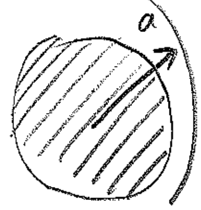
PART 2

EXTERNAL FIELD OF AN ISOLATED SOURCE

NON-LINEARITY (POST MINKOWSKIAN) EXPANSION

In exterior region $(r > a)$



isolated source
(radius a)

$$\begin{cases} \Box\, h^{\mu\nu}_{ext} = \Lambda^{\mu\nu}(h_{ext}) \\ \partial_\nu h^{\mu\nu}_{ext} = 0 \end{cases}$$

of order $\mathcal{O}(h^2_{ext})$

harmonic coordinate condition

We solve these equations by means of <u>post-Minkowskian</u> (PM) or <u>non-linearity</u> expansion

$$\boxed{h^{\mu\nu}_{ext} = \sum_{n=1}^{+\infty} G^n\, h^{\mu\nu}_{(n)}}$$

G = Newton's constant
(viewed here as a "bookeeping" parameter to label the successive PM orders)

Insert PM expansion into <u>vacuum</u> Einstein field eqs.

$$\Box \left(G\, h^{\mu\nu}_{(1)} + G^2\, h^{\mu\nu}_{(2)} + \dots \right) = G^2 \Lambda^{\mu\nu}_{(2)}(h_{(1)}) + G^3 \Lambda^{\mu\nu}_{(3)}(h_{(1)}, h_{(2)}) + \dots$$

$$\partial_\nu \left(\rule{4cm}{0.4pt} \right) = 0$$

where
$$\Lambda_{(2)} \sim h_{(1)} \partial^2 h_{(1)} + \partial h_{(1)} \partial h_{(1)}$$
$$\Lambda_{(3)} \sim h_{(1)} \partial h_{(1)} \partial h_{(1)} + h_{(1)} \partial^2 h_{(2)} + h_{(2)} \partial^2 h_{(1)} + \partial h_{(1)} \partial h_{(2)}$$
$$\dots$$

Hierarchy of PM equations equivalent to Einstein eqs.

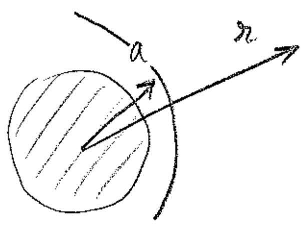
$\forall n \geqslant 1$

$$\boxed{\begin{aligned} \Box h^{\mu\nu}_{(n)} &= \Lambda^{\mu\nu}_{(n)}(h_{(1)}, h_{(2)} \dots h_{(n-1)}) \\ \partial_\nu h^{\mu\nu}_{(n)} &= 0 \end{aligned}}$$

The source term $\Lambda_{(n)}$ is known from previous iterations

LINEARIZED SOLUTION

Solve $\Box h_{(1)} = 0$ by means of multipole expansion (valid in exterior $r > a$)



"Monopolar" general solution

$$h^{\text{Mono.}}_{(1)}(\vec{x}, t) = \frac{R(t - r/c) + A(t + r/c)}{r}$$

Impose no incoming rad. cond.

$$0 = \lim_{\substack{t \to -\infty \\ t + \frac{r}{c} = \text{const}}} \left[\partial_r (r\, h_{(1)}) + \partial_t (r\, h_{(1)}) \right] = 2 A'(t + \tfrac{r}{c})$$ hence $A(u)$ is constant and can be included into definition of $R(t - \frac{r}{c})$.

$$h^{\text{Mono.}}_{(1)} = \frac{R(t - r/c)}{r}$$

$(i = 1, 2, 3)$

"Dipolar" solution is obtained by applying $\partial_i \equiv \frac{\partial}{\partial x^i}$

hence $h^{\text{Dip.}}_{(1)} = \partial_i\left(\dfrac{R_i(t-r/c)}{r}\right)$. General multipolar solution is obtained by applying l spatial derivatives

$$\boxed{h^{\mu\nu}_{(1)}(\vec{x},t) = \sum_{l=0}^{+\infty} \partial_L\left(\frac{R^{\mu\nu}_L(u)}{r}\right)} \quad \left(u \equiv t-\frac{r}{c}\right)$$

$L = i_1 i_2 \cdots i_l$ a multi-index with l spatial indices

$$\partial_L \equiv \partial_{i_1 i_2 \cdots i_l} = \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_l}}$$

Without loss of generality we can assume that R_L is <u>symmetric and trace-free</u> (STF)

STF tensors

$$R_L = \hat{R}_L + \sum_{j \leqslant l-1} \varepsilon \, \underbrace{\delta\delta\ldots\delta}_{1 \text{ to } [\frac{l}{2}]} \, \hat{U}_J$$

0 or 1 Levi-Civita symbol

Kronecker symbol

where the $\hat{U}_J$'s are linear in the $\varepsilon\,\delta\ldots\delta\,R_K$'s.

For example:

$$\begin{cases} R_{ij} = \hat{R}_{ij} + \varepsilon_{ijk}\,\hat{U}_k + \delta_{ij}\,\hat{U} \\ \hat{U}_k = \frac{1}{2}\varepsilon_{kab} R_{ab} \\ \hat{U} = \frac{1}{3} R_{kk} \end{cases}$$

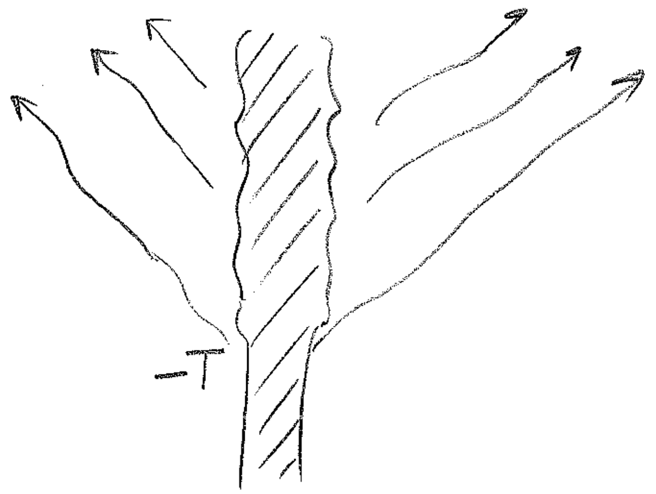
$\hat{R}_{ij} = \dfrac{R_{ij}+R_{ji}}{2} - \dfrac{1}{3}\delta_{ij} R_{kk}$ is the STF part of R_{ij}.

$$\partial_L\left(\frac{1}{r}R_L\right) = \partial_L\left(\frac{1}{r}\hat{R}_L\right) + \sum_{k\geq 1} \Delta^k \, \partial_{L-2K}\left(\frac{1}{r}\hat{U}_{L-2K}\right)$$

because of k Kronecker δs
(terms with one ε cancelled by symmetry of ∂_L)

$$\Delta^k \partial\left(\frac{1}{r}\hat{U}(u)\right) = \partial\left(\frac{1}{r}\frac{d^{2k}\hat{U}}{c^{2k}du^{2k}}(u)\right)$$ takes same structure

For simplicity assume that source emits GWs only from some finite instant $-T$ in the past (<u>stationarity in the past</u>)



$h^{\mu\nu}_{ext}(\vec{x})$ is independent of time when $t \leq -T$

(and even when
$$\underbrace{t - \frac{r}{c} - \frac{2GM}{c^3}\ln\left(\frac{r}{r_0}\right) + \cdots}_{\text{"light cone" in coordinates } (t,r)} \leq -T)$$

There are 10 independent functions $R^{\mu\nu}_L(u)$ (for each multi-index L) at this stage.

We impose now the harmonicity condition $\partial_\nu h^{\mu\nu}_{(1)} = 0$ which gives 4 differential relations between the R_L's. Hence we end up with <u>6 independent functions</u> (6 types of "source" multipole moments).

Most general solution of $\Box h_{(1)} = 0 = \partial h_{(1)}$ is (Thorne 1980)

$$\boxed{h^{\mu\nu}_{(1)} = k^{\mu\nu}_{(1)} + \underbrace{\partial^\mu \varphi^\nu_{(1)} + \partial^\nu \varphi^\mu_{(1)} - \eta^{\mu\nu}\partial_\rho \varphi^{\rho}_{(1)}}_{\text{linearized gauge transformation}}}$$

where $k^{\mu\nu}_{(1)}$ depends on two sets of <u>STF multipole moments</u>

$$\boxed{I_L(u) \quad \text{and} \quad J_L(u)}$$

I_L: <u>mass-moment of order l</u> $\qquad$ J_L: <u>current-moment of order l</u>

and $\varphi^\mu_{(1)}$ depends on four sets of moments (for its four components $\mu = 0,1,2,3$)

$$W_L(u) \quad X_L(u) \quad Y_L(u) \quad \text{and} \quad Z_L(u)$$

$$\boxed{\begin{aligned}
k^{00}_{(1)} &= -\frac{4}{c^2}\sum_{l=0}^{+\infty} \frac{(-)^l}{l!}\,\partial_L\left(\frac{1}{r} I_L(u)\right) \\
k^{0i}_{(1)} &= \frac{4}{c^3}\sum_{l=1}^{+\infty} \frac{(-)^l}{l!}\left\{\partial_{L-1}\left(\frac{1}{r}\dot{I}_{iL-1}(u)\right) + \frac{l}{l+1}\varepsilon_{iab}\,\partial_{aL-1}\left(\frac{1}{r}J_{bL-1}(u)\right)\right\} \\
k^{ij}_{(1)} &= -\frac{4}{c^4}\sum_{l=2}^{+\infty} \frac{(-)^l}{l!}\left\{\partial_{L-2}\left(\frac{1}{r}\ddot{I}_{ijL-2}\right) + \frac{2l}{l+1}\partial_{aL-2}\left(\frac{1}{r}\varepsilon_{ab(i}\dot{J}_{j)bL-2}\right)\right\}
\end{aligned}}$$

Dots mean derivative w.r.t. time $u = t - r/c$

$I_L(u)$ and $J_L(u)$ are arbitrary functions of time u except for the <u>conservation laws</u> (directly issued from the harmonicity condition $\partial h_{(1)} = 0$)

$$\boxed{\begin{array}{lll} M \equiv I = \text{const} & & \text{total mass} \\ X_i \equiv \dfrac{I_i}{I} = \text{const} & & \text{center-of-mass position} \\ P_i \equiv \dot{I}_i = 0 & & \text{linear momentum} \\ S_i \equiv J_i = \text{const} & & \text{angular momentum} \end{array}}$$

These conservation laws are exact (by definition of the moments) and refer to the total quantities associated with the source and <u>including the contributions of GWs emitted by the source</u>

They describe the <u>initial</u> state of the source before emisson of GWs.

In particular <u>$M = I$ is the total ADM mass of source</u>

Finally $h_{(1)}$ (and hence $h_{ext} = \sum G^n h_{(n)}$) will be described by

$$\boxed{\underbrace{I_L(u) \quad J_L(u)}_{\substack{\text{main moments} \\ \text{(encode all properties of} \\ \text{source at linear order)}}} \quad \underbrace{W_L(u) \ \dots \ Z_L(u)}_{\substack{\text{gauge moments} \\ \text{(will play a role} \\ \text{at non-linear order)}}} = \text{six source multipole moments}}$$

NON-LINEAR VACUUM SOLUTION

When $r \to 0$ $\quad h_{(1)} \sim \partial\left(\frac{R(t-r)}{r}\right)$ diverges. This is because $h_{(1)}$ is valid only in the exterior $r > a$. Inserting $h_{(1)}$ into $\Lambda_{(2)}$ we get

$$\Lambda_{(2)} \sim \partial\left(\frac{R(t-r)}{r}\right)\partial\left(\frac{S(t-r)}{r}\right)$$

$$\sim \sum_{k\geq 2} \frac{\hat{n}_L}{r^k} F(t-r)$$

STF product of unit vectors n_i $\quad \boxed{\hat{n}_L = n_{\langle i_1} \cdots n_{i_\ell \rangle}}$ is equivalent to spherical harmonics $Y_{\ell m}(\theta,\varphi)$

$$\begin{cases} \hat{n}_L(\theta,\varphi) = \displaystyle\sum_{m=-\ell}^{\ell} \alpha_L^m \, Y_{\ell m}(\theta,\varphi) \\ \alpha_L^m = \displaystyle\int d\Omega \, \hat{n}_L \, Y_{\ell m}^* \end{cases}$$

$\leftarrow$ constant STF tensor

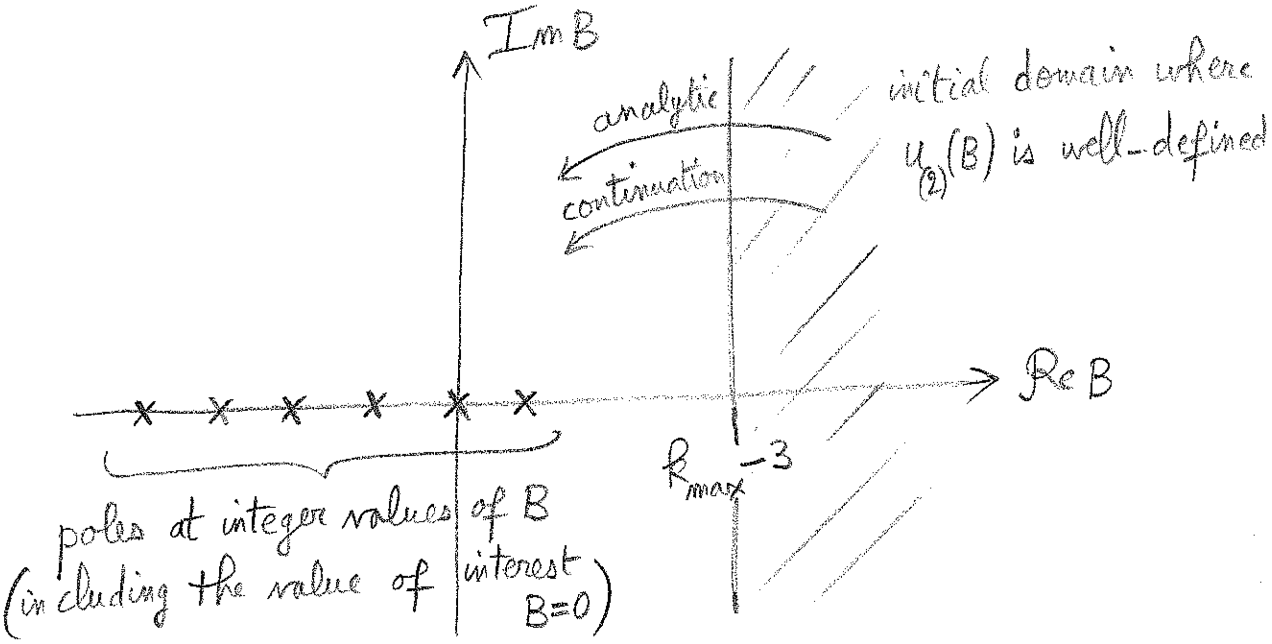
Because of divergence when $r \to 0$ one cannot apply the standard retarded integral.

If we assume $h_{(1)}$ is made of a finite set of moments, say $\ell \leq \ell_{max}$, there is a maximal order of divergencies in $\Lambda_{(2)}$, $k \leq k_{max}$. We can <u>regularize</u> $\Lambda_{(2)}$ by multiplying by some factor r^B (where $B \in \mathbb{C}$).

Next we define:

$$\boxed{u_{(2)}^{\mu\nu}(B) \equiv \Box^{-1}_{\mathrm{Ret}}\left[\left(\frac{r}{r_0}\right)^B \Lambda_{(2)}^{\mu\nu}\right]}$$

The retarded integral is convergent when $\mathrm{Re}\, B > k_{max} - 3$



$$u_{(2)}(B) = \sum_{p=p_0}^{+\infty} \lambda_p B^p$$

Laurent expansion when $B \to 0$ ($p \in \mathbb{Z}$)

Applying $\Box$ we get $\left(\frac{r}{r_0}\right)^B \Lambda_{(2)} = \sum (\Box \lambda_p) B^p$

$$p_0 \leq p \leq -1 \quad \Rightarrow \quad \Box \lambda_p = 0$$

$$p \geq 0 \quad \Rightarrow \quad \Box \lambda_p = \frac{(\ln(r/r_0))^p}{p!} \Lambda_{(2)}$$

In particular when $p=0$ we obtain a solution of the eq. we want ($\Box u_{(2)} = \Lambda_{(2)}$). Pose $u_{(2)}^{\mu\nu} \equiv \lambda_0^{\mu\nu}$

$$\boxed{u^{\mu\nu}_{(2)} = \underset{B\to 0}{\text{Finite Part}}\ \Box^{-1}_{Ret}\left[r^B \Lambda^{\mu\nu}_{(2)}\right]} \qquad (r_0 = 1)$$

Thus $\Box u_{(2)} = \Lambda_{(2)}$ is satisfied and $u_{(2)}$ has the same structure $\sim \sum \frac{\hat{n}_L}{r^k} G(t-r)$ as $\Lambda_{(2)}$ but $\partial_\nu u^{\mu\nu}_{(2)} \neq 0$ in general.

$$\boxed{w^{\mu}_{(2)} \equiv \partial_\nu u^{\mu\nu}_{(2)} = \underset{B\to 0}{FP}\ \Box^{-1}_{Ret}\left[B\, n_i\, r^{B-1} \Lambda^{\mu i}_{(2)}\right]}$$

computed from the fact that $\partial_\nu \Lambda^{\mu\nu}_{(2)} = 0$

Because of factor B (coming from $\partial_i r^B = B r^{B-1} n_i$) $w^\mu_{(2)}$ is non zero when the integral <u>develops a pole</u> $\propto \frac{1}{B}$. The structure of the pole is that of a source-free (retarded) solution of d'Alembert's eq.

$$\boxed{w^{\mu}_{(2)} = \sum_{l=0}^{\infty} \partial_L \left(\frac{S^\mu_L(u)}{r}\right)}$$

Indeed $\Box w_{(2)} = \underset{B\to 0}{FP}\left(B n r^{B-1} \Lambda\right) = 0$. From that structure one can construct "<u>algorithmically</u>"

$$v^{\mu\nu}_{(2)} = \mathcal{H}^{\mu\nu}(w_{(2)})$$

an algorithm which gives a unique $v^{\mu\nu}_{(2)}$ starting from any $w^\mu_{(2)}$ (source-free solution) such that (at once) $\Box v_{(2)} = 0$ and $\partial v_{(2)} = - u_{(2)}$

$$\boxed{v^{\mu\nu}_{(2)} = \sum_{l=0}^{\infty} \partial_L \left(\frac{T^{\mu\nu}_L(u)}{r} \right)}$$

where the $T^{\mu\nu}_L$'s are given in terms of the S^{μ}_L's by the algorithm $\mathcal{H}$. Solution is thus

$$\boxed{h^{\mu\nu}_{(2)} = u^{\mu\nu}_{(2)} + v^{\mu\nu}_{(2)}}$$

Same method applies by induction to any n (Blanchet & Damour 1986)

$$\boxed{\begin{aligned} u^{\mu\nu}_{(n)} &= \underset{B \to 0}{\text{Finite Part}}\ \Box^{-1}_{\text{Ret}} \left[\left(\frac{r}{r_0}\right)^B \Lambda_{(n)}(h_{(1)} \cdots h_{(n-1)}) \right] \\ v^{\mu\nu}_{(n)} &= \mathcal{H}^{\mu\nu}(\partial u_{(n)}) \\ h^{\mu\nu}_{(n)} &= u^{\mu\nu}_{(n)} + v^{\mu\nu}_{(n)} \end{aligned}}$$

To $h_{(n)}$ one can still add a <u>homogeneous</u> solution (such that $\Box h^{\text{Hom}}_{(n)} = 0 = \partial h^{\text{Hom}}_{(n)}$) but $h^{\text{Hom}}_{(n)}$ is necessarily of the form $h_{(1)}$[some moments]. Hence

$$h_{(n)}^{gen} = h_{(n)}[I_L \cdots Z_L] + \underbrace{h_{(1)}[\delta I_L \cdots \delta Z_L]}_{\substack{\text{can be re-absorbed into} \\ h_{(1)}[I_L \cdots Z_L] \text{ by posing}}}$$

$$\begin{cases} I_L^{new} = I_L + G^{n-1}\delta I_L \\ \vdots \\ Z_L^{new} = Z_L + G^{n-1}\delta Z_L \end{cases}$$

Hence the previous construction represents the <u>most general</u> solution of Einstein's field eqs. outside the source

Resulting metric $\boxed{g_{\mu\nu}^{ext}(x;\, \underbrace{I_L\, J_L\, \overbrace{W_L\, X_L\, Y_L\, Z_L}^{4 \text{ gauge moments}}}_{6 \text{ source moments}})}$

One can define by coord. transformation $x \longrightarrow x'$ a "canonical" metric which depends only on 2 moments $M_L\, S_L$.

Thus $\boxed{g_{\mu\nu}^{can}(x';\, \underbrace{M_L\, S_L}_{2 \text{ canonical moments}})}$

is <u>isometric</u> to $g_{\mu\nu}^{ext}$ i.e. $g_{\mu\nu}^{can}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}^{ext}(x)$ where

$$\boxed{x'^\mu = x^\mu + G\, \underbrace{\varphi_{(1)}^\mu(x;\, W_L X_L Y_L Z_L)}_{\substack{\text{gauge vector in the} \\ \text{general linear solution}}} + \underbrace{O(G^2)}_{\substack{\text{crucial} \\ \text{non-linear} \\ \text{corrections}}}}$$

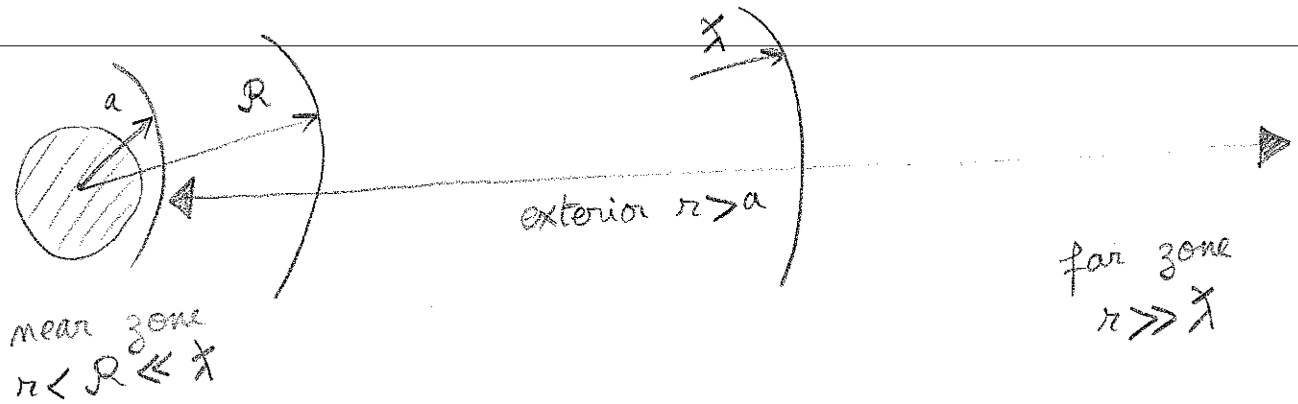
Hence any isolated system can be described by <u>2 sets of moments</u>

$$\underset{\text{mass-type}}{M_L(u)} \qquad \underset{\text{current-type}}{S_L(u)}$$

$$\boxed{\begin{aligned} M_L &= I_L + O(G) \\ S_L &= J_L + O(G) \end{aligned}}$$

complicated non-linear functionals of $I_L\ J_L\ X_L \dots Z_L$

GENERAL STRUCTURE OF THE SOLUTION



The solution $h_{ext} = \sum G^n h_{(n)}$ is physically valid in the exterior $r > a$ but is defined for any $r > 0$. When $r \to 0$

$$\boxed{h_{(n)} = \sum_{p \leq N} \hat{n}_L(\theta,\varphi)\, r^p (\ln r)^q F(t) + O(r^N)}$$

(proved by induction on n in the construction of $h_{(n)}$).

Note appearance of powers of $\ln r$ with $q \leq n-2$.

Since $r \to 0$ means $\frac{r}{c} \to 0$ or $c \to \infty$ we have the general structure of the post-Newtonian (PN) expansion

$$\boxed{h_{(n)}(c) = \sum_{p \leq N} \frac{(\ln c)^q}{c^p} + \mathcal{O}\left(\frac{1}{c^N}\right)}$$

When $r \to \infty$ (wave zone) we find also a "poly-logarithmic" structure

$$h_{(n)} = \sum_{k \leq N} \hat{n}_L \frac{(\ln r)^q}{r^k} G(u) + o\left(\frac{1}{r^N}\right) \quad \text{where } u = t - r/c$$

(expansion at $\mathcal{J}^+$)

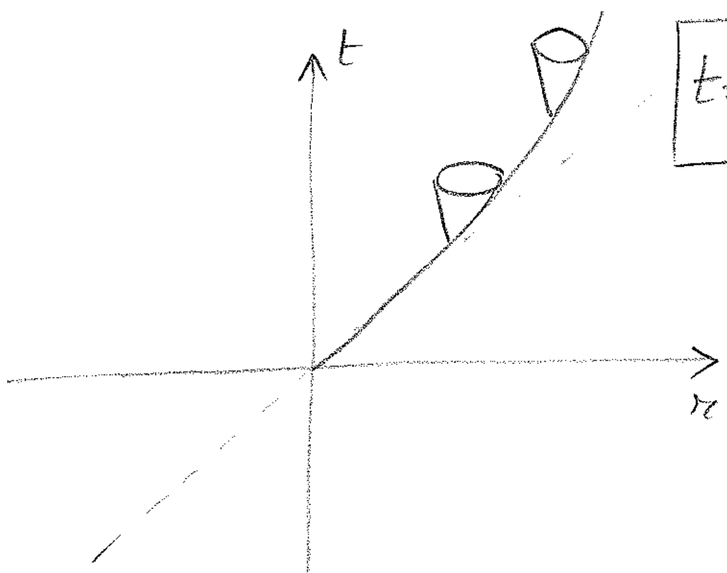
The logs here come from the well-known deviation of light rays in harmonic coordinates.

Schwarzschild in harmonic coord.

$$ds^2 = -\left(\frac{r-M}{r+M}\right)dt^2 + \left(\frac{r+M}{r-M}\right)dr^2 + (r+M)^2 d\Omega^2$$

For an <u>outgoing radial</u> ($\theta = \text{const}$ $\varphi = \text{const}$) photon

$$dt = \frac{r+M}{r-M}\,dr \quad \Rightarrow \quad t = r + 2M \ln\left(\frac{r-M}{\text{const}}\right)$$



$$\boxed{t = \frac{r}{c} + \frac{2GM}{c^3} \ln\left(\frac{r}{r_0}\right) + \mathcal{O}(G^2)}$$

We shall see that all these logs (in the FZ) can be removed by a coord. transformation

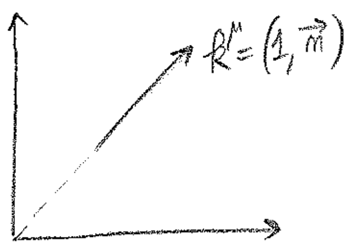
STRUCTURE OF THE QUADRATIC SOLUTION

$$h_{(1)} = \sum \partial_L\left(\frac{1}{r} R(t-r)\right) = \sum \frac{(-)^l n_L}{r} \overset{(l)}{R}(u) + \mathcal{O}(r^{-2})$$

Pose $\boxed{h^{\mu\nu}_{(1)} = \frac{1}{r} z^{\mu\nu}(\vec{n}, u) + \mathcal{O}(r^{-2})}$

Inserting $h_{(1)}$ into $\Lambda_{(2)}(h_{(1)}) \sim h_{(1)} \partial^2 h_{(1)} + \partial h_{(1)} \partial h_{(1)}$ obtain

$$\boxed{\Lambda^{\mu\nu}_{(2)} = \frac{1}{r^2}\left[4M \overset{(2)}{z}{}^{\mu\nu} + k^\mu k^\nu \sigma\right] + \mathcal{O}(r^{-3})}$$



where $\sigma(\vec{n}, u) = \frac{1}{2} \overset{(1)}{z}{}^{\mu\nu} \overset{(1)}{z}_{\mu\nu} - \frac{1}{4} \overset{(1)}{z}{}^{\mu}_{\ \mu} \overset{(1)}{z}{}^{\nu}_{\ \nu}$

$$\boxed{\sigma \propto \left(\frac{dE}{du\, d\Omega}\right)^{GW}}$$

σ is generated by distribution of energy of linearized GWs.

Structure of the remainder can be proved to be

$$\mathcal{O}(r^{-3}) = \sum_{3 \leq k \leq l+2} \partial_P\left(\frac{\hat{n}_L}{r^k} H(u)\right)$$

We need the retarded integrals of source terms of the type $\frac{\hat{n}_L}{r^2} F(u)$ or $\frac{\hat{n}_L}{r^k} H(u)$ with $3 \leq k \leq l+2$

$$\Box^{-1}_{Ret}\left(\frac{\hat{n}_L}{r^2}F(u)\right) = -\frac{\hat{n}_L}{r}\int_r^{+\infty} dz\, F(t-z)\, Q_\ell\left(\frac{z}{r}\right)$$

Legendre function of second kind

When $r \longrightarrow +\infty$

$$\boxed{\Box^{-1}_{Ret}\left(\frac{\hat{n}_L}{r^2}F(u)\right) = \frac{\hat{n}_L}{2r}\int_0^{+\infty} dy\, F(u-y)\left[\ln\left(\frac{y}{r}\right)+2\sum_{i=1}^{\ell}\frac{1}{i}\right] + O\left(\frac{\ln r}{r^2}\right)}$$

integral over the entire "past" of the source (so called hereditary terms)

appearance of $\ln r$ (linked with deviation of light cones)

When $3 \leq k \leq \ell+2$ the result is simple

$$\boxed{\text{F.P.}\,\Box^{-1}_{Ret}\left(\frac{\hat{n}_L}{r^k}F(u)\right) = \hat{n}_L\sum_{j=0}^{k-3} C_{j,\ell,k}\,\frac{F^{(k-3-j)}(u)}{r^{j+1}}}$$

composed only of instantaneous terms

$$u^{\mu\nu}_{(2)} = \text{F.P.}\,\Box^{-1}_{Ret}\Lambda^{\mu\nu}_{(2)} = \underbrace{\Box^{-1}_{Ret}\left[\frac{4M}{r^2}z^{(2)\mu\nu}\right]}_{\text{produces the so-called tails}} + \underbrace{\Box^{-1}_{Ret}\left[\frac{k^\mu k^\nu}{r^2}\sigma\right]}_{\text{responsible for non-linear memory integral}} + \left(\begin{matrix}\text{instantaneous}\\ \text{terms}\end{matrix}\right)$$

(Blanchet & Damour 1992)

(Thorne 91 Christodoulou 91 Will & Wiseman 92 BD 92)

SHOW STRUCTURE OF TAILS AND MEMORY

We have also the other piece

$$v_{(2)}^{\mu\nu} = \mathcal{H}^{\mu\nu}(w_{(2)}) \quad \text{where} \quad w_{(2)}^{\mu} = \partial_\nu u_{(2)}^{\mu\nu}$$

If
$$w_{(2)}^{0} = \frac{1}{r} A(u) + \partial_i\left(\frac{1}{r} A_i(u)\right) + \begin{pmatrix}\text{contributions}\\ l \geq 2\end{pmatrix}$$

$$w_{(2)}^{i} = \frac{1}{r} C_i(u) + \varepsilon_{iab}\, \partial_a\left(\frac{1}{r} D_b(u)\right) + \begin{pmatrix}\text{other}\\ \text{contributions}\end{pmatrix}$$

$$v_{(2)}^{00} = \underbrace{-\frac{1}{r}\int_{-\infty}^{u} dv\, A(v)}_{\text{"hereditary" modification of the mass}} \underbrace{- \partial_i\left[\frac{1}{r}\int_{-\infty}^{u} dv \int_{-\infty}^{v} dw\, A_i(w)\right]}_{\text{hered. modif. of mass dipole}} + \begin{pmatrix}\text{instantaneous}\\ \text{terms}\end{pmatrix}$$

$$v_{(2)}^{0i} = \underbrace{-\frac{1}{r}\int_{-\infty}^{u} dv\, C_i(v)}_{\text{hered. modif. linear momentum}} \underbrace{- \varepsilon_{iab}\, \partial_a\left[\frac{1}{r}\int_{-\infty}^{u} dv\, D_b(u)\right]}_{\text{hered. modif. angular momentum (spin)}} + \begin{pmatrix}\text{inst.}\\ \text{terms}\end{pmatrix}$$

$$v_{(2)}^{ij} = \begin{pmatrix}\text{inst.}\\ \text{terms}\end{pmatrix}$$

These hereditary modifications account for the losses of mass, etc... by GW emission.

$$h_{ext}^{00} = G\, h_{(1)}^{00} + G^2 h_{(2)}^{00} + \dots = \frac{4 M_{Bondi}}{r} + \begin{pmatrix}\text{other}\\ \text{moments}\end{pmatrix}$$

where M_{Bondi} is the mass measured at $\mathcal{J}^+$

$$\boxed{M_{Bondi}(u) = M_{ADM} - \frac{1}{5}\int_{-\infty}^{u} dv\, \dddot{I}_{ij}(v)\, \dddot{I}_{ij}(v) + \begin{pmatrix}\text{other } l \geq 3\\ \text{and}\\ \text{higher PM}\end{pmatrix}}$$

in agreement with quadrupole formula

RADIATIVE MULTIPOLE MOMENTS

From $h_{ext} = \sum G^n h_{(n)}$ (in harmonic coordinates) we can eliminate all the log terms at infinity $r \gg \lambda\!\!\!^{-}$

$$(t, \vec{x}) \longrightarrow (T, \vec{X})$$

harmonic coordinates → radiative coordinates

$$\boxed{U \equiv T - \frac{R}{c} \text{ is null in rad. coordinates } g^{\mu\nu}\partial_\mu U \partial_\nu U = 0}$$

At each PM order we correct from the "logarithmic" deviation of light cones. At linearized order

$$\boxed{H^{\mu\nu}_{(1)} = h^{\mu\nu}_{(1)} + \underbrace{\partial^\mu \xi^\nu_{(1)} + \partial^\nu \xi^\mu_{(1)} - \eta^{\mu\nu} \partial_\rho \xi^\rho_{(1)}}_{\text{gauge transformation at linear order } \mathcal{O}(G)}}$$

where $\xi^\mu_{(1)} = 2M \, \eta^{\mu 0} \ln\left(\frac{r}{r_0}\right)$

This gauge transformation is non-harmonic

$$\partial_\nu H^{\mu\nu}_{(1)} = \Box \xi^\mu_{(1)} = \frac{2M}{r^2} \eta^{\mu 0}$$

Need to control the term r^{-2} in $\Lambda_{(2)}$

$$\boxed{\Lambda^{\mu\nu}_{(2)}(H_{(1)}) = \frac{k^\mu k^\nu}{r^2} \sigma(\vec{n}, u) + \mathcal{O}\left(\frac{1}{r^3}\right)}$$

has the structure of the energy-momentum tensor of massless particles (gravitons)

Apply same "algorithm" as in harm. coord.

$$U^{\mu\nu}_{(2)} = \mathrm{FP}\, \Box^{-1}_{R} \Lambda^{\mu\nu}_{(2)}$$

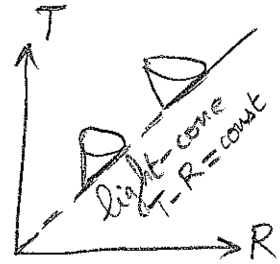
$$V^{\mu\nu}_{(2)} = \mathcal{H}^{\mu\nu}\left(W_{(2)} \equiv \partial U_{(2)}\right)$$

$$\boxed{H^{\mu\nu}_{(2)} = U^{\mu\nu}_{(2)} + V^{\mu\nu}_{(2)} + \underbrace{\partial^\mu \xi^\nu_{(2)} + \partial^\nu \xi^\mu_{(2)} - \eta^{\mu\nu}\partial_\rho \xi^\rho_{(2)}}_{\substack{\text{gauge transformation} \\ \text{at quadratic order } \mathcal{O}(G^2)}}}$$

where $$\boxed{\xi^\mu_{(2)} = \mathrm{FP}\, \Box^{-1}_{Ret}\left[\frac{k^\mu}{2r^2}\int_{-\infty}^{u} dv\, \sigma(\vec{n}, v)\right]}$$

Thanks to the structure of the r^{-2} term in $\Lambda_{(2)}$ $(\propto k^\mu k^\nu \sigma)$ this term is cancelled by the gauge transformation

$$U^{\mu\nu}_{(2)} + \partial \xi^{\mu\nu}_{(2)} = \underbrace{\Box^{-1}\left(\frac{k^\mu k^\nu \sigma}{r^2}\right) + \partial^{(\mu} \Box^{-1}\left(\frac{k^{\nu)}}{r^2}\int \sigma\right)}_{\substack{\text{cancel at leading order } r^{-2} \\ \text{since } \partial^\mu = -k^\mu \partial_u}} + \cdots$$



Hence no logs are produced

$$\xi^\mu = G\, \xi^\mu_{(1)} + G^2 \xi^\mu_{(2)} + \cdots$$

gives the (full non-linear) coordinate transformation between harmonic and radiative coordinates $(\vec{x}, t) \longrightarrow (\vec{X}, T)$ (Blanchet 1987)

We find

$$U^{\mu\nu}_{(2)} + \partial \xi^{\mu\nu}_{(2)} = \underbrace{\frac{1}{r}\int_{-\infty}^{u} dv\, K^{\mu\nu}(v,\vec{n})}_{\text{non-linear memory integral}} \quad \text{where} \quad K \approx \sum \hat{n}_L \overset{(p)}{I}_{L_1} \overset{(p)}{I}_{L_2}$$

In rad. coord. $(T, \vec{X})$ the metric admits a Bondi-type expansion $R \longrightarrow +\infty$ $U = T - R/c =$ const $(\mathscr{I}^+)$

$$\boxed{H_{(n)}(T,\vec{X}) = \sum_{k \leq N} \frac{\hat{N}_L}{R^k} K(U) + \mathcal{O}\left(\frac{1}{R^N}\right)}$$

One can then prove the "asymptotic simplicity" (Penrose 1963, 1965)

i.e. existence of a conformal transformation such that

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \quad \text{is } C^\infty \text{ at } \mathscr{I}^+$$

The radiative moments are then defined by (Thorne 1980)

$$\boxed{H^{TT}_{ij} = \frac{4G}{c^2 R} \mathcal{P}_{ijkl}(\vec{N}) \sum_{l=2}^{\infty} \frac{1}{c^l l!} \left\{ \underset{\uparrow \text{ mass-type}}{N_{L-2}\, U_{klL-2}(T-R)} + \underset{\uparrow \text{ current-type}}{\frac{1}{c} N_{aL-2}\, \varepsilon_{ab(k} V_{l)bL-2}(T-R)} \right\}} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

Purely an "algebraic" definition from the $1/R$ term of the metric in radiative coordinates

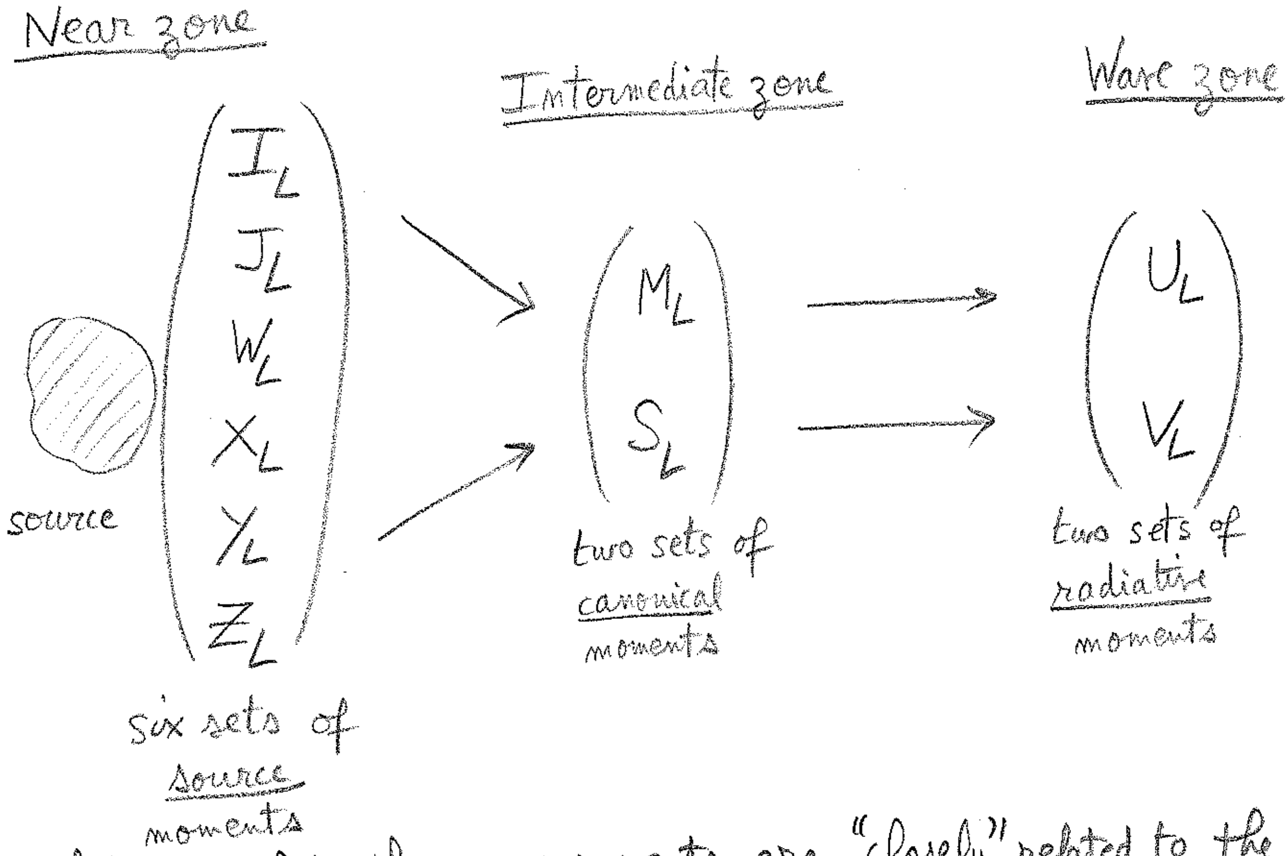
Energy flux in GWs is

$$\boxed{\mathcal{F} \equiv \left(\frac{dE}{dT}\right)^{GW} = \sum_{l=2}^{\infty} \frac{1}{c^{2l+1}} \left\{ a_l \overset{(1)}{U}_L \overset{(1)}{U}_L + \frac{b_l}{c^2} \overset{(1)}{V}_L \overset{(1)}{V}_L \right\}}$$

The rad. moments agree with the canonical ones at leading PM order

$$\boxed{\begin{aligned} U_L &= \overset{(l)}{M}_L + \mathcal{O}(G) \\ V_L &= \overset{(l)}{S}_L + \mathcal{O}(G) \end{aligned}}$$

non-linear corrections including tails, tails-of-tails non-linear memory etc...



We shall see that the source moments are "closely" related to the source in the sense that they admit closed form expressions in terms of the source's parameters.