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New theoretical approaches to black holes

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Abstract

Quite recently, some new mathematical approaches to black holes have appeared in the literature. They do not rely on the classical concept of event horizon—which is very global, but on the local concept of hypersurfaces foliated by trapped surfaces. After a brief introduction to these new horizons, we focus on a viscous fluid analogy that can be developed to describe their dynamics, in a fashion similar to the membrane paradigm introduced for event horizons in the seventies, but with a significant change of sign of the bulk viscosity.

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1. Introduction

1.1. What is a black hole?

The standard mathematical definition of a *black hole* is (Hawking and Ellis, 1973)

$$\mathscr{B} := \mathscr{M} - J^{-}(\mathscr{I}^{+}), \tag{1}$$

where $\mathscr{M}$ is a 4-dimensional manifold, endowed with a Lorentzian metric $\boldsymbol{g}$ such that $(\mathscr{M}, \boldsymbol{g})$ is asymptotically flat, $\mathscr{I}^{+}$ is the future null infinity, and $J^{-}(\mathscr{I}^{+})$ is the causal past of $\mathscr{I}^{+}$ (cf. Fig. 1). In common language, this means that a black hole is the region of spacetime where light rays cannot escape to infinity. The *event horizon* $\mathscr{H}$ is then defined as the boundary of $\mathscr{B}$. Provided that it is smooth, it is well-known that $\mathscr{H}$ is a null hypersurface (hence it is appears as a line inclined at 45° in Fig. 1).

1.2. Drawbacks of the classical definition

As noticed by Jean-Pierre Lasota and Marek Demiański long time ago (Demiański and Lasota, 1973), definition (1) is not applicable is cosmology, for usually a cosmological spacetime $(\mathscr{M}, \boldsymbol{g})$ is not asymptotically flat.

Moreover, even when applicable, definition (1) is highly non-local: the determination of $J^{-}(\mathscr{I}^{+})$ requires the knowledge of the entire future null infinity. In addition this definition has no direct relation with the notion of strong gravitational field: as shown by Ashtekar and Krishnan (2004) and Krishnan (in press) on an example based on the Vaidya metric, an event horizon can form in a flat region of spacetime, where by *flat* it is meant a vanishing Riemann tensor, i.e. no gravitational field at all. This means that no local physical experiment whatsoever can locate an event horizon.

Another non-local feature of event horizons is their *teleological* nature (Hawking and Hartle, 1972; Damour, 1979; Thorne et al., 1986). The classical black hole boundary, i.e. the event horizon, responds in advance to what will happen in the future. This is shown by Booth (2005) on the explicit example of a black hole formed by the collapse of two successive matter shells: after the first shell has

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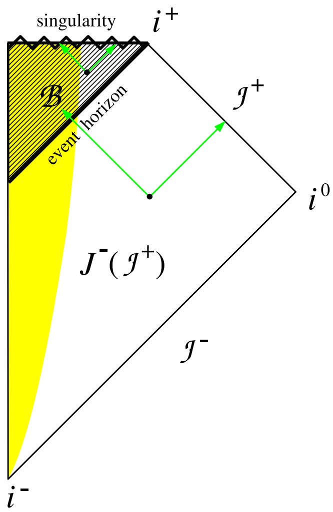

Fig. 1. Carter–Penrose diagram of a black hole $\mathscr{B}$ (cross-hatched region) formed by gravitational collapse of a star (colored region). In this conformal diagram, light rays appear as straight lines inclined at $\pm 45^\circ$.

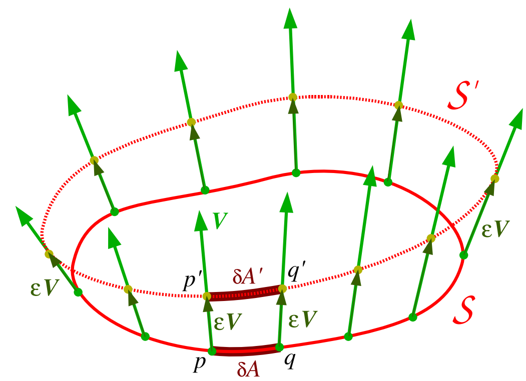

Fig. 2. Lie dragging of a spacelike 2-surface $\mathscr{S}$ along a normal vector field $\boldsymbol{V}$. In this plot, $\mathscr{S}$ appears as a closed line, whereas it is actually 2-dimensional.

collapsed to form the event horizon, the latter remains stationary for a while and then starts to grow *before* the second collapsing shell reaches it, as if it was anticipating its arrival.

If one would like to deal with black holes as "ordinary" physical objects, like for instance in quantum gravity or numerical relativity, the non-local (both in space and time) behavior of the event horizon mentioned above would be problematic. This has motivated the search for local characterizations of black holes.

2. New approaches to black holes

2.1. Local characterizations of black holes

The local definitions of black holes can be traced back to the "perfect horizons" of Hájíček (1973). However this applied only to equilibrium black holes. More recently, the local approach has been extended to black holes out of equilibrium, with the introduction of

– *trapping horizons* by Hayward (1994),
– *isolated horizons* by Ashtekar et al. (1999),
– *dynamical horizons* by Ashtekar and Krishnan (2002),
– *slowly evolving horizons* by Booth and Fairhurst (2004).

All these horizons are 3-dimensional submanifolds, as the event horizon. But contrary to the latter, they rely on *local* concepts. More precisely they are all based on the notion of *trapped surfaces*, which we examine now.

2.2. Trapped surface

Before defining a trapped surface, let us start by the general concept of the expansion of a surface along a normal vector field. Consider a spacelike 2-surface $\mathscr{S}$, as in Fig. 2. Take a vector field $\boldsymbol{V}$ defined on $\mathscr{S}$ and normal to $\mathscr{S}$ at each point. For a given small parameter $\varepsilon \in \mathbb{R}$, displace the point $p \in \mathscr{S}$ by the vector $\varepsilon \boldsymbol{V}$ to the point p'. Repeat this for each point in $\mathscr{S}$, keeping the value of ε fixed. This defines a new surface $\mathscr{S}'$. This process is called *Lie dragging* along the vector field $\boldsymbol{V}$. At each point, the *expansion of $\mathscr{S}$ along $\boldsymbol{V}$* is defined from the relative change in the area element δA around that point (cf. Fig. 2):

$$\theta^{(V)} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \frac{\delta A' - \delta A}{\delta A} = \mathscr{L}_V \ln \sqrt{q} = q^{\mu\nu} \nabla_\mu V_\nu, \tag{2}$$

where $\boldsymbol{q}$ denotes the metric induced on $\mathscr{S}$ by the spacetime metric $\boldsymbol{g}$, q the determinant of $q_{\mu\nu}$, $\mathscr{L}_V$ the Lie derivative along the vector field $\boldsymbol{V}$ and ∇ the spacetime covariant derivative.

With this definition of the expansion in hand, we are ready to define a trapped surface as follows. Consider a *closed* (i.e. compact without boundary) and *spacelike* 2-dimensional surface $\mathscr{S}$ embedded in the spacetime $(\mathscr{M}, \boldsymbol{g})$. Being spacelike, $\mathscr{S}$ lies outside the light cone (cf. Fig. 3), which means that there exist two future-directed null directions orthogonal to $\mathscr{S}$: $\boldsymbol{\ell}$, the so-called *outgoing null normal*, and $\boldsymbol{k}$, the so-called *ingoing null normal*. Note that $\boldsymbol{\ell}$ and $\boldsymbol{k}$ are defined up to a rescaling: $\boldsymbol{\ell}' = \lambda \boldsymbol{\ell}$ and $\boldsymbol{k}' = \mu \boldsymbol{k}$.

In flat space, the expansion of $\mathscr{S}$ along $\boldsymbol{\ell}$ is always positive: $\theta^{(\ell)} > 0$, whereas that along $\boldsymbol{k}$ is negative: $\theta^{(k)} < 0$. Now the surface $\mathscr{S}$ is called *trapped* iff both expansions are negative: $\theta^{(\ell)} < 0$ and $\theta^{(k)} < 0$. The limiting case, $\theta^{(\ell)} = 0$ and $\theta^{(k)} < 0$, is called a *marginally trapped surface*. These definitions have been introduced by Penrose (1965). They clearly constitute a local concept.[1] Moreover, this concept is related to very strong gravitational fields, since for weak fields, one has clearly $\theta^{(\ell)} > 0$.

It is worth noticing that in the previously mentioned work by Demiański and Lasota (1973), the "local event

[1] Some authors say *quasilocal* instead of local, because the definition relies on the notion of a surface, and not merely a point.

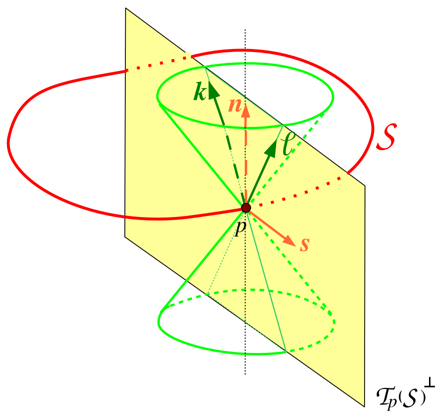

Fig. 3. Null directions $\boldsymbol{\ell}$ and $\boldsymbol{k}$ normal to a closed spacelike 2-surface $\mathscr{S}$. As in Fig. 2, $\mathscr{S}$ is drawn as a 1-dimensional contour, instead of a 2-dimensional surface. $\mathscr{T}_p(\mathscr{S})^{\perp}$ is the 2-plane normal to $\mathscr{S}$ at the point p. Its intersection with the light cone emanating from p defines the null directions $\boldsymbol{\ell}$ and $\boldsymbol{k}$. The unit vectors $\boldsymbol{n}$ and $\boldsymbol{s}$ are respectively a timelike normal and a spacelike normal to $\mathscr{S}$.

horizon" defined by the authors is nothing but a marginally trapped surface.

2.3. Link with apparent horizons

A closed spacelike 2-surface $\mathscr{S}$ is said to be *outer trapped* (resp. *marginally outer trapped*) if, and only if, (Hawking and Ellis, 1973)

- the notions of *interior* and *exterior* of $\mathscr{S}$ can be defined (for instance spacetime asymptotically flat);
- the *outgoing* null normal $\boldsymbol{\ell}$ satisfies $\theta^{(\boldsymbol{\ell})} < 0$ (resp. $\theta^{(\boldsymbol{\ell})} = 0$).

Notice that no condition is imposed on the expansion $\theta^{(\boldsymbol{k})}$ along the ingoing null normal.

Let us then consider a spacelike hypersurface Σ extending to spatial infinity (Cauchy surface) (cf. Fig. 4). The *outer trapped region* of Σ is defined as the set Ω of points $p \in \Sigma$ through which there is a outer trapped surface $\mathscr{S}$ lying in Σ. An *apparent horizon* in Σ is then a connected component $\mathscr{A}$ of the boundary of Ω (Hawking and Ellis, 1973). Then a classical result by Hawking and Ellis (1973) states that the apparent horizon is a marginally outer trapped surface (see also the recent study by Andersson and Metzger (2008)).

2.4. Connection with singularities and black holes

A famous theorem by Penrose (1965) makes the link with the trapped surfaces introduced above and spacetime singularities: provided that the weak energy condition holds, if there exists a trapped surface $\mathscr{S}$, then there exists a singularity in $(\mathscr{M}, \boldsymbol{g})$ (in the form of a future inextendible null geodesic). Another theorem by Hawking and Ellis (1973) states that, provided that the cosmic censorship

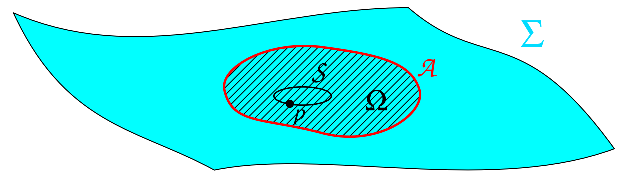

Fig. 4. Spacelike hypersurface Σ containing an outer trapped region Ω (i.e. a set of points through which there is at least one outer trapped surface). The apparent horizon $\mathscr{A}$ is then the boundary of Ω.

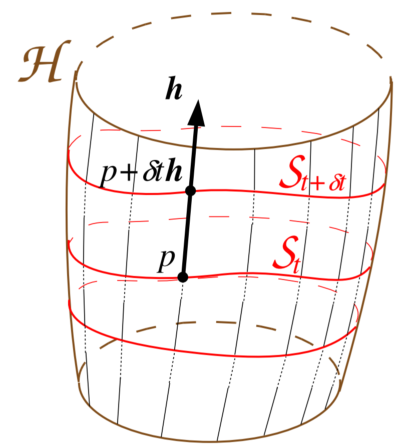

Fig. 5. Hypersurface $\mathscr{H}$ foliated by a 1-parameter family of 2-surfaces $(\mathscr{S}_t)_{t\in\mathbb{R}}$. $\boldsymbol{h}$ is the canonical evolution vector associated with the parameter t.

conjecture holds, if the spacetime contains a trapped surface $\mathscr{S}$, then it necessarily contains a black hole $\mathscr{B}$ and $\mathscr{S} \subset \mathscr{B}$.

2.5. Local definitions of "black holes"

Having recalled the previous classical results about trapped surfaces and apparent horizons, we now state the local definitions of black hole horizons, alternative to the event horizon, that have appeared quite recently in the literature. A hypersurface $\mathscr{H}$ of $(\mathscr{M}, \boldsymbol{g})$ is said to be

- a *future outer trapping horizon (FOTH)* iff (i) $\mathscr{H}$ foliated by marginally trapped 2-surfaces: $\mathscr{H} = \bigcup_{t\in\mathbb{R}} \mathscr{S}_t$ with $\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} = 0$ (cf. Fig. 5), and (ii) the outermost condition $\mathscr{L}_{\boldsymbol{k}}\theta^{(\boldsymbol{\ell})} < 0$ is satisfied (Hayward, 1994);
- a *dynamical horizon (DH)* iff (i) $\mathscr{H}$ is foliated by marginally trapped 2-surfaces and (ii) $\mathscr{H}$ is spacelike (Ashtekar and Krishnan, 2002);
- a *non-expanding horizon*[2] iff (i) $\mathscr{H}$ is a null hypersurface (with null normal $\boldsymbol{\ell}$ say) and (ii) $\theta^{(\boldsymbol{\ell})} = 0$ (Hájíček, 1973);
- an *isolated horizon* iff (i) $\mathscr{H}$ is a non-expanding horizon (it has then a well defined geometry, with a unique connection $\hat{\nabla}$, despite the induced metric is degenerate) and (ii) $\mathscr{H}$'s geometry is not evolving along the null generators: $[\mathscr{L}_{\boldsymbol{\ell}}, \hat{\boldsymbol{\nabla}}] = 0$ (Ashtekar et al., 1999).

[2] The non-expanding horizons were called *perfect horizons by* Hájíček (1973).

Note that in generic dynamical situations, the notions of FOTH and DH are equivalent (Booth, 2005). In stationary situations, a FOTH becomes a null hypersurface, whereas a DH (which by definition is spacelike) cannot exist; it should be replaced by the notion of isolated horizon (Ashtekar et al., 1999; Ashtekar and Krishnan, 2004; Booth, 2005; Gourgoulhon and Jaramillo, 2006a). If $\mathscr{H}$ is an event horizon, the 2-surfaces $\mathscr{S}_t$ are not marginally trapped, except in stationary configurations (Kerr black hole). On the contrary they are expanding, by the famous Hawking (1972) area increase law: $\theta^{(\ell)} > 0$.

These "new" horizons have their own dynamics, ruled by Einstein equations. In particular, one can establish for them existence and (partial) uniqueness theorems (Andersson et al., 2005; Ashtekar and Galloway, 2005), first and second laws analogous to the classical laws of black hole mechanics (Ashtekar and Krishnan, 2003; Hayward, 2004), a viscous fluid bubble analogy ("membrane paradigm" as for the event horizon), leading to a Navier–Stokes-like equation (Gourgoulhon, 2005; Gourgoulhon and Jaramillo, 2006b). Recent review articles on the subject are Ashtekar and Krishnan (2004), Booth (2005), Gourgoulhon and Jaramillo (2006a) and Krishnan (in press). Notice that the FOTH and DH proved to be useful in numerical relativity (Schnetter et al., 2006; Jaramillo et al., 2007, 2008, in press not only in the "follow-up" of the horizon, but also in the prescription of excised black hole initial data in quasi-equilibrium (Jaramillo et al., 2004; Dain et al., 2005; Cook and Pfeiffer, 2004; Ansorg, 2005; Caudill et al., 2006).

3. Geometry of hypersurface foliations by spacelike 2-surfaces

Since the trapping and dynamical horizons are based on a foliation $(\mathscr{S}_t)_{t\in\mathbb{R}}$ by closed spacelike 2-surfaces of a hypersurface $\mathscr{H}$, let us first discuss the geometrical properties of such foliations.

3.1. Relevant vectors

We shall call *evolution vector* the unique vector field $\boldsymbol{h}$ that is tangent to $\mathscr{H}$, orthogonal to $\mathscr{S}_t$ and such that $\mathscr{L}_{\boldsymbol{h}} t = 1$, where $\mathscr{L}_{\boldsymbol{h}}$ denotes the Lie derivative along $\boldsymbol{h}$: $\mathscr{L}_{\boldsymbol{h}} t = h^\mu \partial_\mu t$ (cf. Fig. 5). The latter property implies that the 2-surfaces $\mathscr{S}_t$ are Lie-dragged to each other by $\boldsymbol{h}$. Let C be half the scalar square of $\boldsymbol{h}$ with respect to the metric $\boldsymbol{g}$:

$$\boldsymbol{h}\cdot\boldsymbol{h} = 2C \tag{3}$$

(we systematically denote the scalar product corresponding to the spacetime metric $\boldsymbol{g}$ with a dot). It is easy to see that the sign of C gives the signature of the hypersurface $\mathscr{H}$: C is positive, zero and negative for respectively spacelike, null and timelike hypersurfaces. There exists a unique pair $(\boldsymbol{\ell}, \boldsymbol{k})$ of null vectors normal to $\mathscr{S}_t$ and a unique vector $\boldsymbol{m}$ normal to $\mathscr{H}$ such that (cf. Fig. 6)

$$\boldsymbol{h} = \boldsymbol{\ell} - C\boldsymbol{k}, \quad \boldsymbol{m} = \boldsymbol{\ell} + C\boldsymbol{k} \quad \text{and} \quad \boldsymbol{\ell}\cdot\boldsymbol{k} = -1. \tag{4}$$

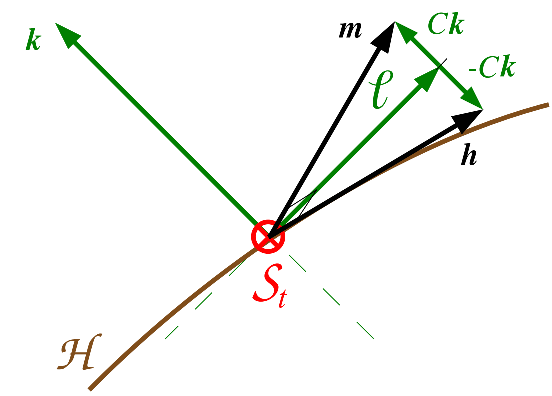

Fig. 6. Evolution vector $\boldsymbol{h}$, null normals $\boldsymbol{\ell}$ and $\boldsymbol{k}$ and normal vector $\boldsymbol{m}$ along a foliated hypersurface $\mathscr{H}$. The figure is drawn in the plane normal to $\mathscr{S}_t$, which is reduced to a point, $\mathscr{H}$ being reduced to a line.

For any vector field $\boldsymbol{v}$ normal to $\mathscr{S}_t$, such as $\boldsymbol{h}$, $\boldsymbol{m}$, $\boldsymbol{\ell}$ or $\boldsymbol{k}$, we define the *shear tensor* $\boldsymbol{\sigma}^{(\boldsymbol{v})}$ of the surface $\mathscr{S}_t$ when Lie-dragged along $\boldsymbol{v}$ by

$$\mathscr{L}_{\boldsymbol{v}}\boldsymbol{q} = \theta^{(\boldsymbol{v})}\boldsymbol{q} + 2\boldsymbol{\sigma}^{(\boldsymbol{v})} \quad \text{and} \quad \mathrm{tr}\,\boldsymbol{\sigma}^{(\boldsymbol{v})} = 0, \tag{5}$$

where, as before, $\boldsymbol{q}$ is the induced metric on $\mathscr{S}_t$ ($\boldsymbol{q}$ is positive definite since $\mathscr{S}_t$ is assumed to be spacelike) and $\mathscr{L}_{\boldsymbol{v}}\boldsymbol{q}$ is its Lie derivative resulting from the dragging of the surface $\mathscr{S}_t$ along the normal vector $\boldsymbol{v}$. The vanishing of the trace of $\boldsymbol{\sigma}^{(\boldsymbol{v})}$ with respect to the metric $\boldsymbol{q}$ is a consequence of definition (2) of $\theta^{(\boldsymbol{v})}$.

Let us denote by κ the component along $\boldsymbol{\ell}$ of the "acceleration" of $\boldsymbol{h}$ in the decomposition (Gourgoulhon and Jaramillo, 2006b)

$$\nabla_{\boldsymbol{h}}\boldsymbol{h} = \kappa\boldsymbol{\ell} + (C\kappa - \mathscr{L}_{\boldsymbol{h}}C)\boldsymbol{k} - \mathscr{D}C. \tag{6}$$

If $\mathscr{H}$ is an event horizon, then it is a null hypersurface, so that $\boldsymbol{h} = \boldsymbol{\ell}$, $C = 0$ and the above relation reduces to

$$\nabla_{\boldsymbol{\ell}}\boldsymbol{\ell} = \kappa\boldsymbol{\ell}, \tag{7}$$

showing that, in this case, κ is nothing but the surface gravity of the black hole.

3.2. Extrinsic geometry of the 2-surfaces

On each 2-surface $\mathscr{S}_t$, let us denote by $\mathscr{D}$ the connection compatible with the induced metric $\boldsymbol{q}$ (this connection is unique since $\boldsymbol{q}$ is not degenerate). $\mathscr{D}$ is related to the spacetime connection ∇ via the the *second fundamental tensor* of $\mathscr{S}_t$, $\mathscr{K}$ (Carter, 1992; Senovilla, 2005):

$$\forall(\boldsymbol{u}, \boldsymbol{v}) \in \mathscr{T}(\mathscr{S}_t)^2, \quad \nabla_{\boldsymbol{u}}\boldsymbol{v} = \mathscr{D}_{\boldsymbol{u}}\boldsymbol{v} + \mathscr{K}(\boldsymbol{u}, \boldsymbol{v}). \tag{8}$$

$\mathscr{K}$ is a type (1,2) tensor, which is expressible in term of the covariant derivative of $\boldsymbol{q}$, according to

$$\mathscr{K}^{\alpha}_{\ \beta\gamma} = \nabla_\mu q^\alpha_{\ \nu} q^\mu_{\ \beta} q^\nu_{\ \gamma}. \tag{9}$$

Contrary to the case of a hypersurface,[3] the extrinsic geometry of the 2-surface $\mathscr{S}_t$ is not entirely specified by $\mathscr{K}$. The

[3] For a non-degenerate hypersurface, $\mathscr{K}$ is related to the second fundamental *form* $\boldsymbol{K}$ (also called *extrinsic curvature tensor*) via $\mathscr{K}^{\alpha}_{\ \beta\gamma} = -n^\alpha K_{\beta\gamma}$, where $\boldsymbol{n}$ is the unit normal to the hypersurface.

latter encodes only the part of the variation of $\mathcal{S}_t$'s normals which is parallel to $\mathcal{S}_t$. The remaining part, i.e. the variation of the two normals with respect to each other, is encoded by the *normal fundamental forms* (also called *external rotation coefficients* or *connection on the normal bundle*, or if $\mathcal{H}$ is null, Hájíček 1-form), defined by (see e.g. Hayward, 1994)

$$\boldsymbol{\Omega}^{(\ell)} := -\boldsymbol{k}\cdot\boldsymbol{\nabla}_{\vec{q}}\boldsymbol{\ell} \tag{10}$$

$$\boldsymbol{\Omega}^{(k)} := -\boldsymbol{\ell}\cdot\boldsymbol{\nabla}_{\vec{q}}\boldsymbol{k}, \tag{11}$$

where $\vec{\boldsymbol{q}}$ denotes the orthogonal projector on the surface $\mathcal{S}_t$. In terms of components, Eq. (10) is written

$$\Omega^{(\ell)}_{\alpha} := -k_\mu \nabla_\nu \ell^\mu q^\nu_{\ \alpha}, \tag{12}$$

with a similar relation for $\Omega^{(\ell)}_{\alpha}$. Thanks to the relation $\boldsymbol{\ell}\cdot\boldsymbol{k} = -1$ [Eq. (4)], we have $\boldsymbol{\Omega}^{(k)} = -\boldsymbol{\Omega}^{(\ell)}$. Note that contrary to the second fundamental tensor $\mathcal{K}$, the normal fundamental forms are not unique: any rescaling $\boldsymbol{\ell}' = \lambda\boldsymbol{\ell}$ of the null normal results in

$$\boldsymbol{\Omega}^{(\ell')} = \boldsymbol{\Omega}^{(\ell)} + \mathcal{D}\ln\lambda. \tag{13}$$

4. A Navier–Stokes-like equation

4.1. Concept of black hole viscosity

When studying the response of the event horizon to external perturbations in the early seventies, Hawking and Hartle (1972) and Hartle (1973) introduced the concept of *black hole viscosity*. This fluid analogy took its full significance when Damour (1979, 1982) derived from Einstein equation a 2-dimensional Navier–Stokes-like equation governing the evolution of the event horizon, and letting appear some *shear viscosity* and well as some *bulk viscosity*. The 2-dimensional fluid (membrane) point of view has been further developed in the famous *Membrane Paradigm* book by Thorne et al. (1986).

A natural question which then arises is: shall we restrict the analysis to the event horizon? In other words, can we extend the concept of viscosity to the local characterizations of black hole recently introduced, i.e. FOTH and DH?

A priori this does not seem obvious because, from a pure geometrical point of view, the event horizon and the "local" horizons are of different type: the event horizon is always a null hypersurface (hence is endowed with a degenerate metric), whereas a DH is always a spacelike surface (hence with a positive definite metric) and a FOTH can be either null or spacelike. We shall see that nevertheless the fluid analogy can also be extended to these horizons, with some significant change in the sign of the bulk viscosity.

4.2. Original Damour–Navier–Stokes equation

Damour considered the case where $\mathcal{H}$ is a black hole event horizon. In particular it is a null hypersurface and the null vector $\boldsymbol{\ell}$ is normal to it. From the Einstein equation, he has derived the relation (Damour, 1979, 1982) (see also Damour and Lilley, 2008)

$$^{\mathcal{S}}\mathcal{L}_{\ell}\boldsymbol{\pi} + \theta^{(\ell)}\boldsymbol{\pi} = -\mathcal{D}P + 2\mu\mathcal{D}\cdot\vec{\boldsymbol{\sigma}}^{(\ell)} + \zeta\mathcal{D}\theta^{(\ell)} + \boldsymbol{f}, \tag{14}$$

where

- $\boldsymbol{\pi} := -1/(8\pi)\,\boldsymbol{\Omega}^{(\ell)}$ is analogous to some momentum surface density,
- $P := \kappa/(8\pi)$ is analogous to the pressure [κ being defined by Eq. (7)],
- $\mu := 1/(16\pi)$ is analogous to the shear viscosity,
- $\zeta := -1/(16\pi)$ is analogous to the bulk viscosity,
- $\boldsymbol{f} := -\boldsymbol{T}(\boldsymbol{\ell},\vec{\boldsymbol{q}})$ is the external force surface density, $\boldsymbol{T}$ being the stress-energy tensor of any matter or electromagnetic field present around the horizon.

Eq. (14) is structurally identical to a Navier–Stokes equation for a 2-dimensional fluid. The reader is referred to Chap. VI of Thorne et al. (1986) for an extended discussion of this analogy with a viscous fluid (see also Section 2.3 of Damour and Lilley, 2008).

A striking feature of the above Navier–Stokes Eq. (14) is that the bulk viscosity is negative:

$$\zeta = \zeta_{\rm EH} = -\frac{1}{16\pi} < 0, \tag{15}$$

where the subscript EH stands for "event horizon". For an ordinary fluid, this negative value would yield to a dilation or contraction instability. This is in agreement with the well-known tendency of a null hypersurface to continually contract or expand. However the event horizon is stabilized by the teleological condition that its expansion must vanish in the far future, when an equilibrium state has been reached (Damour, 1979).

4.3. Generalization to the non-null case

In order to generalize Eq. (14) to the case where $\mathcal{H}$ is not necessarily a null hypersurface, it is worth to notice that in Eq. (14), the vector $\boldsymbol{\ell}$ plays two role: it is the natural evolution vector along $\mathcal{H}$ and it is also the normal to $\mathcal{H}$. In the non null case, these two role are played respectively by the vectors $\boldsymbol{h}$ and $\boldsymbol{m}$ introduced in Section 3.1. Of course, at the null limit $(C = 0)$, $\boldsymbol{h} = \boldsymbol{m} = \boldsymbol{\ell}$ [cf. Eq. (4)].

Having realized this, the starting point of the calculation is the contracted Ricci identity applied to the vector $\boldsymbol{m}$ and projected onto $\mathcal{S}_t$:

$$(\nabla_\mu\nabla_\nu m^\mu - \nabla_\nu\nabla_\mu m^\mu)q^\nu_{\ \alpha} = R_{\mu\nu}m^\mu q^\nu_{\ \alpha}, \tag{16}$$

where $R_{\mu\nu}$ is the spacetime Ricci tensor, to be replaced ultimately by its expression in terms of the matter stress-energy tensor $T_{\mu\nu}$ according to Einstein equation. After some manipulations, one arrives at (Gourgoulhon, 2005)

$$^{\mathcal{S}}\mathcal{L}_{\boldsymbol{h}}\boldsymbol{\Omega}^{(\ell)} + \theta^{(\boldsymbol{h})}\boldsymbol{\Omega}^{(\ell)} = \mathcal{D}\kappa - \mathcal{D}\cdot\vec{\boldsymbol{\sigma}}^{(\boldsymbol{m})} + \frac{1}{2}\mathcal{D}\theta^{(\boldsymbol{m})} - \theta^{(\boldsymbol{k})}\mathcal{D}C + 8\pi\boldsymbol{T}(\boldsymbol{m},\vec{\boldsymbol{q}}). \tag{17}$$

In the null limit, $C=0$, $\boldsymbol{h}=\boldsymbol{m}=\boldsymbol{\ell}$, and the above equation reduces to the original Damour–Navier–Stokes Eq. (14). On the other side, if $\mathscr{H}$ is a FOTH or a DH, then $\theta^{(\boldsymbol{m})}=-\theta^{(\boldsymbol{h})}$ (since in this case $\theta^{(\boldsymbol{\ell})}=0$ and we can deduce from Eq. (4) the relation $\theta^{(\boldsymbol{m})}=-\theta^{(\boldsymbol{h})}+2\theta^{(\boldsymbol{\ell})}$) and Eq. (17) can be written

$$ {}^{\mathscr{S}}\mathscr{L}_{\boldsymbol{\ell}}\boldsymbol{\pi}+\theta^{(\boldsymbol{h})}\boldsymbol{\pi}=-\mathscr{D}P+\frac{1}{8\pi}\mathscr{D}\cdot\vec{\boldsymbol{\sigma}}^{(\boldsymbol{m})}+\zeta\mathscr{D}\theta^{(\boldsymbol{h})}+\boldsymbol{f}, \tag{18} $$

where $\boldsymbol{f}:=-\boldsymbol{T}(\boldsymbol{m},\vec{\boldsymbol{q}})+\theta^{(\boldsymbol{k})}/(8\pi)\mathscr{D}C$ and, as in Eq. (14), $\boldsymbol{\pi}:=-1/(8\pi)\,\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$, $P:=\kappa/(8\pi)$, but contrary to Eq. (14),

$$ \zeta=\zeta_{\mathrm{FOTH}}:=\frac{1}{16\pi}>0. \tag{19} $$

This positive value of the bulk viscosity shows that FOTHs and DHs behave as "ordinary" physical objects.

4.4. Angular momentum flux law

In general relativity, the angular momentum is usually well defined only if there exists a Killing vector field $\boldsymbol{\varphi}$ which generates a symmetry around some axis. To generalize the definition of angular momentum to the cases where no symmetry is present, let us follow Booth and Fairhurst (2005) and introduce a vector field $\boldsymbol{\varphi}$ on $\mathscr{H}$ which

- is tangent to $\mathscr{S}_t$
- has closed orbits
- has vanishing divergence with respect to the induced metric on $\mathscr{S}_t$:

$$ \mathscr{D}\cdot\boldsymbol{\varphi}=0. \tag{20} $$

Notice that (20) is a condition weaker than being a Killing vector of $(\mathscr{S}_t,\boldsymbol{q})$, which would write $\mathscr{D}_\alpha\varphi_\beta+\mathscr{D}_\beta\varphi_\alpha=0$. For dynamical horizons, $\theta^{(\boldsymbol{h})}\neq 0$ and there is a unique choice of $\boldsymbol{\varphi}$ as the generator (conveniently normalized) of the curves of constant $\theta^{(\boldsymbol{h})}$ (Hayward, 2006).

The *generalized angular momentum associated with* $\boldsymbol{\varphi}$ is then defined by

$$ J(\boldsymbol{\varphi}):=-\frac{1}{8\pi}\oint_{\mathscr{S}_t}\langle\boldsymbol{\Omega}^{(\boldsymbol{\ell})},\boldsymbol{\varphi}\rangle\boldsymbol{\epsilon}_{\mathscr{S}}, \tag{21} $$

where $\langle\boldsymbol{\Omega}^{(\boldsymbol{\ell})},\boldsymbol{\varphi}\rangle$ stands for the normal fundamental form $\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$ applied to the vector $\boldsymbol{\varphi}$ and $\boldsymbol{\epsilon}_{\mathscr{S}}$ is the volume element on $\mathscr{S}_t$ associated with the metric $\boldsymbol{q}$. Note that the definition of $J(\boldsymbol{\varphi})$ does not depend upon the choice of null vector $\boldsymbol{\ell}$, thanks to the divergence-free property of $\boldsymbol{\varphi}$ and to the transformation law (13) of $\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$ under a change of $\boldsymbol{\ell}$. Formula (21) coincides with Ashtekar and Krishnan's (2003) definition. It also coincides with Brown–York angular momentum (Brown and York, 1993) if $\mathscr{H}$ is timelike and $\boldsymbol{\varphi}$ a Killing vector.

Under the supplementary hypothesis that $\boldsymbol{\varphi}$ is transported along the evolution vector $\boldsymbol{h}:\mathscr{L}_{\boldsymbol{h}}\boldsymbol{\varphi}=0$, Eq. (17) leads to (Gourgoulhon, 2005)

$$ \frac{\mathrm{d}}{\mathrm{d}t}J(\boldsymbol{\varphi})=-\oint_{\mathscr{S}_t}\boldsymbol{T}(\boldsymbol{m},\boldsymbol{\varphi})\boldsymbol{\epsilon}_{\mathscr{S}} -\frac{1}{16\pi}\oint_{\mathscr{S}_t}[\vec{\vec{\boldsymbol{\sigma}}}^{(\boldsymbol{m})}:\mathscr{L}_{\boldsymbol{\varphi}}\boldsymbol{q}-2\theta^{(\boldsymbol{k})}\boldsymbol{\varphi}\cdot\mathscr{D}C]\boldsymbol{\epsilon}_{\mathscr{S}}, \tag{22} $$

where a double arrow stands for the double "index raising" via the metric $\boldsymbol{q}$ and the colon denotes a double contraction. There are two interesting limiting cases for this equation. First of all, if $\mathscr{H}$ is a null hypersurface ($C=0$ and $\boldsymbol{m}=\boldsymbol{\ell}$), it reduces to

$$ \frac{\mathrm{d}}{\mathrm{d}t}J(\boldsymbol{\varphi})=-\oint_{\mathscr{S}_t}\boldsymbol{T}(\boldsymbol{\ell},\boldsymbol{\varphi})\boldsymbol{\epsilon}_{\mathscr{S}}-\frac{1}{16\pi}\oint_{\mathscr{S}_t}\vec{\vec{\boldsymbol{\sigma}}}^{(\boldsymbol{\ell})}:\mathscr{L}_{\boldsymbol{\varphi}}\boldsymbol{q}\boldsymbol{\epsilon}_{\mathscr{S}}, \tag{23} $$

i.e. we recover Eq. (6.134) of the *Membrane Paradigm* book (Thorne et al., 1986). Second, if $\mathscr{H}$ is a FOTH, one may show that the last term in Eq. (22) vanishes (Gourgoulhon, 2005), so that it reduces to

$$ \frac{\mathrm{d}}{\mathrm{d}t}J(\boldsymbol{\varphi})=-\oint_{\mathscr{S}_t}\boldsymbol{T}(\boldsymbol{m},\boldsymbol{\varphi})\boldsymbol{\epsilon}_{\mathscr{S}}-\frac{1}{16\pi}\oint_{\mathscr{S}_t}\vec{\vec{\boldsymbol{\sigma}}}^{(\boldsymbol{m})}:\mathscr{L}_{\boldsymbol{\varphi}}\boldsymbol{q}\boldsymbol{\epsilon}_{\mathscr{S}}. \tag{24} $$

Thus for a FOTH, the Navier–Stokes-like equation (18) leads to an evolution equation for the angular momentum which is as simple as Eq. (23) for an event horizon. In particular the r.h.s. has only two terms, which are interpretable as respectively (i) the flux of angular momentum due to some matter or electromagnetic field near the horizon and (ii) the flux of angular momentum due to "gravitational radiation". The latter interpretation is pretty vague and relies on the fact that $\mathscr{L}_{\boldsymbol{\varphi}}\boldsymbol{q}=0$ in axisymmetry, where gravitational radiation does not carry any angular momentum.

5. Area evolution and energy equation

5.1. Evolution of the expansion

Let us search for an evolution equation for the expansion $\theta^{(\boldsymbol{h})}$, which governs the evolution of the area of the surfaces $\mathscr{S}_t$ via Eq. (2). The starting point turns out to be the Ricci identity applied to the normal vector $\boldsymbol{m}$, as in Section 4.3, but instead of projecting it onto $\mathscr{S}_t$ [Eq. (16)], we shall project it along the normal direction to $\mathscr{S}_t$ lying in $\mathscr{H}$, namely $\boldsymbol{h}$:

$$ (\nabla_\mu\nabla_\nu m^\mu-\nabla_\nu\nabla_\mu m^\mu)h^\nu=R_{\mu\nu}m^\mu h^\nu. \tag{25} $$

By means of the Einstein equation, and after some computations, we arrive at (Gourgoulhon and Jaramillo, 2006b):

$$ \mathscr{L}_{\boldsymbol{h}}\theta^{(\boldsymbol{m})}=\kappa\theta^{(\boldsymbol{h})}-\frac{1}{2}\theta^{(\boldsymbol{h})}\theta^{(\boldsymbol{m})}-\boldsymbol{\sigma}^{(\boldsymbol{h})}:\boldsymbol{\sigma}^{(\boldsymbol{m})}+\theta^{(\boldsymbol{k})}\mathscr{L}_{\boldsymbol{h}}C +\mathscr{D}\cdot(2C\vec{\boldsymbol{\Omega}}^{(\boldsymbol{\ell})}-\vec{\mathscr{D}}C)-8\pi\boldsymbol{T}(\boldsymbol{m},\boldsymbol{h}), \tag{26} $$

where an upper arrow indicates "index raising" with the metric $\boldsymbol{q}$ and the colon stands for the double contraction, i.e. $\boldsymbol{\sigma}^{(\boldsymbol{h})}:\boldsymbol{\sigma}^{(\boldsymbol{m})}:=\sigma^{(\boldsymbol{h})}_{ab}\sigma^{(\boldsymbol{m})ab}$. If we specialize Eq. (26) to the cases of (i) an event horizon and (ii) a FOTH or a DH, we obtain respectively

$$\mathscr{L}_{\ell}\theta^{(\ell)} + (\theta^{(\ell)})^2 - \kappa\theta^{(\ell)} = \frac{1}{2}(\theta^{(\ell)})^2 - \boldsymbol{\sigma}^{(\ell)} : \boldsymbol{\sigma}^{(\ell)} - 8\pi \boldsymbol{T}(\boldsymbol{\ell}, \boldsymbol{\ell}), \tag{27}$$

$$\mathscr{L}_{\boldsymbol{h}}\theta^{(\boldsymbol{h})} + (\theta^{(\boldsymbol{h})})^2 + \kappa\theta^{(\boldsymbol{h})} = \frac{1}{2}(\theta^{(\boldsymbol{h})})^2 + \boldsymbol{\sigma}^{(\boldsymbol{h})} : \boldsymbol{\sigma}^{(\boldsymbol{m})} - \theta^{(\boldsymbol{k})}\mathscr{L}_{\boldsymbol{h}}C + \mathscr{D}\cdot(\vec{\mathscr{D}}C - 2C\vec{\boldsymbol{\Omega}}^{(\ell)}) + 8\pi\boldsymbol{T}(\boldsymbol{m}, \boldsymbol{h}). \tag{28}$$

For the event horizon, we have used the null character of $\mathscr{H}$, which implies $C = 0$ and $\boldsymbol{h} = \boldsymbol{m} = \boldsymbol{\ell}$, yielding Eq. (27). It is nothing but the null Raychaudhuri equation for a surface-orthogonal congruence (Hawking and Hartle, 1972). For the FOTH/DH case [Eq. (28)], we have used the property $\theta^{(\boldsymbol{m})} = -\theta^{(\boldsymbol{h})}$ already encountered in Section 4.3. Notice the change of some signs between Eqs. (27) and (28).

5.2. Energy dissipation and bulk viscosity

In the fluid membrane approach to black holes, Price and Thorne (1986) and Thorne et al. (1986) defined the *surface energy density* of an event horizon as $\varepsilon := -\theta^{(\ell)}/8\pi$ and interpreted Eq. (27) as an energy balance law, with heat production resulting from viscous stresses. By analogy, let us define the *surface energy density* of a FOTH/DH as $\varepsilon := -\theta^{(\boldsymbol{m})}/8\pi$, where the role of the normal to $\mathscr{H}$ is now taken by $\boldsymbol{m}$ instead of $\boldsymbol{\ell}$. Since $\theta^{(\boldsymbol{m})} = -\theta^{(\boldsymbol{h})}$ for a FOTH/DH, we have

$$\varepsilon = \frac{\theta^{(\boldsymbol{h})}}{8\pi} \tag{29}$$

and we may rewrite Eq. (28) as

$$\mathscr{L}_{\boldsymbol{h}}\varepsilon + \theta^{(\boldsymbol{h})}\varepsilon = -\frac{\kappa}{8\pi}\theta^{(\boldsymbol{h})} + \frac{1}{8\pi}\boldsymbol{\sigma}^{(\boldsymbol{h})} : \boldsymbol{\sigma}^{(\boldsymbol{m})} + \frac{(\theta^{(\boldsymbol{h})})^2}{16\pi} - \mathscr{D}\cdot\boldsymbol{Q} + \boldsymbol{T}(\boldsymbol{m}, \boldsymbol{h}) - \frac{\theta^{(\boldsymbol{k})}}{8\pi}\mathscr{L}_{\boldsymbol{h}}C, \tag{30}$$

with $\boldsymbol{Q} := \frac{1}{4\pi}[C\vec{\boldsymbol{\Omega}}^{(\ell)} - 1/2\vec{\mathscr{D}}C] = -\frac{C}{4\pi}\vec{\varpi}$, where ϖ is the *anholonomicity 1-form* (or *twist 1-form*) of the 2-surface $\mathscr{S}_t$ (Hayward, 1994) (see also Section IV.A of Gourgoulhon, 2005) and $\vec{\varpi}$ denotes its vector dual.

It is striking that Eqs. (18) and (30) are fully analogous to the equations that govern a two-dimensional non-relativistic fluid of internal energy density ε, momentum density $\boldsymbol{\pi}$, pressure $\kappa/8\pi$, shear stress tensor $\boldsymbol{\sigma}^{(\boldsymbol{m})}/8\pi$, bulk viscosity $\zeta = 1/16\pi$, shear strain tensor $\boldsymbol{\sigma}^{(\boldsymbol{h})}$, expansion $\theta^{(\boldsymbol{h})}$, subject to the external force density $-\boldsymbol{T}(\boldsymbol{m}, \vec{\boldsymbol{q}}) + \theta^{(\boldsymbol{k})}/8\pi\mathscr{D}C$, external energy production rate $\boldsymbol{T}(\boldsymbol{m}, \boldsymbol{h}) - \theta^{(\boldsymbol{k})}/8\pi\mathscr{L}_{\boldsymbol{h}}C$ and heat flux $\boldsymbol{Q}$ (see e.g. Rieutord, 1997). In particular the value of the bulk viscosity read on Eq. (30) (the coefficient of $(\theta^{(\boldsymbol{h})})^2$) is the same as that obtained from the Navier–Stokes equation (18) and given by Eq. (19).

Besides, let us notice that the shear viscosity μ does not appear in Eqs. (18) and (30), because the standard Newtonian-fluid relation between the shear stress tensor $\boldsymbol{\sigma}^{(\boldsymbol{m})}/8\pi$ and the shear strain tensor $\boldsymbol{\sigma}^{(\boldsymbol{h})}$, namely $\boldsymbol{\sigma}^{(\boldsymbol{m})}/8\pi = 2\mu\boldsymbol{\sigma}^{(\boldsymbol{h})}$, does not hold. Here we have $\boldsymbol{\sigma}^{(\boldsymbol{m})}/8\pi = [\boldsymbol{\sigma}^{(\boldsymbol{h})} + 2C\boldsymbol{\sigma}^{(\boldsymbol{k})}]/8\pi$, so that the Newtonian-fluid assumption is fulfilled only if $C = 0$ (isolated horizon limit). On the contrary, it appears from Eqs. (18) and (30) that the trace part of the viscous stress tensor $\boldsymbol{S}_{\rm visc}$ does obey the Newtonian-fluid law, being proportional to the trace part of the strain tensor (i.e. the expansion $\theta^{(\boldsymbol{h})}$): $\mathrm{tr}\boldsymbol{S}_{\rm visc} = 3\zeta\theta^{(\boldsymbol{h})}$.

We may point out two differences with the event horizon case (Damour, 1979, 1982; Price and Thorne, 1986; Thorne et al., 1986). First the heat flux $\boldsymbol{Q}$ is not vanishing for a FOTH/DH, whereas it was zero for an EH. Notice that $\boldsymbol{Q}$ is a vector tangent to $\mathscr{S}_t$ so that the integration of Eq. (30) over the closed surface $\mathscr{S}_t$ to get a global internal energy balance law would not contain any net heat flux. The second major difference is that, as already stressed in Section 4.3, the bulk viscosity ζ is positive, being equal to $1/16\pi$ [Eq. (19)], whereas it was found to be negative, being equal to $-1/16\pi$ [Eq. (15)], for an event horizon. As commented in Section 4.2, this negative value is related to the teleological character of event horizons. On the contrary the positive value of the bulk viscosity for FOTHs and DHs that these objects behave as "ordinary" physical objects and is in perfect agreement with their local nature.

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