Full constrained formulations of Einstein's equations

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based on a collaboration with

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Numerical modelling of astrophysical sources of gravitational radiation
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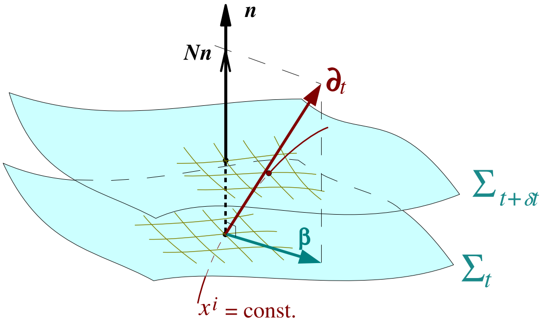
Plan

Outline

1. Constrained and free evolution schemes for 3+1 Einstein equations
2. The Meudon-Valencia FCF scheme
3. Extended CFC approximation
4. Conclusion

3+1 foliation of spacetime



Spacetime $(\mathscr{M}, \boldsymbol{g})$ assumed to be **globally hyperbolic**: ∃ a **foliation** (or **slicing**) of the spacetime manifold $\mathscr{M}$ by a family of spacelike hypersurfaces Σ_t :

$$\mathscr{M} = \bigcup_{t\in\mathbb{R}} \Sigma_t$$

$n_\alpha = -N\nabla_\alpha t$

$\boldsymbol{n}$: unit normal to Σ_t

N : lapse function

shift vector $\boldsymbol{\beta}$: $\boldsymbol{\partial}_t = N\boldsymbol{n} + \boldsymbol{\beta}$

Metric tensor in terms of lapse and shift :

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -N^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

3+1 Einstein system

Thanks to the Gauss, Codazzi and Ricci equations, the Einstein equation

$$ {}^4R_{\alpha\beta} - \frac{1}{2}\,{}^4R\, g_{\alpha\beta} = 8\pi T_{\alpha\beta} $$

is equivalent to the system

- $\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}}\right)\gamma_{ij} = -2NK_{ij}$ kinematical relation $\boldsymbol{K} = -\frac{1}{2}\boldsymbol{\mathcal{L}_n}\,\boldsymbol{\gamma}$
- $\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}}\right)K_{ij} = -D_iD_jN + N\Big\{R_{ij} + KK_{ij} - 2K_{ik}K^k_{\ \,j}$ $+4\pi\left[(S-E)\gamma_{ij} - 2S_{ij}\right]\Big\}$ dynamical part of Einstein equation
- $R + K^2 - K_{ij}K^{ij} = 16\pi E$ Hamiltonian constraint
- $D_jK^j_{\ \,i} - D_iK = 8\pi p_i$ momentum constraint

$$ T_{\alpha\beta} = S_{\alpha\beta} + n_\alpha p_\beta + p_\alpha n_\beta + E n_\alpha n_\beta $$

The full PDE system

Supplementary equations:

$$D_i D_j N = \frac{\partial^2 N}{\partial x^i \partial x^j} - \Gamma^k_{\ ij} \frac{\partial N}{\partial x^k}$$

$$D_j K^j_{\ i} = \frac{\partial K^j_{\ i}}{\partial x^j} + \Gamma^j_{\ jk} K^k_{\ i} - \Gamma^k_{\ ji} K^j_{\ k}$$

$$D_i K = \frac{\partial K}{\partial x^i}$$

$$\mathcal{L}_{\boldsymbol{\beta}}\, \gamma_{ij} = \frac{\partial \beta_i}{\partial x^j} + \frac{\partial \beta_j}{\partial x^i} - 2\Gamma^k_{\ ij} \beta_k$$

$$\mathcal{L}_{\boldsymbol{\beta}}\, K_{ij} = \beta^k \frac{\partial K_{ij}}{\partial x^k} + K_{kj} \frac{\partial \beta^k}{\partial x^i} + K_{ik} \frac{\partial \beta^k}{\partial x^j}$$

$$R_{ij} = \frac{\partial \Gamma^k_{\ ij}}{\partial x^k} - \frac{\partial \Gamma^k_{\ ik}}{\partial x^j} + \Gamma^k_{\ ij} \Gamma^l_{\ kl} - \Gamma^l_{\ ik} \Gamma^k_{\ lj}$$

$$R = \gamma^{ij} R_{ij}$$

$$\Gamma^k_{\ ij} = \frac{1}{2} \gamma^{kl} \left(\frac{\partial \gamma_{lj}}{\partial x^i} + \frac{\partial \gamma_{il}}{\partial x^j} - \frac{\partial \gamma_{ij}}{\partial x^l} \right)$$

A few words of history about the 3+1 Einstein system

- G. Darmois (1927): 3+1 Einstein equations in terms of (γ_{ij}, K_{ij}) with $N = 1$ and $\boldsymbol{\beta} = 0$ (Gaussian normal coordinates)
- A. Lichnerowicz (1939) : $N \neq 1$ and $\boldsymbol{\beta} = 0$ (normal coordinates)
- Y. Choquet-Bruhat (1948) : $N \neq 1$ and $\boldsymbol{\beta} \neq 0$ (general coordinates)
- R. Arnowitt, S. Deser & C.W. Misner (1962) : *Hamiltonian formulation* of GR based on a 3+1 decomposition in terms of (γ_{ij}, π^{ij})
  NB: spatial projection of *Einstein tensor* instead of *Ricci tensor* in previous works
- J. Wheeler (1964) : coined the terms *lapse* and *shift*
- J.W. York (1979) : modern 3+1 decomposition based on spatial projection of *Ricci tensor*

The Cauchy problem

The first two equations of the 3+1 Einstein system can be put in the form of a **Cauchy problem:**

$$\frac{\partial^2 \gamma_{ij}}{\partial t^2} = F_{ij}\left(\gamma_{kl}, \frac{\partial \gamma_{kl}}{\partial x^m}, \frac{\partial \gamma_{kl}}{\partial t}, \frac{\partial^2 \gamma_{kl}}{\partial x^m \partial x^n}\right) \quad (1)$$

Cauchy problem: given initial data at $t = 0$: γ_{ij} and $\frac{\partial \gamma_{ij}}{\partial t}$, find a solution for $t > 0$

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Cauchy problem: given initial data at $t = 0$: γ_{ij} and $\frac{\partial \gamma_{ij}}{\partial t}$, find a solution for $t > 0$

But this Cauchy problem is subject to the constraints

- $R + K^2 - K_{ij}K^{ij} = 16\pi E$ Hamiltonian constraint
- $D_j K^j{}_i - D_i K = 8\pi p_i$ momentum constraint

Preservation of the constraints

Thanks to the Bianchi identities, it can be shown that if the constraints are satisfied at $t = 0$, they are preserved by the evolution system (1), provided that $\nabla_\beta T^{\alpha\beta} = 0$ is maintained

Existence and uniqueness of solutions

Question:

Given a set $(\Sigma_0, \gamma, \boldsymbol{K}, E, \boldsymbol{p})$, where

Σ_0 is a three-dimensional manifold,

γ a Riemannian metric on Σ_0,

$\boldsymbol{K}$ a symmetric bilinear form field on Σ_0,

E a scalar field on Σ_0

$\boldsymbol{p}$ a 1-form field on Σ_0,

which obeys the constraint equations, does there exist a spacetime $(\mathscr{M}, \boldsymbol{g}, \boldsymbol{T})$ such that $(\boldsymbol{g}, \boldsymbol{T})$ fulfills Einstein equation and Σ_0 can be embedded as an hypersurface of $\mathscr{M}$ with induced metric γ and extrinsic curvature $\boldsymbol{K}$?

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Answer:

- the solution exists and is unique in a vicinity of Σ_0 for **analytic** initial data (Cauchy-Kovalevskaya theorem) [Darmois (1927)], [Lichnerowicz (1939)]
- the solution exists and is unique in a vicinity of Σ_0 for **generic** (i.e. smooth) initial data [Choquet-Bruhat (1952)]
- there exists a unique maximal solution [Choquet-Bruhat & Geroch (1969)]

Free vs. constrained evolution schemes

Taking into account the *constraint preservation property*, various schemes can be contemplated[1]:

- **free evolution scheme:** the constraints are not solved during the evolution (they are employed only to get valid initial data or to monitor the solution); *example:* BSSN scheme
- **partially constrained scheme:** some of the constraints are solved along with the evolution equation
- **fully constrained scheme:** the four constraints are solved at each step of the evolution

[1] for a review see [Jaramillo, Valiente Kroon & Gourgoulhon, CQG **25**, 093001 (2008)]

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NB: the constraint preservation is a property of the exact mathematical system: it may not hold in actual numerical implementations of free schemes, due to the appearance of unstable constraint-violating modes
cf. Miguel Alcubierre's talk

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Constrained schemes

2D (axisymmetric) codes:

- **partially constrained** (Hamiltonian constraint enforced):
  - [Bardeen & Piran (1983)], [Stark & Piran (1985)], [Evans (1986)] : gravitational collapse of a stellar core
  - [Abrahams & Evans (1993)], [Garfinkle & Duncan, PRD **63**, 044011 (2001)] : evolution of Brill waves
- **fully constrained:**
  - [Evans (1989)], [Shapiro & Teukolsky (1992)], [Abrahams, Cook, Shapiro & Teukolsky (1994)] : gravitational collapse
  - [Choptuik, Hirschmann, Liebling & Pretorius, CQG **20**, 1857 (2003)] : critical collapse
  - [Rinne, CQG **25**, 135009 (2008)] : gravitational collapse of of Brill waves

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3D codes: fully constrained schemes:

- **Isenberg-Wilson-Mathews approximation to GR:** CFC
  [Isenberg (1978)], [Wilson & Mathews (1989)]
- **full GR:**
  - [Anderson & Matzner, Found. Phys. **35**, 1477 (2005)] : evolution of a black hole
  - [Bonazzola, Gourgoulhon, Grandclément & Novak, PRD **70**, 104007 (2004)], [Cordero-Carrión, Ibáñez, Gourgoulhon, Jaramillo & Novak, PRD **77**, 084007 (2008)] [Cordero-Carrión, Cerdá-Durán, Dimmelmeier, Jaramillo, Novak & Gourgoulhon, arXiv:0809...]: *the Meudon-Valencia FCF scheme*

Outline

Original formulation

Constrained scheme built upon **maximal slicing** and **Dirac gauge**

[Bonazzola, Gourgoulhon, Grandclément & Novak, PRD **70**, 104007 (2004)]

Motivations

- to maximize the number of *elliptic* equations and minimize that of *hyperbolic* equations (elliptic equations usually more stable)
- no constraint-violating mode by construction
- recover at the steady-state limit, the equations describing stationary spacetimes

Conformal metric and dynamics of the gravitational field

Dynamical degrees of freedom of the gravitational field:

York (1972) : they are carried by the conformal "metric"

$$\hat{\gamma}_{ij} := \gamma^{-1/3}\,\gamma_{ij} \qquad \text{with } \gamma := \det \gamma_{ij}$$

$\hat{\gamma}_{ij}$ = *tensor density* of weight $-2/3$

To work with *tensor fields* only, introduce an *extra structure* on Σ_t: a *flat metric* $\boldsymbol{f}$ such that $\dfrac{\partial f_{ij}}{\partial t} = 0$ and $\gamma_{ij} \sim f_{ij}$ at spatial infinity *(asymptotic flatness)*

Define $\tilde{\gamma}_{ij} := \Psi^{-4}\,\gamma_{ij}$ or $\gamma_{ij} =: \Psi^4\,\tilde{\gamma}_{ij}$ with $\Psi := \left(\frac{\gamma}{f}\right)^{1/12}$, $f := \det f_{ij}$

$\tilde{\gamma}_{ij}$ is invariant under any conformal transformation of γ_{ij} and verifies $\det \tilde{\gamma}_{ij} = f$

Notations:
- $\tilde{\gamma}^{ij}$: inverse conformal metric : $\tilde{\gamma}_{ik}\,\tilde{\gamma}^{kj} = \delta_i^{\ j}$
- $\tilde{D}_i$: covariant derivative associated with $\tilde{\gamma}_{ij}$, $\tilde{D}^i := \tilde{\gamma}^{ij}\,\tilde{D}_j$
- $\mathcal{D}_i$: covariant derivative associated with f_{ij}, $\mathcal{D}^i := f^{ij}\,\mathcal{D}_j$

Dirac gauge: definition

Conformal decomposition of the metric γ_{ij} of the spacelike hypersurfaces Σ_t:

$$\gamma_{ij} =: \Psi^4 \, \tilde{\gamma}_{ij} \qquad \text{with} \qquad \tilde{\gamma}^{ij} =: f^{ij} + h^{ij}$$

where f_{ij} is a flat metric on Σ_t, h^{ij} a symmetric tensor and Ψ a scalar field defined by $\Psi := \left(\frac{\det \gamma_{ij}}{\det f_{ij}} \right)^{1/12}$

Dirac gauge (Dirac, 1959) = *divergence-free* condition on $\tilde{\gamma}^{ij}$:

$$\mathcal{D}_j \tilde{\gamma}^{ij} = \mathcal{D}_j h^{ij} = 0$$

where $\mathcal{D}_j$ denotes the covariant derivative with respect to the flat metric f_{ij}.
Compare

- minimal distortion (Smarr & York 1978) : $D_j \left(\partial \tilde{\gamma}^{ij} / \partial t \right) = 0$
- pseudo-minimal distortion (Nakamura 1994) : $\mathcal{D}^j \left(\partial \tilde{\gamma}^{ij} / \partial t \right) = 0$

Notice: Dirac gauge $\Longleftrightarrow$ BSSN connection functions vanish: $\tilde{\Gamma}^i = 0$

Dirac gauge: motivation

Expressing the Ricci tensor of conformal metric as a second order operator:
In terms of the covariant derivative $\mathcal{D}_i$ associated with the flat metric $\boldsymbol{f}$:

$$\tilde{\gamma}^{ik}\tilde{\gamma}^{jl}\tilde{R}_{kl} = \frac{1}{2}\left(\tilde{\gamma}^{kl}\mathcal{D}_k\mathcal{D}_l h^{ij} - \tilde{\gamma}^{ik}\mathcal{D}_k H^j - \tilde{\gamma}^{jk}\mathcal{D}_k H^i\right) + \mathcal{Q}(\tilde{\gamma}, \boldsymbol{\mathcal{D}}\tilde{\gamma})$$

with $H^i := \mathcal{D}_j h^{ij} = \mathcal{D}_j \tilde{\gamma}^{ij} = -\tilde{\gamma}^{kl}\Delta^i{}_{kl} = -\tilde{\gamma}^{kl}(\tilde{\Gamma}^i{}_{kl} - \bar{\Gamma}^i{}_{kl})$

and $\mathcal{Q}(\tilde{\gamma}, \boldsymbol{\mathcal{D}}\tilde{\gamma})$ is quadratic in first order derivatives $\boldsymbol{\mathcal{D}h}$

Dirac gauge: $H^i = 0 \Longrightarrow$ Ricci tensor becomes an elliptic operator for h^{ij}
Similar property as harmonic coordinates for the 4-dimensional Ricci tensor:

$$^4R_{\alpha\beta} = -\frac{1}{2} g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} g_{\alpha\beta} + \text{quadratic terms}$$

Dirac gauge: motivation (con't)

- spatial harmonic coordinates: $\mathcal{D}_j \left[\left(\frac{\gamma}{f} \right)^{1/2} \gamma^{ij} \right] = 0$

  $\Longrightarrow$ makes the Ricci tensor R_{ij} (associated with the **physical** 3-metric γ_{ij}) an elliptic operator for γ^{ij} [Andersson & Moncrief, Ann. Henri Poincaré **4**, 1 (2003)]

- Dirac gauge: $\mathcal{D}_j \left[\left(\frac{\gamma}{f} \right)^{1/3} \gamma^{ij} \right] = 0$

  $\Longrightarrow$ makes the Ricci tensor $\tilde{R}_{ij}$ (associated with the **conformal** 3-metric $\tilde{\gamma}_{ij}$) an elliptic operator for $\tilde{\gamma}^{ij}$

Dirac gauge: discussion

- introduced by Dirac (1959) in order to fix the coordinates in some *Hamiltonian formulation* of general relativity; originally defined for Cartesian coordinates only: $\frac{\partial}{\partial x^j}\left(\gamma^{1/3}\,\gamma^{ij}\right) = 0$

  but trivially extended by us to more general type of coordinates (e.g. spherical) thanks to the introduction of the flat metric f_{ij}:
  $\mathcal{D}_j\left((\gamma/f)^{1/3}\gamma^{ij}\right) = 0$
- first discussed in the context of numerical relativity by Smarr & York (1978), as a candidate for a radiation gauge, but disregarded for not being covariant under coordinate transformation $(x^i) \mapsto (x^{i'})$ in the hypersurface Σ_t, contrary to the *minimal distortion gauge* proposed by them
- fully specifies (up to some boundary conditions) the coordinates in each hypersurface Σ_t, including the initial one $\Rightarrow$ allows for the search for *stationary solutions*
- Shibata, Uryu & Friedman [PRD **70**, 044044 (2004)] propose to use Dirac gauge to compute quasiequilibrium configurations of binary neutron stars beyond the IWM approximation

Dirac gauge: discussion (con't)

Dirac gauge

- leads asymptotically to transverse-traceless (TT) coordinates (same as minimal distortion gauge). Both gauges are analogous to *Coulomb gauge* in electrodynamics
- turns the Ricci tensor of conformal metric $\tilde{\gamma}_{ij}$ into an elliptic operator for h^{ij} $\Longrightarrow$ the dynamical Einstein equations become a *wave equation* for h^{ij}
- insures that the Ricci scalar $\tilde{R}$ (arising in the Hamiltonian constraint) does not contain any second order derivative of h^{ij}
- results in a *vector elliptic equation* for the shift: vector β^i
- is fulfilled by conformally flat initial data : $\tilde{\gamma}_{ij} = f_{ij} \Longrightarrow h^{ij} = 0$: this allows for the direct use of many currently available initial data sets

Maximal slicing + Dirac gauge

Our choice of coordinates to solve numerically the Cauchy problem:

- choice of Σ_t foliation: maximal slicing: $K := \mathrm{tr}\, \boldsymbol{K} = 0$
- choice of (x^i) coordinates within Σ_t: Dirac gauge: $\mathcal{D}_j h^{ij} = 0$

Note: the Cauchy problem has been shown to be locally strongly well posed for a similar coordinate system, namely *constant mean curvature* $(K = t)$ and *spatial harmonic coordinates* $\left(\mathcal{D}_j \left[(\gamma/f)^{1/2} \gamma^{ij} \right] = 0\right)$
[Andersson & Moncrief, Ann. Henri Poincaré **4**, 1 (2003)]

Decomposition of the extrinsic curvature

$K^{ij} = \Psi^{-10} \hat{A}^{ij}$ $(K = 0)$ (Lichnerowicz rescaling)

$\hat{A}^{ij} = (LW)^{ij} + \hat{A}^{ij}_{\rm TT}$ (York longitudinal/transverse decomposition)

$(LW)^{ij} := \mathcal{D}^i W^j + \mathcal{D}^j W^i - \frac{2}{3} \mathcal{D}_k W^k f^{ij}$ (conformal Killing operator)

$f_{ij} \hat{A}^{ij}_{\rm TT} = 0$ and $\mathcal{D}_j \hat{A}^{ij}_{\rm TT} = 0$ (TT tensor)

NB: expression of $\hat{A}^{ij}$ in terms of the shift vector β^i:

$\hat{A}^{ij} = \frac{\Psi^6}{2N} \left[(\tilde{L}\beta)^{ij} + \frac{\partial \tilde{\gamma}^{ij}}{\partial t} \right]$ $(\tilde{L}\beta)^{ij} := \tilde{D}^i \beta^j + \tilde{D}^j \beta^i - \frac{2}{3} \tilde{D}_k \beta^k \tilde{\gamma}^{ij}$

Rescaled matter quantities

- From the energy-momentum tensor:

$\hat{E} := \Psi^6 E$ $\quad$ $\hat{p}_i := \Psi^6 p_i$ $\quad$ $\hat{S} := \Psi^6 S$, $\quad S := \gamma^{ij} S_{ij}$

- Baryon number:

$\hat{D} := \Psi^6 \Gamma n$, $\quad$ n : proper number density of baryons

$\Gamma = N u^0$: fluid Lorentz factor w.r.t Eulerian observer

Equation of state: $P = P(n, \epsilon)$

Perfect fluid:

$E = \Gamma^2(\epsilon + P) - P$

$S = 3P + (E + P) U_i U^i$, with $U^i = \frac{1}{N}\left(\frac{dx^i}{dt} + \beta^i\right) = (E + P)^{-1} \gamma^{ij} p_j$

$\Gamma = (1 - U_i U^i)^{-1/2}$

Part 1 of FCF scheme: evolution equations

[Cordero-Carrión, Cerdá-Durán, Dimmelmeier, Jaramillo, Novak & Gourgoulhon, arXiv:0809...]

- **Fluid equations** (conservation of baryon number and energy-momentum):
  $\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}^j}{\partial x^j} = \boldsymbol{\mathcal{S}} \qquad \boldsymbol{U} := (\hat{D}, \hat{E}, \hat{p}_i) \quad \Longrightarrow \hat{D},\ \hat{E},\ \hat{p}_i$
- **Dynamical Einstein equations** :

$$\begin{cases} \frac{\partial h^{ij}}{\partial t} = \frac{2N}{\Psi^6} \hat{A}^{ij} + \cdots \\ \frac{\partial \hat{A}^{ij}}{\partial t} = \frac{N\Psi^2}{2} \Delta h^{ij} + \cdots \end{cases}$$

Constraints:

- $\det(f^{ij} + h^{ij}) = \det f^{ij}$ (unimodular) and $\mathcal{D}_j h^{ij} = 0$ (Dirac gauge)
- $f_{ij}\hat{A}^{ij} = 0$ and $\mathcal{D}_j \hat{A}^{ij} = 8\pi \tilde{\gamma}^{ij}\hat{p}_j - \Delta^i{}_{kl}\hat{A}^{kl}$ (momentum constraint)

$\Longrightarrow$ $(h^{ij}, \hat{A}^{ij})$ have only 2 degrees of freedom

$\Longrightarrow$ solve only for the TT part of the above system

$\Longrightarrow$ this involves two scalar potentials A and $\tilde{B}$ [Novak et al., in preparation], from which one can reconstruct h^{ij} ($\Longrightarrow \tilde{\gamma}^{ij}$) and $\hat{A}^{ij}_{\mathrm{TT}}$

Part 2 of FCF scheme: elliptic equations

[Cordero-Carrión, Cerdá-Durán, Dimmelmeier, Jaramillo, Novak & Gourgoulhon, arXiv:0809...]

1. Momentum constraint[2]:

$$\Delta W^i + \frac{1}{3}\mathcal{D}^i\mathcal{D}_j W^j + \Delta^i{}_{kl}(LW)^{kl} = 8\pi\tilde{\gamma}^{ij}\hat{p}_j - \Delta^i{}_{kl}\hat{A}^{kl}_{\rm TT}$$

$$\Longrightarrow W^i \Longrightarrow \hat{A}^{ij} = (LW)^{ij} + \hat{A}^{ij}_{\rm TT}$$

2. Hamiltonian constraint :

$$\tilde{\gamma}^{kl}\mathcal{D}_k\mathcal{D}_l\Psi = -2\pi\frac{\hat{E}}{\Psi} - \frac{\tilde{\gamma}_{il}\tilde{\gamma}_{jm}\hat{A}^{lm}\hat{A}^{ij}}{8\Psi^7} + \frac{\Psi\tilde{R}}{8} \quad \Longrightarrow \Psi \Longrightarrow P \Longrightarrow \hat{S}$$

3. Maximal slicing condition (+ Ham. constraint) :

$$\tilde{\gamma}^{kl}\mathcal{D}_k\mathcal{D}_l(N\Psi) = N\Psi\left[2\pi\Psi^{-2}(\hat{E}+2\hat{S}) + \left(\frac{7\tilde{\gamma}_{il}\tilde{\gamma}_{jm}\hat{A}^{lm}\hat{A}^{ij}}{8\Psi^8} + \frac{\tilde{R}}{8}\right)\right]$$

$$\Longrightarrow N\Psi \Longrightarrow N$$

4. Preservation of Dirac gauge in time (+ momentum constraint) :

$$\tilde{\gamma}^{kl}\mathcal{D}_k\mathcal{D}_l\beta^i + \frac{1}{3}\tilde{\gamma}^{ik}\mathcal{D}_k\mathcal{D}_l\beta^l = \frac{N}{\Psi^6}\left(16\pi\tilde{\gamma}^{ij}\hat{p}_j - 2\Delta^i_{kl}\hat{A}^{kl}\right) + 2\hat{A}^{ij}\mathcal{D}_j\left(\frac{N}{\Psi^6}\right)$$

$$\Longrightarrow \beta^i$$

[2] $\Delta^i{}_{kl} := \tilde{\Gamma}^i{}_{kl} - \bar{\Gamma}^i{}_{kl} = \tilde{\gamma}^{im}\left(\mathcal{D}_k\tilde{\gamma}_{ml} + \mathcal{D}_l\tilde{\gamma}_{km} - \mathcal{D}_m\tilde{\gamma}_{kl}\right)/2$

Mathematical analysis of the evolution part of the FCF system

If $\frac{\partial}{\partial t}$ is timelike and h^{ij} obeys to the Dirac gauge, then the evolution equations

$$\begin{cases} \frac{\partial h^{ij}}{\partial t} = \frac{2N}{\Psi^6}\hat{A}^{ij} + \cdots \\ \frac{\partial \hat{A}^{ij}}{\partial t} = \frac{N\Psi^2}{2}\Delta h^{ij} + \cdots \end{cases}$$

form a **strongly hyperbolic** system

[Cordero-Carrión, Ibáñez, Gourgoulhon, Jaramillo & Novak, PRD **77**, 084007 (2008)]

Outline

Conformally flat limit of the FCF scheme

Hypotheses

- $\tilde{\gamma}_{ij} = f_{ij}$ ($\iff$ $h^{ij} = 0$) $\implies$ $\hat{A}^{ij} = \frac{\Psi^6}{2N}(L\beta)^{ij}$
- $\hat{A}^{ij}_{\mathrm{TT}} = 0$

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$\implies$ evolution equations only for matter quantities $\implies \hat{D}$, $\hat{E}$, $\hat{p}_i$

For the gravitational field, the elliptic FCF equations reduce to

- (XCFC0) $\Delta W^i + \dfrac{1}{3}\mathcal{D}^i\mathcal{D}_j W^j = 8\pi f^{ij}\hat{p}_j$ $\implies W^i \implies \hat{A}^{ij} = (LW)^{ij}$
- (XCFC1) $\Delta\Psi = -2\pi\dfrac{\hat{E}}{\Psi} - \dfrac{f_{il}f_{jm}\hat{A}^{lm}\hat{A}^{ij}}{8\Psi^7}$ $\implies \Psi \implies P \implies \hat{S}$
- (XCFC2) $\Delta(N\Psi) = \left[2\pi\Psi^{-2}(\hat{E}+2\hat{S}) + \dfrac{7f_{il}f_{jm}\hat{A}^{lm}\hat{A}^{ij}}{8\Psi^8}\right](N\Psi)$ $\implies N\Psi$
- (XCFC3) $\Delta\beta^i + \dfrac{1}{3}\mathcal{D}^i\mathcal{D}_l\beta^l = \dfrac{N}{\Psi^6}\left(16\pi f^{ij}\hat{p}_j\right) + 2\hat{A}^{ij}\mathcal{D}_j\left(\dfrac{N}{\Psi^6}\right)$ $\implies \beta^i$

Conformally flat limit of the FCF scheme

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For the gravitational field, the elliptic FCF equations reduce to

- (XCFC0) $\Delta W^i + \frac{1}{3}\mathcal{D}^i\mathcal{D}_j W^j = 8\pi f^{ij}\hat{p}_j \implies W^i \implies \hat{A}^{ij} = (LW)^{ij}$
- (XCFC1) $\Delta\Psi = -2\pi\frac{\hat{E}}{\Psi} - \frac{f_{il}f_{jm}\hat{A}^{lm}\hat{A}^{ij}}{8\Psi^7} \implies \Psi \implies P \implies \hat{S}$
- (XCFC2) $\Delta(N\Psi) = \left[2\pi\Psi^{-2}(\hat{E} + 2\hat{S}) + \frac{7f_{il}f_{jm}\hat{A}^{lm}\hat{A}^{ij}}{8\Psi^8}\right](N\Psi) \implies N\Psi$
- (XCFC3) $\Delta\beta^i + \frac{1}{3}\mathcal{D}^i\mathcal{D}_l\beta^l = \frac{N}{\Psi^6}\left(16\pi f^{ij}\hat{p}_j\right) + 2\hat{A}^{ij}\mathcal{D}_j\left(\frac{N}{\Psi^6}\right) \implies \beta^i$

Except for the rescaling of matter quantities, similar to Shibata & Uryu's system devised to compute BH-NS binary initial data [PRD **74**, 121503(R) (2006)]

Comparison with the standard CFC scheme

- $\Delta\Psi = -2\pi\Psi^5 E - \frac{\Psi^5}{32N^2} f_{il} f_{jm} (L\beta)^{lm} (L\beta)^{ij}$ (CFC1)
- $\Delta(N\Psi) = 2\pi\Psi^4 (E + 2S)(N\Psi) + \frac{7\Psi^6}{32} f_{il} f_{jm} (L\beta)^{lm} (L\beta)^{ij} \, (N\Psi)^{-1}$ (CFC2)
- $\Delta\beta^i + \frac{1}{3}\mathcal{D}^i\mathcal{D}_l\beta^l = 16\pi N f^{ij} p_j + \frac{\Psi^6}{N}(L\beta)^{ij}\mathcal{D}_j\left(\frac{N}{\Psi^6}\right)$ (CFC3)

[Isenberg (1978)], [Wilson & Mathews (1989)]

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NB: **CFC** = same system as the **Extended Conformal Thin Sandwich (XCTS)** for quasiequilibrium initial data [Pfeiffer & York, PRD **67**, 044022 (2003)]

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Differences between CFC/XCTS and XCFC

- CFC/XCTS = 5-components system ↔ XCFC = 8-components system
- CFC/XCTS = coupled system ↔ XCFC = hierarchically decoupled
- CFC/XCTS : $\hat{A}^{ij}_{\rm TT} \neq 0$ ↔ XCFC: $\hat{A}^{ij}_{\rm TT}$ set to zero as an additional approximation (consistent with $\tilde{\gamma}_{ij} = f_{ij}$)
- XCFC involves the rescaled matter variables $(\hat{E}, \hat{S}, \hat{p}_i)$
- power -1 of $(N\Psi)$ in rhs (CFC2) ↔ power $+1$ in (XCFC2) ← a key feature

Non-uniqueness issue in XCTS-like schemes

Local uniqueness theorem

Consider the elliptic equation

$$\Delta u + h\, u^p = g \qquad (*)$$

where $p \in \mathbb{R}$ and h and g are a smooth functions independent of u.
If $ph \leq 0$, any solution of $(*)$ is locally unique.

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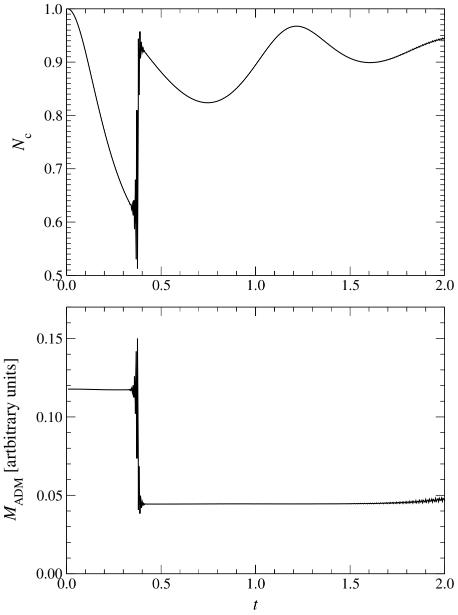
Application: Eqs. (CFC2) and (XCFC2) for $u = N\Psi$ (all other fields fixed)

- (CFC2) : $h = -\frac{7\Psi^6}{32} f_{il} f_{jm} (L\beta)^{lm} (L\beta)^{ij} \leq 0$ and $p = -1 \Longrightarrow hp \geq 0$: the theorem is not applicable: the solution may be not unique $\Longrightarrow$ well known property of XCTS [Pfeiffer & York, PRL **95**, 091101 (2005)], [Baumgarte, Ó Murchadha & Pfeiffer, PRD **75**, 044009 (2007)], [Walsh, CQG **24**, 1911 (2007)]
- (XCFC2) : $h = -\frac{7 f_{il} f_{jm} \hat{A}^{lm} \hat{A}^{ij}}{8\Psi^8} \leq 0$ and $p = 1 \Longrightarrow hp \leq 0$: the solution is unique !

Illustration of the non-uniqueness issue

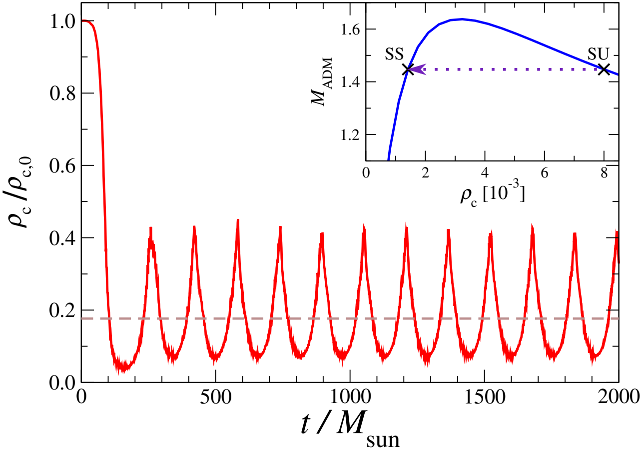


Collapse of a large amplitude Teukolsky wave computed using the *original version* of the FCF scheme (which did not introduce the vector W^i) [Bonazzola, Gourgoulhon, Grandclément & Novak, PRD **70**, 104007 (2004)]

Numerical code based on spectral methods (C++ library LORENE)

At $t \simeq 0.4$, the code jumped to a second solution: the black hole formation could not be computed

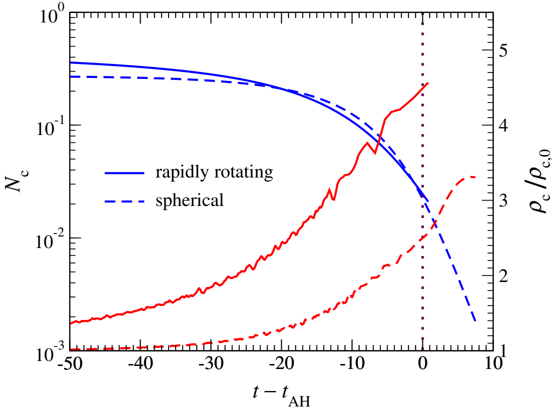
Unstable neutron star migration in XCFC

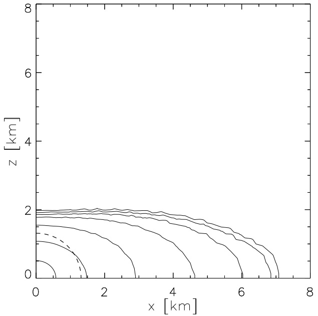


Numerical computation with the XCFC version of CoCoNuT code [Dimmelmeier, Novak, Font, Ibáñez & Müller, PRD **71**, 064023 (2005)], [Cordero-Carrión et al. arXiv:0809...]

Due to the non-uniqueness issue, such a calculation was not possible in CFC

Gravitational collapse to a black hole in XCFC





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Due to the non-uniqueness issue, such a calculation was not possible in CFC, even in spherical symmetry

Relation to previous works

- [Shapiro & Teukolsky, ApJ **235**, 199 (1980)] : full constrained code in spherical symmetry with conformal decomposition (isotropic coordinates): could get black formation, whereas CFC cannot !
  Shapiro and Teukolsky solved the momentum constraint for $\Psi^6 K^r{}_r = \hat{A}^{rr}$, as in XCFC (except that in XCFC the momentum constraint is solved for W^i first, leading to $\hat{A}^{ij} = (LW)^{ij}$)
  On the contrary, in CFC the momentum constraint is solved for the shift vector β^i, leading to the wrong sign in the equation for $N\Psi$
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- [Rinne, CQG **25**, 135009 (2008)] : fully constrained code for full GR (not conformally flat) in axisymmetry and vacuum
  Also adds a vector W^i to solve the momentum constraint, in addition to the elliptic equations for the shift
  Meudon-Valencia FCF : 3D generalisation of Rinne scheme (albeit in different spatial gauge)

Outline

Conclusions and future prospects

- A new fully constrained scheme, based on the Meudon (2004) one, has been introduced to address certain non-uniqueness of the solution of the elliptic part: the Meudon-Valencia FCF
- The mathematical analysis of the hyperbolic part has been performed; that of the entire scheme remains to be done
- Assuming a conformally flat 3-metric, the new scheme gives rise to the XCFC system, which cures the non-uniqueness issue of standard CFC in the strong relativistic regime
- Numerical implementation of XCFC has been performed, demonstrating its capability to compute unstable NS migration and BH formation, contrary to CFC
- Numerical implementation of the complete FCF is underway:
  - see **P. Grandclément**'s talk for a new computational infrastructure
  - see **I. Cordero**'s talk for numerical solutions of the hyperbolic part (at fixed Ψ, N, β^i and matter sources)
  - see **N. Vasset**'s talk for treatment of black holes as trapping horizons within the FCF formulation