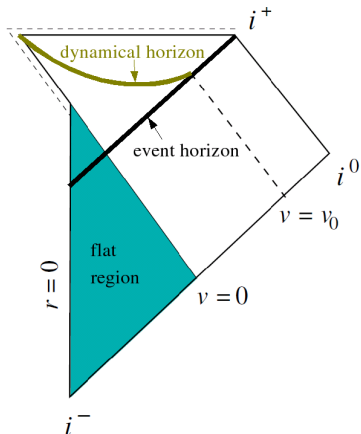
$$ds^2 = - \left( 1 - \frac{2m(v)}{r} \right) dv^2 + 2dv dr + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

$$\text{with } \begin{aligned} m(v) &= 0 & \text{for } v < 0 \\ dm/dv &> 0 & \text{for } 0 \leq v \leq v_0 \\ m(v) &= M_0 & \text{for } v > v_0 \end{aligned}$$

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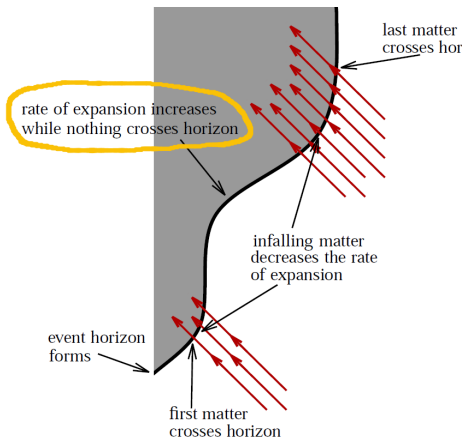
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$\Rightarrow$  no local physical experiment whatsoever can locate the event horizon

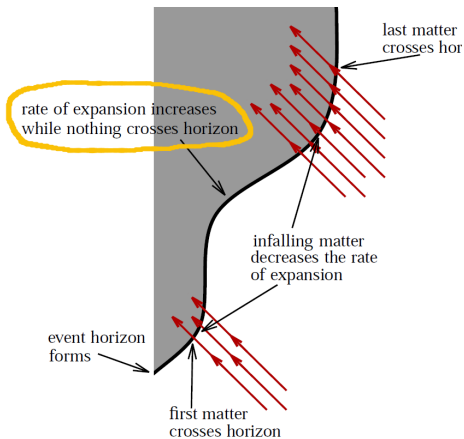
# Another non-local feature: teleological nature of event horizons



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[Booth, *Can. J. Phys.* **83**, 1073 (2005)]

# Another non-local feature: teleological nature of event horizons



The classical black hole boundary, i.e. the **event horizon**, responds in advance to what will happen in the future.

To deal with black holes as ordinary physical objects, a **local** definition would be desirable

→ quantum gravity, numerical relativity

[Booth, Can. J. Phys. **83**, 1073 (2005)]

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# Local characterizations of black holes

Recently a **new paradigm** has appeared in the theoretical approach of black holes: instead of *event horizons*, black holes are described by

- **trapping horizons** (Hayward 1994)
- **isolated horizons** (Ashtekar et al. 1999)
- **dynamical horizons** (Ashtekar and Krishnan 2002)
- **slowly evolving horizons** (Booth and Fairhurst 2004)

All these concepts are **local** and are based on the notion of **trapped surfaces**

# What is a trapped surface ?

1/ Expansion of a surface along a normal vector field

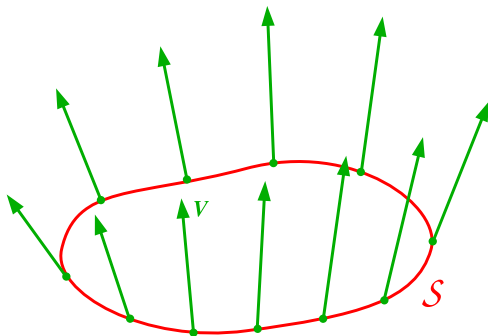
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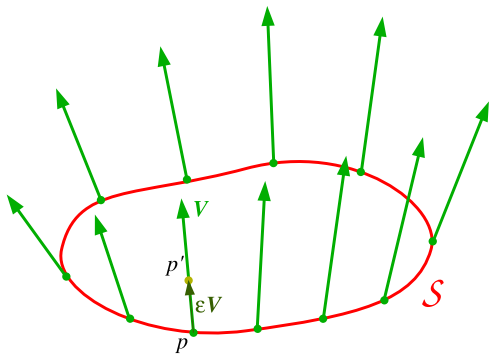
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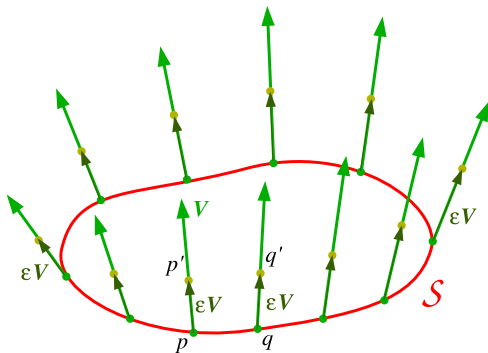
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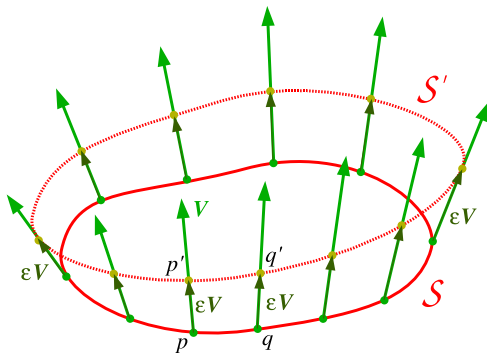
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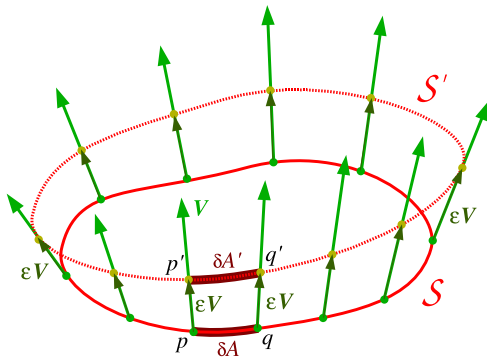
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At each point, the **expansion of  $S$  along  $v$**  is defined from the relative change in

the area element  $\delta A$ :

$$\theta^{(v)} := \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \frac{\delta A' - \delta A}{\delta A} = \mathcal{L}_v \ln \sqrt{q} = q^{\mu\nu} \nabla_\mu v_\nu$$

# What is a trapped surface ?

## 2/ The definition

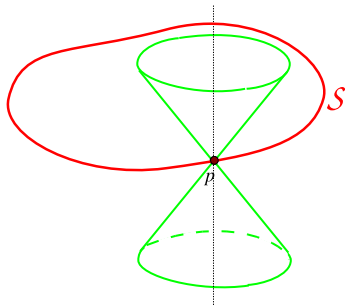
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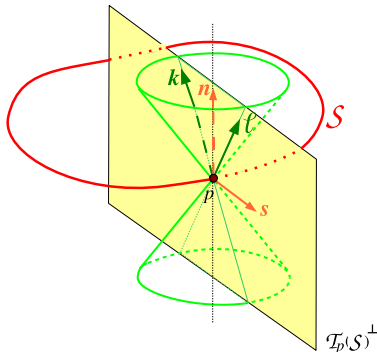


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$\ell$  = outgoing, expansion  $\theta^{(\ell)}$

$k$  = ingoing, expansion  $\theta^{(k)}$

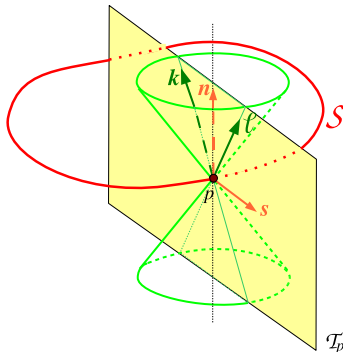
In flat space,  $\theta^{(k)} < 0$  and  $\theta^{(\ell)} > 0$



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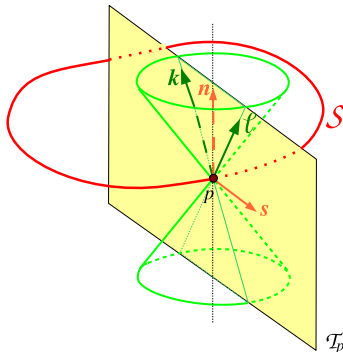
$\mathcal{S}$  is **marginally trapped**  $\iff \theta^{(k)} < 0$  and  $\theta^{(\ell)} = 0$

[Penrose 1965]

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*trapped surface* = **local** concept characterizing very strong gravitational fields

# Link with apparent horizons

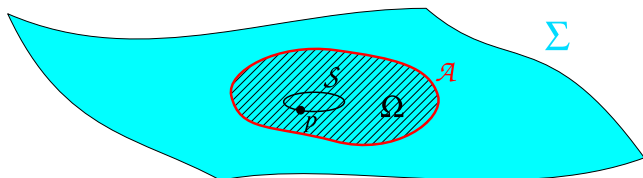
A closed spacelike 2-surface  $\mathcal{S}$  is said to be **outer trapped** (resp. **marginally outer trapped (MOTS)**) iff [Hawking & Ellis 1973]

- the notions of *interior* and *exterior* of  $\mathcal{S}$  can be defined (for instance spacetime asymptotically flat)  $\Rightarrow \ell$  is chosen to be the *outgoing* null normal and  $k$  to be the *ingoing* one
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$\Sigma$ : spacelike hypersurface extending to spatial infinity (Cauchy surface)

**outer trapped region** of  $\Sigma$ :  $\Omega =$  set of points  $p \in \Sigma$  through which there is an outer trapped surface  $\mathcal{S}$  lying in  $\Sigma$

**apparent horizon** in  $\Sigma$ :  $\mathcal{A} =$  connected component of the boundary of  $\Omega$

**Proposition** [Hawking & Ellis 1973]:  $\mathcal{A}$  smooth  $\implies \mathcal{A}$  is a MOTS

# Connection with singularities and black holes

*Proposition* [Penrose (1965)]:

provided that the weak energy condition holds,

$\exists$  a trapped surface  $\mathcal{S} \implies \exists$  a singularity in  $(\mathcal{M}, g)$  (in the form of a future inextendible null geodesic)

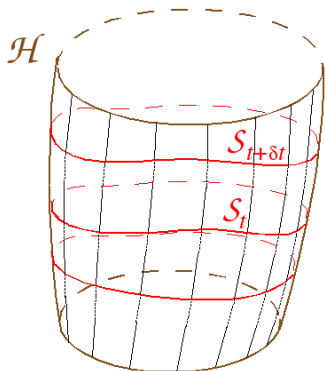
*Proposition* [Hawking & Ellis (1973)]:

provided that the cosmic censorship conjecture holds,

$\exists$  a trapped surface  $\mathcal{S} \implies \exists$  a black hole  $\mathcal{B}$  and  $\mathcal{S} \subset \mathcal{B}$

# Local definitions of “black holes”

A hypersurface  $\mathcal{H}$  of  $(\mathcal{M}, g)$  is said to be

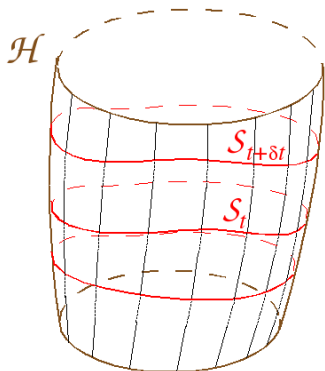


- a **future outer trapping horizon (FOTH)** iff
  - $\mathcal{H}$  foliated by marginally trapped 2-surfaces ( $\theta^{(k)} < 0$  and  $\theta^{(\ell)} = 0$ )
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[Hayward, PRD **49**, 6467 (1994)]

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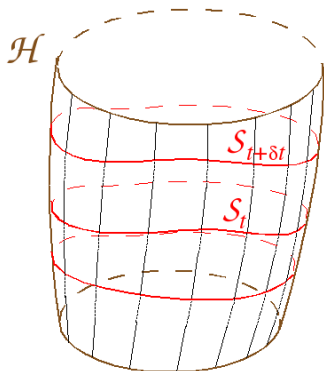
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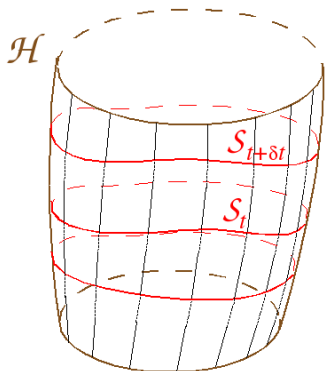


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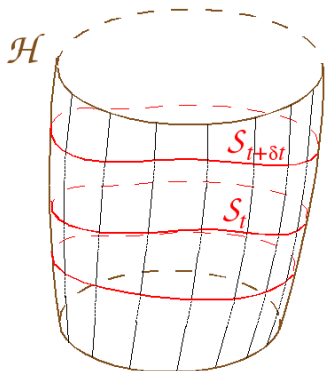
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BH in equilibrium = IH  
(e.g. Kerr)

BH out of equilibrium = DH  
generic BH = FOTH

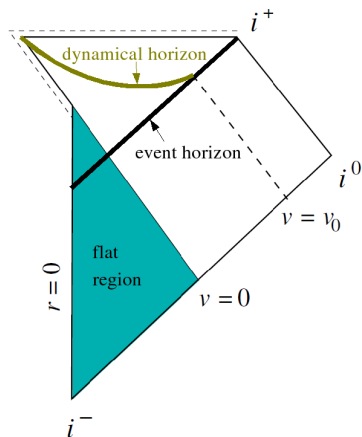
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# Example: Vaidya spacetime



- The **event horizon** crosses the flat region
- The **dynamical horizon** lies entirely outside the flat region

[Ashtekar & Krishnan, LRR 7, 10 (2004)]

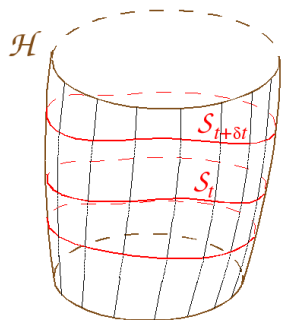
# Dynamics of these new horizons

The *trapping horizons* and *dynamical horizons* have their **own dynamics**, ruled by Einstein equations.

In particular, one can establish for them

- existence and (partial) uniqueness theorems  
[Andersson, Mars & Simon, PRL **95**, 111102 (2005)],  
[Ashtekar & Galloway, Adv. Theor. Math. Phys. **9**, 1 (2005)]
- first and second laws of black hole mechanics  
[Ashtekar & Krishnan, PRD **68**, 104030 (2003)], [Hayward, PRD **70**, 104027 (2004)]
- a viscous fluid bubble analogy (“membrane paradigm”, as for the event horizon)  
[EG, PRD **72**, 104007 (2005)], [EG & Jaramillo, PRD **74**, 087502 (2006)]

# Foliation of a hypersurface by spacelike 2-surfaces



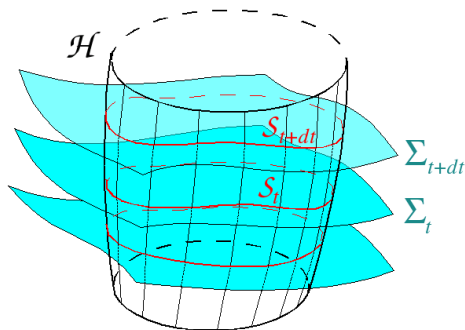
hypersurface  $\mathcal{H}$  = submanifold of spacetime  $(\mathcal{M}, g)$  of codimension 1

$\mathcal{H}$  can be  $\left\{ \begin{array}{l} \text{spacelike} \\ \text{null} \\ \text{timelike} \end{array} \right.$

$$\mathcal{H} = \bigcup_{t \in \mathbb{R}} S_t$$

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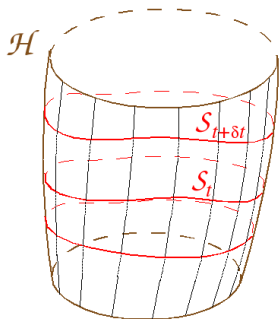
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$\Leftarrow$  3+1 perspective

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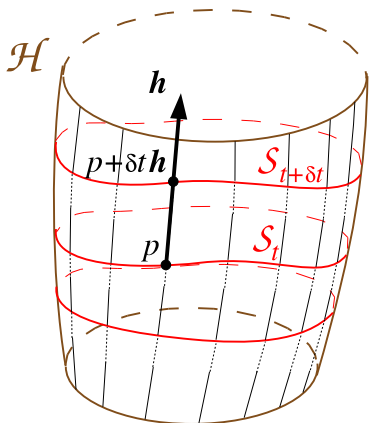
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**intrinsic viewpoint** adopted here (i.e. not relying on extra-structure such as a 3+1 foliation)

$q$  : induced metric on  $\mathcal{S}_t$  (positive definite)

$\mathcal{D}$  : connection associated with  $q$

# Evolution vector on the horizon



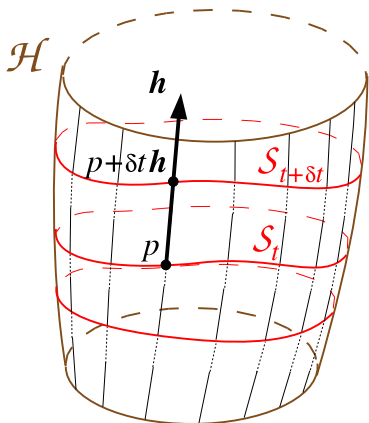
Vector field  $h$  on  $\mathcal{H}$  defined by

- (i)  $h$  is tangent to  $\mathcal{H}$
- (ii)  $h$  is orthogonal to  $\mathcal{S}_t$
- (iii)  $\mathcal{L}_h t = h^\mu \partial_\mu t = \langle dt, h \rangle = 1$

NB: (iii)  $\implies$  the 2-surfaces  $\mathcal{S}_t$  are **Lie-dragged** by  $h$



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Vector field  $\mathbf{h}$  on  $\mathcal{H}$  defined by

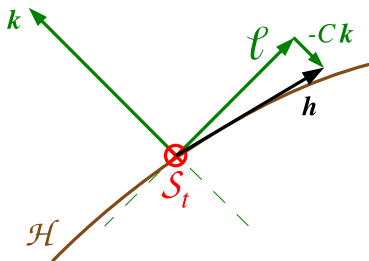
- (i)  $\mathbf{h}$  is tangent to  $\mathcal{H}$
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- (iii)  $\mathcal{L}_{\mathbf{h}} t = h^\mu \partial_\mu t = \langle dt, \mathbf{h} \rangle = 1$

NB: (iii)  $\implies$  the 2-surfaces  $\mathcal{S}_t$  are Lie-dragged by  $\mathbf{h}$

Define  $C := \frac{1}{2} \mathbf{h} \cdot \mathbf{h}$

$\mathcal{H}$ is spacelike	$\iff$	$C > 0$	$\iff$	$\mathbf{h}$ is spacelike
$\mathcal{H}$ is null	$\iff$	$C = 0$	$\iff$	$\mathbf{h}$ is null
$\mathcal{H}$ is timelike	$\iff$	$C < 0$	$\iff$	$\mathbf{h}$ is timelike.

# Normal null frame associated with the evolution vector



The foliation  $(S_t)_{t \in \mathbb{R}}$  entirely fixes the ambiguities in the choice of the null normal frame  $(\ell, k)$ , via the evolution vector  $h$ : there exists a **unique normal null frame**  $(\ell, k)$  such that

$$h = \ell - Ck \quad \text{and} \quad \ell \cdot k = -1$$

Normal fundamental form:  $\Omega^{(\ell)} := -k \cdot \nabla_{\bar{q}} \ell$  or  $\Omega_{\alpha}^{(\ell)} := -k_{\mu} \nabla_{\nu} \ell^{\mu} q^{\nu}_{\alpha}$

Evolution of  $h$  along itself:  $\nabla_h h = \kappa \ell + (C\kappa - \mathcal{L}_h C)k - \mathcal{D}C$

NB: null limit :  $C = 0, h = \ell \implies \nabla_{\ell} \ell = \kappa \ell \implies \kappa = \text{surface gravity}$

# Outline

- 1 Concept of black hole and event horizon
- 2 Local approaches to black holes
- 3 Viscous fluid analogy**
- 4 Angular momentum and area evolution laws
- 5 Applications to numerical relativity

# Concept of black hole viscosity

- **Hartle and Hawking (1972, 1973)**: introduced the concept of **black hole viscosity** when studying the response of the *event horizon* to external perturbations
- **Damour (1979)**: 2-dimensional **Navier-Stokes** like equation for the event horizon  $\implies$  *shear viscosity* and *bulk viscosity*
- **Thorne and Price (1986)**: **membrane paradigm** for black holes

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- **Thorne and Price (1986)**: **membrane paradigm** for black holes

Shall we restrict the analysis to the event horizon ?

Can we extend the concept of viscosity to the local characterizations of black hole recently introduced, i.e. **future outer trapping horizons** and **dynamical horizons** ?

**NB:** *event horizon* = null hypersurface  
*future outer trapping horizon* = null or spacelike hypersurface  
*dynamical horizon* = spacelike hypersurface

# Navier-Stokes equation in Newtonian fluid dynamics

$$\rho \left( \frac{\partial v^i}{\partial t} + v^j \nabla_j v^i \right) = -\nabla^i P + \mu \Delta v^i + \left( \zeta + \frac{\mu}{3} \right) \nabla^i (\nabla_j v^j) + f^i$$

or, in terms of fluid momentum density  $\pi_i := \rho v_i$ ,

$$\frac{\partial \pi_i}{\partial t} + v^j \nabla_j \pi_i + \theta \pi_i = -\nabla_i P + 2\mu \nabla^j \sigma_{ij} + \zeta \nabla_i \theta + f_i$$

where  $\theta$  is the fluid expansion:

$$\theta := \nabla_j v^j$$

and  $\sigma_{ij}$  the velocity shear tensor:

$$\sigma_{ij} := \frac{1}{2} (\nabla_i v_j + \nabla_j v_i) - \frac{1}{3} \theta \delta_{ij}$$

$P$  is the pressure,  $\mu$  the shear viscosity,  $\zeta$  the bulk viscosity and  $f_i$  the density of external forces

# Original Damour-Navier-Stokes equation

*Hyp*:  $\mathcal{H}$  = null hypersurface (particular case: black hole **event horizon**)

Then  $\mathbf{h} = \boldsymbol{\ell}$  ( $C = 0$ )

Damour (1979) has derived from **Einstein equation** the relation

$${}^S\mathcal{L}_{\boldsymbol{\ell}}\boldsymbol{\Omega}^{(\ell)} + \theta^{(\ell)}\boldsymbol{\Omega}^{(\ell)} = \mathcal{D}\kappa - \mathcal{D} \cdot \boldsymbol{\sigma}^{(\ell)} + \frac{1}{2}\mathcal{D}\theta^{(\ell)} + 8\pi\bar{q}^* \boldsymbol{T} \cdot \boldsymbol{\ell}$$

or equivalently

$${}^S\mathcal{L}_{\boldsymbol{\ell}}\boldsymbol{\pi} + \theta^{(\ell)}\boldsymbol{\pi} = -\mathcal{D}P + 2\mu\mathcal{D} \cdot \boldsymbol{\sigma}^{(\ell)} + \zeta\mathcal{D}\theta^{(\ell)} + \boldsymbol{f} \quad (*)$$

with  $\boldsymbol{\pi} := -\frac{1}{8\pi}\boldsymbol{\Omega}^{(\ell)}$  momentum surface density

$P := \frac{\kappa}{8\pi}$  pressure

$\mu := \frac{1}{16\pi}$  shear viscosity

$\zeta := -\frac{1}{16\pi}$  bulk viscosity

$\boldsymbol{f} := -\bar{q}^* \boldsymbol{T} \cdot \boldsymbol{\ell}$  external force surface density ( $\boldsymbol{T}$  = stress-energy tensor)

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(\*) is identical to a 2-dimensional Navier-Stokes equation



# Original Damour-Navier-Stokes equation (con't)

Introducing a coordinate system  $(t, x^1, x^2, x^3)$  such that

- $t$  is compatible with  $\ell$ :  $\mathcal{L}_\ell t = 1$
- $\mathcal{H}$  is defined by  $x^1 = \text{const}$ , so that  $x^a = (x^2, x^3)$  are coordinates spanning  $\mathcal{S}_t$

then

$$\ell = \frac{\partial}{\partial t} + V$$

with  $V$  tangent to  $\mathcal{S}_t$ : velocity of  $\mathcal{H}$ 's null generators with respect to the coordinates  $x^a$  [Damour, PRD 18, 3598 (1978)].

Then

$$\theta^{(\ell)} = \mathcal{D}_a V^a + \frac{\partial}{\partial t} \ln \sqrt{q} \quad q := \det q_{ab}$$

$$\sigma_{ab}^{(\ell)} = \frac{1}{2} (\mathcal{D}_a V_b + \mathcal{D}_b V_a) - \frac{1}{2} \theta^{(\ell)} q_{ab} + \frac{1}{2} \frac{\partial q_{ab}}{\partial t}$$

◀ compare

# Negative bulk viscosity of event horizons

From the Damour-Navier-Stokes equation,  $\zeta = -\frac{1}{16\pi} < 0$

This negative value would yield to a *dilation or contraction instability* in an ordinary fluid

It is in agreement with the tendency of a null hypersurface to continually contract or expand

The event horizon is stabilized by the **teleological condition** imposing its expansion to vanish in the far future (equilibrium state reached)

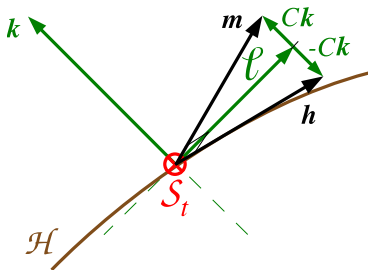
# Generalization to the non-null case

Starting remark: in the null case (event horizon),  $\ell$  plays two different roles:

- evolution vector along  $\mathcal{H}$  (e.g. term  ${}^S\mathcal{L}_\ell$ )
- normal to  $\mathcal{H}$  (e.g. term  $\bar{q}^*T \cdot \ell$ )

When  $\mathcal{H}$  is no longer null, these two roles have to be taken by two different vectors:

- **evolution vector**: obviously  $h$
- **vector normal to  $\mathcal{H}$** : a natural choice is  $m := \ell + Ck$



# Generalized Damour-Navier-Stokes equation

From the contracted Ricci identity applied to the vector  $m$  and projected onto  $\mathcal{S}_t$ :  $(\nabla_\mu \nabla_\nu m^\mu - \nabla_\nu \nabla_\mu m^\mu) q^\nu{}_\alpha = R_{\mu\nu} m^\mu q^\nu{}_\alpha$  and using Einstein equation to express  $R_{\mu\nu}$ , one gets an evolution equation for  $\Omega^{(\ell)}$  along  $\mathcal{H}$ :

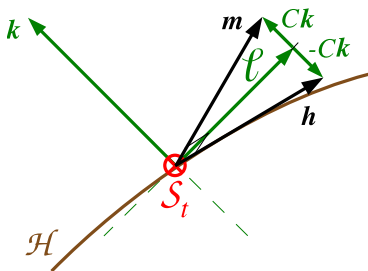
$$\mathcal{L}_h \Omega^{(\ell)} + \theta^{(h)} \Omega^{(\ell)} = \mathcal{D}\kappa - \mathcal{D} \cdot \sigma^{(m)} + \frac{1}{2} \mathcal{D}\theta^{(m)} - \theta^{(k)} \mathcal{D}C + 8\pi \vec{q}^* T \cdot m$$

- $\Omega^{(\ell)}$  : normal fundamental form of  $\mathcal{S}_t$  associated with null normal  $\ell$
- $\theta^{(h)}$ ,  $\theta^{(m)}$  and  $\theta^{(k)}$ : expansion scalars of  $\mathcal{S}_t$  along the vectors  $h$ ,  $m$  and  $k$  respectively
- $\mathcal{D}$  : covariant derivative within  $(\mathcal{S}_t, q)$
- $\kappa$  : component of  $\nabla_h h$  along  $\ell$
- $\sigma^{(m)}$  : shear tensor of  $\mathcal{S}_t$  along the vector  $m$
- $C$  : half the scalar square of  $h$

# Null limit (event horizon)

If  $\mathcal{H}$  is a null hypersurface,

$$h = m = \ell \quad \text{and} \quad C = 0$$



and we recover the original Damour-Navier-Stokes equation:

$${}^S\mathcal{L}_\ell \Omega^{(\ell)} + \theta^{(\ell)} \Omega^{(\ell)} = \mathcal{D}_\kappa - \mathcal{D} \cdot \sigma^{(\ell)} + \frac{1}{2} \mathcal{D} \theta^{(\ell)} + 8\pi \bar{q}^* T \cdot \ell$$

# Case of future trapping horizons

Definition [Hayward, PRD 49, 6467 (1994)] :

$\mathcal{H}$  is a **future trapping horizon** iff  $\theta^{(\ell)} = 0$  and  $\theta^{(k)} < 0$ .

The generalized Damour-Navier-Stokes equation reduces then to

$${}^S\mathcal{L}_h \Omega^{(\ell)} + \theta^{(h)} \Omega^{(\ell)} = \mathcal{D}\kappa - \mathcal{D} \cdot \sigma^{(m)} - \frac{1}{2} \mathcal{D}\theta^{(h)} - \theta^{(k)} \mathcal{D}C + 8\pi \bar{q}^* T \cdot m$$

[EG, PRD 72, 104007 (2005)]

*NB*: Notice the change of sign in the  $-\frac{1}{2} \mathcal{D}\theta^{(h)}$  term with respect to the original Damour-Navier-Stokes equation

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The explanation: it is  $\theta^{(m)}$  which appears in the general equation and

$$\theta^{(m)} + \theta^{(h)} = 2\theta^{(\ell)} \implies \begin{cases} \text{event horizon } (m = h) & : \theta^{(m)} = \theta^{(\ell)} \\ \text{trapping horizon } (\theta^{(\ell)} = 0) & : \theta^{(m)} = -\theta^{(h)} \end{cases}$$

## Viscous fluid form

$${}^S\mathcal{L}_h \pi + \theta^{(h)} \pi = -\mathcal{D}P + \frac{1}{8\pi} \mathcal{D} \cdot \sigma^{(m)} + \zeta \mathcal{D}\theta^{(h)} + f$$

with  $\pi := -\frac{1}{8\pi} \Omega^{(\ell)}$  momentum surface density

$P := \frac{\kappa}{8\pi}$  pressure

$\frac{1}{8\pi} \sigma^{(m)}$  shear stress tensor

$\zeta := \frac{1}{16\pi}$  bulk viscosity

$f := -\bar{q}^* T \cdot m + \frac{\theta^{(k)}}{8\pi} \mathcal{D}C$  external force surface density

Similar to the Damour-Navier-Stokes equation for an event horizon, except

- the **Newtonian-fluid** relation between *stress* and *strain* does not hold:  $\sigma^{(m)}/8\pi \neq 2\mu\sigma^{(h)}$ , rather  $\sigma^{(m)}/8\pi = [\sigma^{(h)} + 2C\sigma^{(k)}]/8\pi$



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- **positive bulk viscosity**

This positive value of bulk viscosity shows that FOTHs and DHs behave as “ordinary” physical objects, in perfect agreement with their **local nature**

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# Angular momentum of trapping horizons

**Definition** [Booth & Fairhurst, CQG 22, 4545 (2005)]: Let  $\varphi$  be a vector field on  $\mathcal{H}$  which

- is tangent to  $\mathcal{S}_t$
- has closed orbits
- has vanishing divergence with respect to the induced metric:  $\mathcal{D} \cdot \varphi = 0$   
(weaker than being a Killing vector of  $(\mathcal{S}_t, \mathbf{q})$  !)

For dynamical horizons,  $\theta^{(h)} \neq 0$  and there is a unique choice of  $\varphi$  as the generator (conveniently normalized) of the curves of constant  $\theta^{(h)}$

[Hayward, PRD 74, 104013 (2006)]

The *generalized angular momentum associated with  $\varphi$*  is then defined by

$$J(\varphi) := -\frac{1}{8\pi} \oint_{\mathcal{S}_t} \langle \Omega^{(\ell)}, \varphi \rangle s_{\epsilon},$$

**Remark 1:** does not depend upon the choice of null vector  $\ell$ , thanks to the divergence-free property of  $\varphi$

**Remark 2:**

- coincides with **Ashtekar & Krishnan's** definition for a dynamical horizon
- coincides with **Brown-York** angular momentum if  $\mathcal{H}$  is timelike and  $\varphi$  a Killing vector

# Angular momentum flux law

Under the supplementary hypothesis that  $\varphi$  is transported along the evolution vector  $\mathbf{h}$  :  $\mathcal{L}_{\mathbf{h}} \varphi = 0$ , the generalized Damour-Navier-Stokes equation leads to

$$\frac{d}{dt} J(\varphi) = - \oint_{S_t} \mathbf{T}(m, \varphi) \cdot \mathbf{s}_\epsilon - \frac{1}{16\pi} \oint_{S_t} \left[ \boldsymbol{\sigma}^{(m)} : \mathcal{L}_\varphi \mathbf{q} - 2\theta^{(k)} \varphi \cdot \mathcal{D}\mathbf{C} \right] \cdot \mathbf{s}_\epsilon$$

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Two interesting limiting cases:

- $\mathcal{H}$  = null hypersurface :  $C = 0$  and  $\mathbf{m} = \boldsymbol{\ell}$  :

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i.e. Eq. (6.134) of the *Membrane Paradigm* book (Thorne, Price & MacDonald 1986)

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- $\mathcal{H}$  = future trapping horizon :

$$\frac{d}{dt} J(\varphi) = - \oint_{S_t} \mathbf{T}(\mathbf{m}, \varphi) \cdot \mathbf{s}_\epsilon - \frac{1}{16\pi} \oint_{S_t} \boldsymbol{\sigma}^{(m)} : \mathcal{L}_\varphi \mathbf{q} \cdot \mathbf{s}_\epsilon$$

# Area evolution law for an event horizon

$A(t)$  : area of the 2-surface  $\mathcal{S}_t$ ;  ${}^s\epsilon$  : volume element of  $\mathcal{S}_t$ ;  $\bar{\kappa}(t) := \frac{1}{A(t)} \int_{\mathcal{S}_t} \kappa {}^s\epsilon$

Integrating the null Raychaudhuri equation on  $\mathcal{S}_t$ , one gets

$$\frac{d^2 A}{dt^2} - \bar{\kappa} \frac{dA}{dt} = - \int_{\mathcal{S}_t} \left[ 8\pi T(\ell, \ell) + \sigma^{(\ell)} : \sigma^{(\ell)} - \frac{(\theta^{(\ell)})^2}{2} + (\bar{\kappa} - \kappa)\theta^{(\ell)} \right] {}^s\epsilon \quad (1)$$

[Damour, 1979]



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Simplified analysis : assume  $\bar{\kappa} = \text{const} > 0$  :

- Cauchy problem  $\implies$  diverging solution of the homogeneous equation:

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- correct treatment: impose  $\frac{dA}{dt} = 0$  at  $t = +\infty$  (teleological !)

$$\frac{dA}{dt} = \int_t^{+\infty} D(u) e^{\bar{\kappa}(t-u)} du \quad D(t) : \text{r.h.s. of Eq. (1)}$$

Non causal evolution

# Area evolution law for a dynamical horizon

Dynamical horizon :  $C > 0$ ;  $\kappa' := \kappa - \mathcal{L}_h \ln C$ ;  $\bar{\kappa}'(t) := \frac{1}{A(t)} \int_{S_t} \kappa' s_\epsilon$

From the  $(\mathbf{m}, \mathbf{h})$  component of Einstein equation, one gets

$$\frac{d^2 A}{dt^2} + \bar{\kappa}' \frac{dA}{dt} = \int_{S_t} \left[ 8\pi T(\mathbf{m}, \mathbf{h}) + \sigma^{(h)} : \sigma^{(m)} + \frac{(\theta^{(h)})^2}{2} + (\bar{\kappa}' - \kappa')\theta^{(h)} \right] s_\epsilon \quad (2)$$

[EG & Jaramillo, PRD **74**, 087502 (2006)]

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[EG & Jaramillo, PRD **74**, 087502 (2006)]

Simplified analysis : assume  $\bar{\kappa}' = \text{const} > 0$

(OK for small departure from equilibrium [Booth & Fairhurst, PRL **92**, 011102 (2004)]):

Standard Cauchy problem :

$$\frac{dA}{dt} = \frac{dA}{dt} \Big|_{t=0} + \int_0^t D(u) e^{\bar{\kappa}'(u-t)} du \quad D(t) : \text{r.h.s. of Eq. (2)}$$

Causal evolution, in agreement with local nature of dynamical horizons

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# Applications to numerical relativity

- **Initial data:** isolated horizons (helical symmetry)
  - [EG, Grandclément & Bonazzola, PRD **65**, 044020 (2002)]
  - [Grandclément, EG & Bonazzola, PRD **65**, 044021 (2002)]
  - [Cook & Pfeiffer, PRD **70**, 104016 (2004)]
  
- **A posteriori analysis:** estimating mass, linear and angular momentum of formed black holes
  - [Schnetter, Krishnan & Beyer, PRD **74**, 024028 (2006)]
  - [Cook & Whiting, PRD **76**, 041501 (2007)]
  - [Krishnan, Lousto & Zlochower, PRD **76**, 081501(R) (2007)]
  
- **Numerical construction of spacetime:** inner boundary conditions for a constrained scheme with “black hole excision”
  - [Jaramillo, EG, Cordero-Carrión, & J.M. Ibáñez, PRD **77**, 047501 (2008)]
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