PART 3

MATCHING TO THE FIELD OF A POST-NEWTONIAN SOURCE

THE MATCHING EQUATION

We have contructed the exterior field (physically valid when $r > a$) of any isolated source

$$h_{ext} = \sum_{n=1}^{+\infty} G^n \, h_{(n)} \big[\underbrace{I_L \; J_L \; W_L \dots Z_L}_{\text{source moments (for the moment arbitrary)}} \big]$$

We suppose that h_{ext} comes from the multipole expansion of h defined everywhere inside and outside the source (for any r)

$$\boxed{h_{ext} = \mathcal{M}(h)}$$

operation of taking the multipole expansion

Note that $\mathcal{M}(h)$ is defined of any $r > 0$ but agrees with the "true" field h only when $r > a$

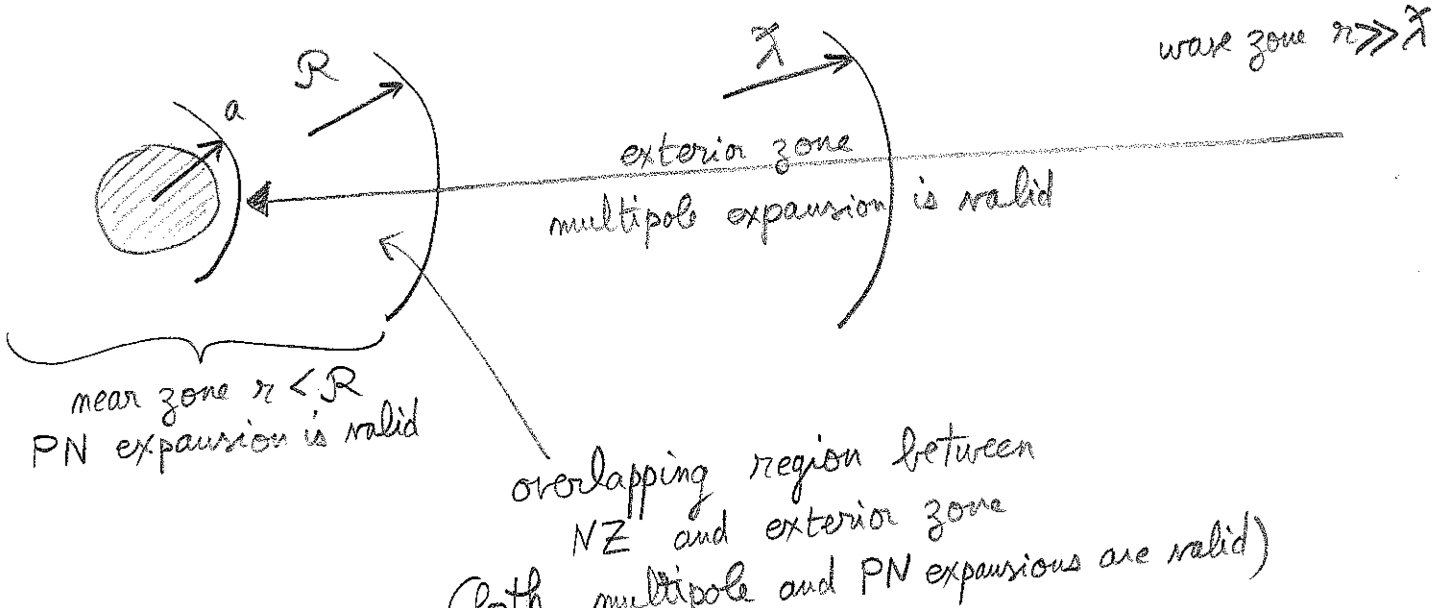
$$\boxed{r > a \quad \Longrightarrow \quad \mathcal{M}(h) = h \quad \text{(numerically)}}$$

But when $r \longrightarrow 0$ $\mathcal{M}(h)$ diverges while h is a perfectly smooth solution Einstein field eqs. inside the matter (of the extended source).

Suppose the source is <u>post-Newtonian</u> (existence of the PN parameter $\varepsilon = \frac{v}{c} \ll 1$). We know that the near zone $r < \mathcal{R}$ where $\mathcal{R} \ll \lambda\!\!\!^{-}$ encloses totally the PN source ($\mathcal{R} > a$).

In the NZ the field h can be expanded as a PN expansion ($\bar{h} = \sum c^{-p} (\ln c)^q$)

$$\boxed{r < \mathcal{R} \quad \Rightarrow \quad h = \bar{h} \quad \text{(numerically)}}$$



$$\boxed{a < r < \mathcal{R} \quad \Rightarrow \quad \mathcal{M}(h) = \bar{h} \quad \text{(numerically)}}$$

The <u>matching equation</u> follows from transforming the latter numerical equality in a <u>functional identity</u> (valid $\forall (\vec{x}, t)$ in $\mathbb{R}^3_* \times \mathbb{R}$) between two <u>formal asymptotic series</u>

Matching equation: $\boxed{\overline{\mathcal{M}(h)} \equiv \mathcal{M}(\overline{h})}$

NZ expansion $\left(\frac{r}{c} \to 0\right)$ of each multipolar coeff. of $\mathcal{M}(h)$

multipole expansion of each PN coefficient of $\overline{h}$

We assume (as part of our fundamental assumptions) that the matching eq. is correct (in the sense of formal series)

$$\boxed{\underset{\frac{r}{c} \to 0}{\text{NZ expansion}} \begin{pmatrix} \text{multipolar} \\ \text{expansion} \\ \frac{a}{r} \to 0 \end{pmatrix} \equiv \underset{r \to \infty}{\text{FZ expansion}} \begin{pmatrix} \text{PN series} \\ c \to \infty \end{pmatrix}}$$

The NZ expansion $\frac{r}{c} \to 0$ is "equivalent" to the PN expansion $c \to +\infty$ for fixed r

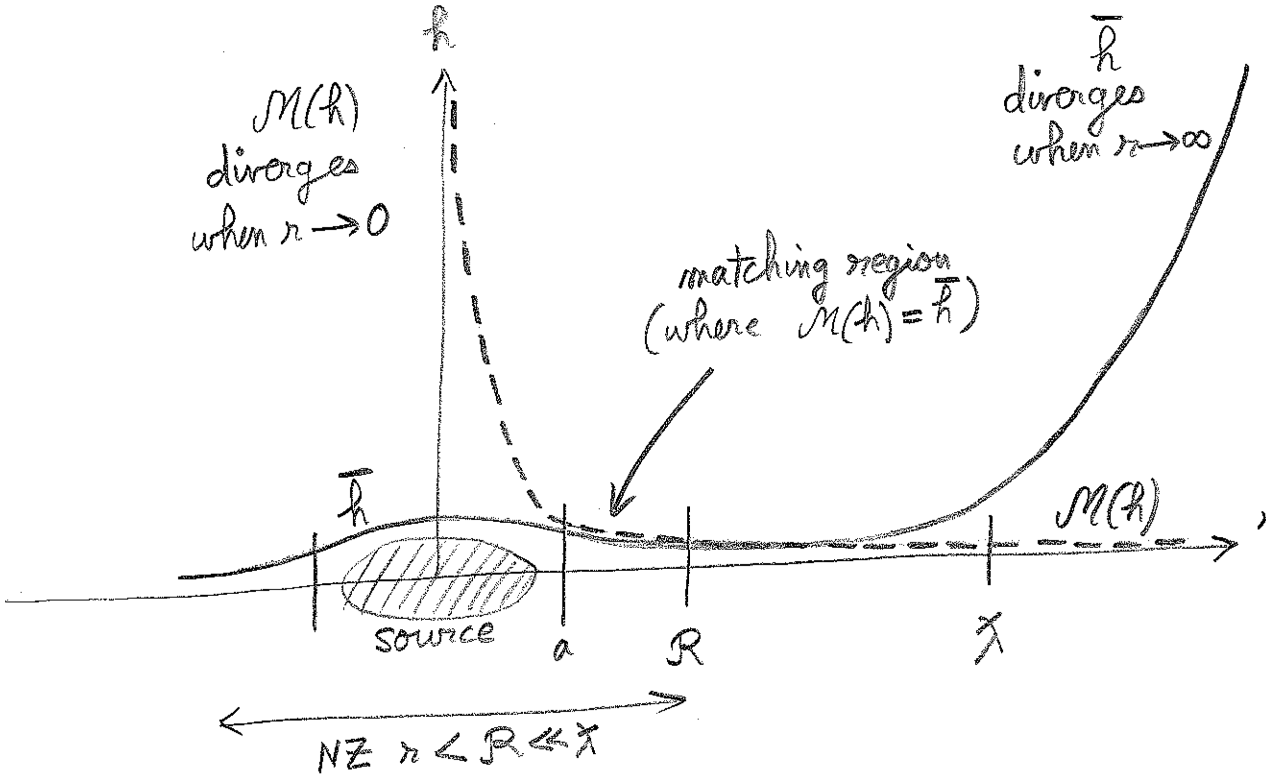
The multipole expansion $\frac{a}{r} \to 0$ is equivalent to the FZ expansion $r \to +\infty$ for a given source (fixed a)

The matching equation says basically the NZ and multipole expansions can be commuted.

Thus there is a common structure for the formal NZ and FZ expansions

$$\overline{\mathcal{M}(h)} \equiv \underbrace{\sum \hat{n}_L r^p (\ln r)^q F(t)}_{\text{can be interpreted either as}} \equiv \mathcal{M}(\bar{h})$$

- NZ singular expansion when $r \to 0$
- FZ ——————— $r \to \infty$



GENERAL EXPRESSION OF THE MULTIPOLE MOMENTS

h is the sol. of Einstein eqs (in harmonic coord. $\partial h = 0$) valid everywhere inside and outside the source

$$h = \frac{16\pi G}{c^4} \Box^{-1}_{Ret} \tau \qquad \begin{pmatrix}\text{suppress} \\ \text{indices } \mu\nu\end{pmatrix}$$

where $$\tau = |g| T + \underbrace{\frac{c^4}{16\pi G}\Lambda}_{\substack{\text{gravitational source-term} \\ \text{(non-linearities in } h)}}$$

Define $\boxed{\Delta \equiv h - \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}} \mathcal{M}(\Lambda)}$

where $\mathcal{M}(\Lambda) = \Lambda[\mathcal{M}(h)] = \Lambda_{ext}$ and FP is the finite part when $B \to 0$ (plays a crucial role because Λ_{ext} diverges when $r \to 0$)

$$\Delta = \underbrace{\frac{16\pi G}{c^4} \Box^{-1}_{\mathrm{Ret}} \tau}_{\substack{\text{no FP here} \\ \text{since } \tau \text{ is regular } (C^\infty)}} - \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}} \mathcal{M}(\Lambda)$$

However we can add FP on the first term (do not change the value because it converges). Using also $\mathcal{M}(\tau) = 0$ since τ has a compact support

$$\Delta = \frac{16\pi G}{c^4} \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}} \left[\tau - \mathcal{M}(\tau)\right]$$

Hence Δ appears as the retarded integral of a source with compact support. Indeed

$$\tau = \mathcal{M}(\tau) \quad \text{when } r > a$$

$$\boxed{\mathcal{M}(\Delta) = -\frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L \left(\frac{1}{r} \mathcal{H}_L(u)\right)}$$

This is standard expression of multipolar expansion outside a compact-support source. Here the moments are

$$\mathcal{H}_L = \mathrm{FP} \int d^3x \, x_L \underbrace{\left[\tau - \mathcal{M}(\tau) \right]}$$

since this has compact support ($r < a$, inside the NZ) we can replace by the NZ or PN expansion

$$\mathcal{H}_L = \mathrm{FP} \int d^3x \, x_L \left[\overline{\tau} - \overline{\mathcal{M}(\tau)} \right]$$

But we know the structure $\overline{\mathcal{M}(\tau)} = \sum \hat{n}_L r^p (\ln r)^q F(t)$ which is sufficient to prove that the second term is <u>zero</u> by analytic continuation

$$\mathrm{FP} \int d^3x \, x_L \, \overline{\mathcal{M}(\tau)} = \sum \mathrm{FP} \int d^3x \, x_L \hat{n}_Q r^p (\ln r)^q$$

$$= \sum \underset{B \to 0}{\text{Finite Part}} \int dr \, r^{B+S} (\ln r)^p$$

integrate over angles

$$= \sum \underset{B \to 0}{\mathrm{FP}} \left(\frac{d}{dB} \right)^p \int_0^{+\infty} dr \, r^{B+S}$$

$$\int_0^{+\infty} dr \, r^{B+S} = \underbrace{\int_0^{R} dr \, r^{B+S}}_{\substack{\text{computed} \\ \text{when } \mathrm{Re}\, B > -S-1}} + \underbrace{\int_R^{+\infty} dr \, r^{B+S}}_{\substack{\text{computed} \\ \text{when } \mathrm{Re}\, B < -S-1}}$$

$$= \underbrace{\frac{R^{B+S+1}}{B+S+1}}_{\text{by analytic continuation}} \qquad = \underbrace{- \frac{R^{B+S+1}}{B+S+1}}_{\text{by analytic continuation}}$$

$$\boxed{\text{Analytic Continuation} \quad \int_0^{+\infty} dr\, r^{B+s} (\ln r)^p = 0 \qquad \forall B \in \mathbb{C}}$$

The general multipole expansion outside the domain of a PN isolated source reads (Blanchet 1995, 1998)

$$\boxed{\mathcal{M}(h) = \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}} \mathcal{M}(\Lambda) - \frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L \left(\frac{1}{r} \mathcal{H}_L(u) \right)}$$

where

$$\boxed{\mathcal{H}_L(u) = \mathrm{FP} \int d^3x\, x_L\, \overline{\tau}(\vec{x}, u)}$$

PN expansion crucial here
(this is where the formalism applies only to PN sources)

Same result but in STF guise

$$\mathcal{M}(h) = \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}} \mathcal{M}(\Lambda) - \frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L \left(\frac{1}{r} \mathcal{F}_L(u) \right)$$

where

$$\mathcal{F}_L(u) = \mathrm{FP} \int d^3x\, \hat{x}_L \int_{-1}^{1} dz\, \delta_l(z)\, \overline{\tau}(\vec{x}, u + z|\vec{x}|/c)$$

$$\delta_l(z) = \frac{(2l+1)!!}{2^{l+1}\, l!} (1-z^2)^l \quad \text{such that} \quad \int_{-1}^{1} dz\, \delta_l(z) = 1, \qquad \lim_{l \to +\infty} \delta_l(z) = \delta(z)$$

Practical way to implement the STF multipole expansion is to use the PN series

$$\boxed{\int_{-1}^{1} dz\, \delta_\ell(z)\, \overline{\tau}\left(\vec{x}, u + z|\vec{x}|/c\right) = \sum_{k=0}^{+\infty} \alpha_k^\ell \left(\frac{|\vec{x}|}{c}\frac{\partial}{\partial u}\right)^{2k} \overline{\tau}(\vec{x}, u)}$$

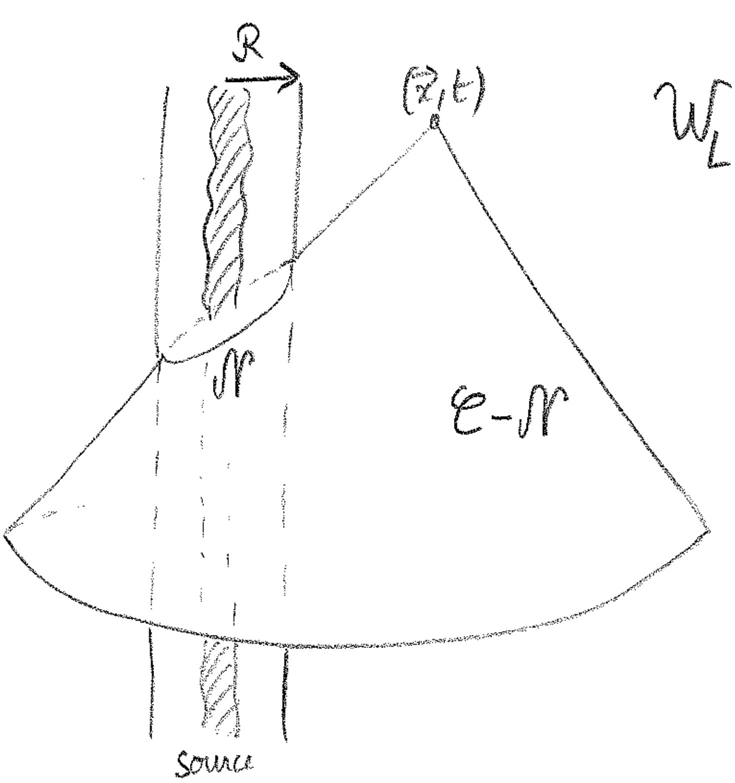
$$\alpha_k^\ell = \frac{(2\ell+1)!!}{(2k)!!\,(2\ell+2k+1)!!}$$

There is an alternative formalism for writing the general multipole expansion (Will & Wiseman 1996)

$$\mathcal{M}(h) = \underbrace{\Box^{-1}_{Ret}\, \mathcal{M}(\Lambda)\Big|_{\mathcal{C}-\mathcal{N}}}_{\text{the retarded integral excludes the NZ of source}} - \frac{4G}{c^4}\sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!}\, \partial_L\left(\frac{1}{r}\mathcal{W}_L(t-r)\right)$$

where

$$\mathcal{W}_L(u) = \underbrace{\int_{r<\mathcal{R}} d^3x\, x_L\, \overline{\tau}(\vec{x}, u)}_{\text{volume integral limited to the NZ of the source } (\mathcal{N})}$$



<u>The two formalisms are equivalent</u>

Next we identify $h_{ext} = \mathcal{M}(h)$ which means

$$G\, h_{(1)}\left[I_L J_L W_L \cdots Z_L\right] + G^2 h_{(2)} + \cdots + G^n h_{(n)} + \cdots$$

$$= \underbrace{-\frac{4G}{c^4}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r}\mathcal{F}_L^1(u)\right)}_{\substack{\text{has the form of the linear}\\ \text{metric } G h_{(1)} \text{ where the } \mathcal{F}_L^1\text{'s}\\ \text{are "equivalent" to } I_L \cdots Z_L}} + \underbrace{\mathrm{FP}\,\Box^{-1}_{Ret}\,\mathcal{M}(\Lambda)}_{\substack{\text{represents the}\\ \text{non-linear corrections}\\ G^2 h_{(2)} + \cdots + G^n h_{(n)} + \cdots}}$$

Note that for the identification to work the $\mathcal{F}_L^1$'s in the right-hand-side should be considered as of <u>zero-th order in G</u>

Then we obtain $I_L \cdots Z_L$ in terms of the components of $\mathcal{F}_L^{\mu\nu}$ and hence of the source's pseudo-tensor $\overline{\tau}^{\mu\nu}$.

Decompose the $\mathcal{F}_L^{\mu\nu}$'s into ten irreducible STF tensors $R_L\ T^{(+)}_{L+1} \cdots U^{(-2)}_{L-2}\ V_L$

$$\begin{cases} \mathcal{F}_L^{1\,00} = R_L \\[2mm] \mathcal{F}_L^{0i} = T^{(+)}_{iL} + \varepsilon_{ai<i_\ell} T^{(0)}_{L-1>a} + \delta_{i<i_\ell} T^{(-)}_{L-1>} \\[2mm] \mathcal{F}_L^{ij} = U^{(+2)}_{ijL} + \underset{L}{\mathrm{STF}}\,\underset{ij}{\mathrm{STF}}\Big[\varepsilon_{aii_\ell} U^{(+1)}_{ajL-1} + \delta_{ii_\ell} U^{(0)}_{jL-1} \\ \qquad + \delta_{ii_\ell}\varepsilon_{aji_{\ell-1}} U^{(-1)}_{aL-2} + \delta_{ii_\ell}\delta_{ji_{\ell-1}} U^{(-2)}_{L-2}\Big] + \delta_{ij} V_L \end{cases}$$

The final result is

$$\boxed{\begin{aligned} I_L &= \mathrm{FP}\int d^3x \int_{-1}^{1} dz \Big\{ \delta_\ell(z)\, \hat{x}_L\, \Sigma - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)}\delta_{\ell+1}\, \hat{x}_{iL}\, \Sigma_i^{(1)} \\ &\qquad + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)}\delta_{\ell+2}\, \hat{x}_{ijL}\, \Sigma_{ij}^{(2)} \Big\}(\vec{x}, u+z|x|/c) \\[2em] J_L &= \mathrm{FP}\int d^3x \int_{-1}^{1} dz\; \varepsilon_{ab\langle i_\ell} \Big\{ \delta_\ell\, \hat{x}_{L-1\rangle a}\, \Sigma_b \\ &\qquad - \frac{2\ell+1}{c^2(\ell+2)(2\ell+3)}\delta_{\ell+1}\, \hat{x}_{L-1\rangle ac}\, \Sigma_{bc}^{(1)} \Big\}(\vec{x}, u+z|\vec{x}|/c) \end{aligned}}$$

where $\begin{cases} \Sigma = \dfrac{\overline{\tau}^{00} + \overline{\tau}^{ii}}{c^2} \\ \Sigma_i = \dfrac{\overline{\tau}^{0i}}{c} \\ \Sigma_{ij} = \overline{\tau}^{ij} \end{cases}$

There are similar expressions for $W_L \cdots Z_L$

These expressions give the source moments of any isolated PN source, <u>up to any PN order</u> (formally).

<u>POST-NEWTONIAN EXPANSION IN THE NEAR ZONE</u>

Consider the PN expansion of the field in the NZ ($r < R$)

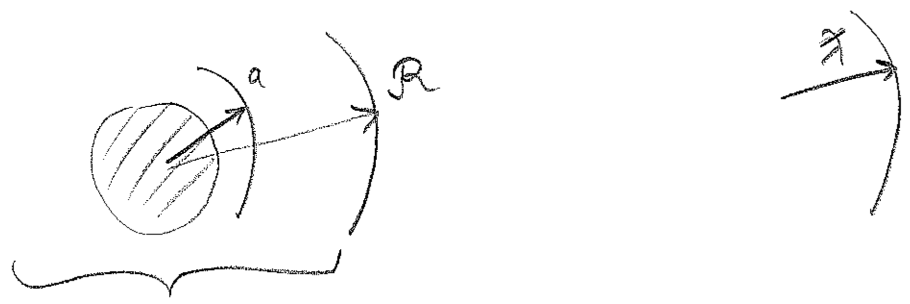
$$\boxed{\overline{h}(\vec{x}, t, c) = \sum_{p=2}^{+\infty} \frac{1}{c^p}\, \overline{h}_p(\vec{x}, t, \ln c)}$$

formal PN series (appearance of $\ln c$'s at 4PN order)

<u>Note</u>: $\overline{h}_p$ denotes the PN coefficient of $\frac{1}{c^p}$ while $h_{(n)}$ denotes the PM coefficient of G^n

To compute iteratively the $\bar{h}_n$'s we meet two problems

① Problem of NZ limitation



$\bar{h}$ is valid only in NZ
(and diverges in the FZ, when $r \to \infty$)

How to incorporate into the PN series the information about boundary conditions at infinity (notably the no-incoming radiation condition which is imposed at $\mathcal{J}^-$) ?

② Problem of divergencies

$$\Delta \bar{h}_p = \begin{pmatrix} \text{source term} \\ \text{with non-compact} \\ \text{support} \\ \text{which blows up when } r \to +\infty \end{pmatrix}$$

Then the usual Poisson integral is divergent

$$\boxed{\bar{h}_p = \int \frac{d^3\vec{x}'}{|\vec{x}-\vec{x}'|}\,(\text{source term})}$$

diverges at the bound $|\vec{x}'| = +\infty$
(for high p)

Problem ① will be solved by matching: $\overline{\mathcal{M}(h)} = \mathcal{M}(\bar{h})$

Problem ② will be solved by finding a suitable solution of the Poisson equation (different from the Poisson integral)

Insert $\overline{h} = \sum \frac{1}{c^p} \overline{h}_p$ into $\begin{cases} \Box \overline{h} = \frac{16\pi G}{c^4} \overline{\tau} \\ \partial \overline{h} = 0 \end{cases}$

Hierarchy of PN equations $(\forall n \geq 2)$

$$\boxed{\begin{aligned} \Delta \overline{h}^{\mu\nu}_p &= 16\pi G \, \overline{\tau}^{\mu\nu}_{p-4} + \partial_t^2 \overline{h}^{\mu\nu}_{p-2} \\ \partial_\nu \overline{h}^{\mu\nu}_p &= 0 \end{aligned}}$$

At any given p the right-hand-side is known from previous iteration (using recursive treatment).

Construct first a particular solution of these equations using the generalized Poisson integral (Poujade & Blanchet 2002)

$$\mathrm{FP}\Delta^{-1}\left[\overline{\tau}_p\right] \equiv \underset{B\to 0}{\text{Finite Part}} \; \underbrace{\frac{-1}{4\pi} \int \frac{d^3\vec{x}' \, |\vec{x}'|^B}{|\vec{x}-\vec{x}'|} \overline{\tau}_p(\vec{x}', t)}_{\text{defined by analytic continuation}}$$

Then we add the general homogeneous solution of Laplace's equation which is regular in the source $(r \to 0)$

$$\Delta \left[a \, \underbrace{\hat{x}_L}_{\substack{\text{solution}\\ \text{regular}\\ \text{when } r\to 0}} + b \, \underbrace{\hat{\partial}_L \frac{1}{r}}_{\substack{\text{solution}\\ \text{regular}\\ \text{when } r\to\infty}} \right] = 0$$

Most general solution is

$$\boxed{\bar{h}'^{\mu\nu}_p = \underbrace{\mathrm{FP}\,\Delta^{-1}\left\{16\pi G\,\bar{\tau}'^{\mu\nu}_{p-4} + \partial_t^2 \bar{h}'^{\mu\nu}_{p-2}\right\}}_{\text{particular solution}} + \underbrace{\sum_{l=0}^{+\infty} B'^{\mu\nu}_{pL}(t)\,\hat{x}_L}_{\text{homogeneous solution}}}$$

particular solution (well-defined thanks to the Finite Part)

homogeneous solution (unknown for the moment)

To compute the homogenous solution we require that it matches the external field in the sense

$$\mathcal{M}\left(\sum \frac{1}{c^p}\bar{h}'^{\mu\nu}_p\right) = \overline{\mathcal{M}(h)} = \overline{\sum G^n h_{(n)}}$$

where $\mathcal{M}(h) = h_{ext} = \sum G^n h_{(n)}$. This fixes uniquely the homogeneous solution which is associated with <u>radiation reaction forces</u> inside the source, appropriate to an isolated system emitting GWs but not receiving GWs from $\mathcal{J}^-$.

Summing up $\bar{h} = \sum \frac{1}{c^p}\bar{h}_p$ we get

$$\boxed{\bar{h}'^{\mu\nu} = \underbrace{\frac{16\pi G}{c^4}\left\{\sum_{k=0}^{\infty}\left(\frac{\partial}{c\partial t}\right)^{2k}\mathrm{FP}\,\Delta^{-k-1}\bar{\tau}'^{\mu\nu}\right\}}_{\text{particular solution}} - \underbrace{\frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\partial_L\left\{\frac{\mathcal{A}^{\mu\nu}_L(t-r)-\mathcal{A}^{\mu\nu}_L(t+r)}{2r}\right\}}_{\text{homogeneous solution}}}$$

particular solution of d'Alembert eq. denoted $\mathrm{FP}\,\mathcal{I}^{-1}\bar{\tau}^{\mu\nu}$

homogeneous solution of d'Alembert eq. which is <u>regular when $r \to 0$</u>

It's an <u>anti-symmetric wave</u> (retarded) − (advanced)

Result of the matching is (Poujade & Blanchet 2002)

$$\mathcal{A}_L^{\mu\nu}(u) = \mathcal{F}_L^{\mu\nu}(u) + \mathcal{R}_L^{\mu\nu}(u)$$

where $\mathcal{F}_L^{\mu\nu}$ is the source's multipole moment $\left(\begin{array}{c}\text{computed}\\ \text{previously}\end{array}\right)$

$$\mathcal{F}_L^{\mu\nu}(u) = \mathrm{FP}\int d^3x\, \hat{x}_L \int_{-1}^{1} dz\, \delta_\ell(z)\, \underbrace{\overline{\tau}^{\mu\nu}}_{\text{PN expansion of } \tau}(\vec{x}, u + z|\vec{x}|/c)$$

and where $\mathcal{R}_L^{\mu\nu}(u)$ is a new type of moment which turns out to parametrize <u>non-linear radiation reaction effects</u> in the source (Blanchet 1993)

$$\boxed{\mathcal{R}_L^{\mu\nu}(u) = \mathrm{FP}\int d^3x\, \hat{x}_L \int_{1}^{+\infty} dz\, \gamma_\ell(z)\, \underbrace{\mathcal{M}(\tau^{\mu\nu})}_{\text{multipole expansion of } \tau}(\vec{x}, u - z|\vec{x}|/c)}$$

where $\gamma_\ell(z) = -2\,\delta_\ell(z)$ satisfies (by analytic continuation in ℓ)

$$\int_1^{+\infty} dz\, \gamma_\ell(z) = 1 \qquad \gamma_\ell(z) = (-)^{\ell+1}\frac{(2\ell+1)!!}{2^\ell \ell!}(z^2-1)^\ell$$

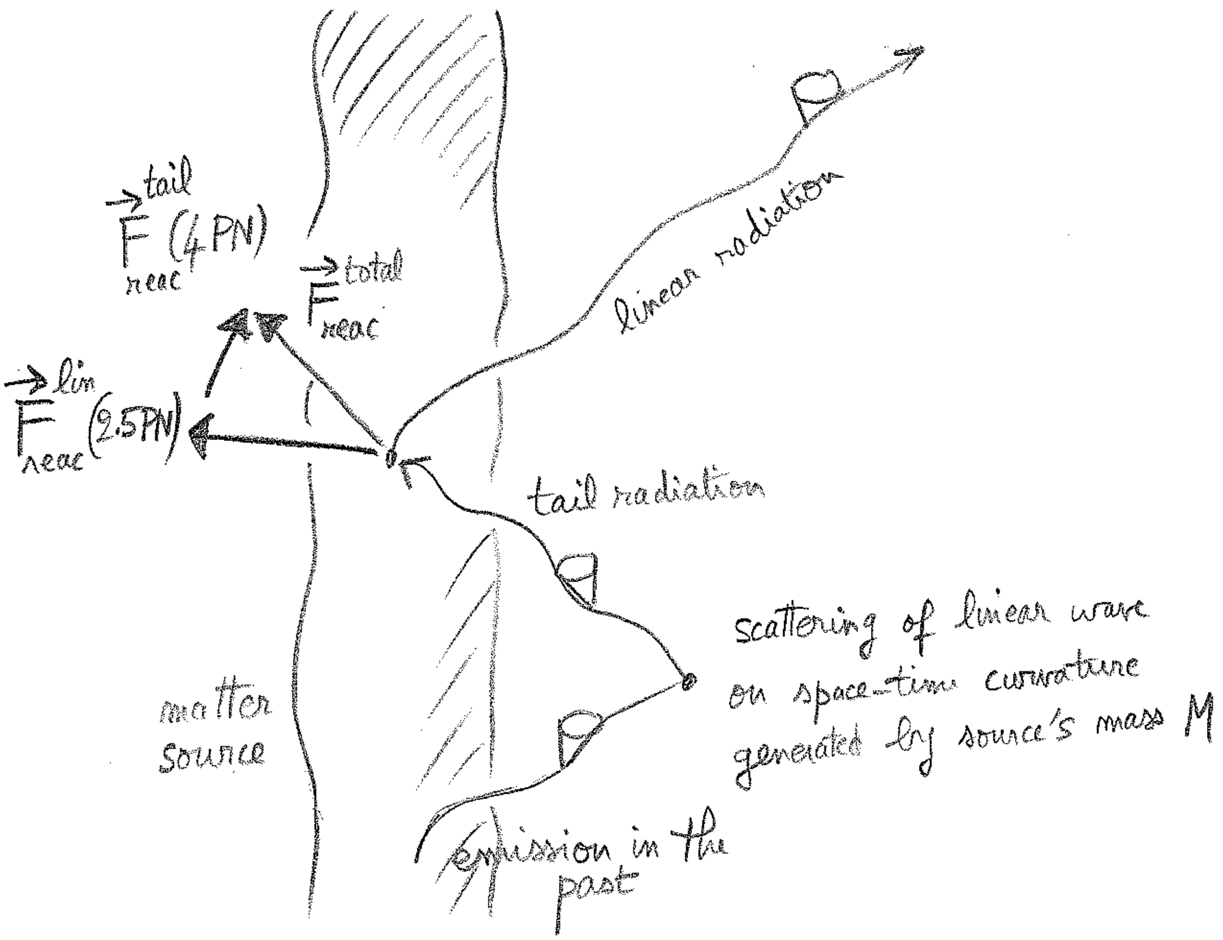
This comes from

$$\underbrace{0 = \int_{-\infty}^{+\infty} dz\, \delta_\ell(z)}_{\text{by analytic continuation in } \ell\in\mathbb{C}} = 2\int_1^{\infty} dz\, \delta_\ell(z) + \int_{-1}^{1} dz\, \delta_\ell(z) = -\int_1^{\infty} dz\, \gamma_\ell(z) + 1$$

Note that the PN expansion in the NZ ($r < \mathcal{R}$) depends on the multipole exp. $\mathcal{M}(\tau^{\mu\nu})$ and therefore on the properties of the field in the FZ ($r \gg \lambda$).

Indeed the PN exp. includes the radiation reaction terms appropriate to an isolated system, satisfying the correct boundary conditions at infinity (notably $\mathcal{I}^-$).

$$\mathcal{A}_L^{\mu\nu} = \underbrace{\mathcal{F}_L^{\mu\nu}}_{\substack{\text{describes "linear"} \\ \text{radiation reaction terms} \\ \text{and \underline{starts at 2.5PN}}}} + \underbrace{\mathcal{R}_L^{\mu\nu}}_{\substack{\text{describes "non-linear"} \\ \text{effects (tails) in the} \\ \text{radiation reaction and} \\ \text{\underline{starts at 4PN}}}}$$



The linear rad. reac. (parametrized by $\mathcal{F}_L^{\mu\nu}$) can be recombined with the particular solution

$$\mathrm{FP}\,\mathcal{I}^{-1}\,\overline{\tau}^{\mu\nu} = \sum_{k=0}^{+\infty}\left(\frac{\partial}{c\partial t}\right)^{2k}\mathrm{FP}\,\Delta^{-k-1}\,\overline{\tau}^{\mu\nu}$$

to give simply the retarded integral

$$\mathrm{FP}\,\Box^{-1}_{\mathrm{Ret}}\,\overline{\tau}^{\mu\nu} = -\frac{1}{4\pi}\underbrace{\sum_{p=0}^{+\infty}\frac{(-)^p}{p!}\left(\frac{\partial}{c\partial t}\right)^p\mathrm{FP}\int d^3x'\,|x-x'|^{p-1}\,\overline{\tau}^{\mu\nu}(\vec{x}',t)}_{\substack{\text{formal expansion } c\to+\infty \\ \text{of the retardation } t-\frac{1}{c}|\vec{x}-\vec{x}'| \\ \text{(well-defined thanks to the FP)}}}$$

The sol. $\mathrm{FP}\,\mathcal{I}^{-1}$ corresponds to the even-parity part $p=2k$.
The odd-parity $p=2k+1$ is exactly given by the terms with $\mathcal{F}_L^{\mu\nu}$

Final result is thus (Blanchet, Faye & Nissanke 2005)

$$\boxed{\overline{h}^{\mu\nu} = \underbrace{\frac{16\pi G}{c^4}\,\mathrm{FP}\,\Box^{-1}_{\mathrm{Ret}}\,\overline{\tau}^{\mu\nu}}_{\substack{\text{corresponds to the old} \\ \text{way of performing the} \\ \text{PN expansion (Anderson \& DeCanio} \\ \text{1975)}}} - \underbrace{\frac{4G}{c^4}\sum_{l=0}^{+\infty}\partial_L\left\{\frac{\mathcal{R}_L^{\mu\nu}(t-r)-\mathcal{R}_L^{\mu\nu}(t+r)}{2r}\right\}}_{\text{starts at 4PN}}}$$

PART 4

APPLICATION TO COMPACT BINARIES

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THE 3PN METRIC

Detailed calculations at 3PN use explicit expressions of the near-zone metric coefficients (in harm. coord.)

$$\boxed{\begin{aligned}
g_{00} &= -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \frac{8}{c^6}\left(\hat{X} + V_iV_i + \frac{V^3}{6}\right) \\
&\quad + \frac{32}{c^8}\left(\hat{T} + \cdots\right) + \mathcal{O}\left(\frac{1}{c^{10}}\right) \\[2ex]
g_{0i} &= -\frac{4}{c^3}V_i - \frac{8}{c^5}\hat{R}_i - \frac{16}{c^7}\left(\hat{Y}_i + \cdots\right) + \mathcal{O}\left(\frac{1}{c^9}\right) \\
g_{ij} &= \delta_{ij}\left[1 + \frac{2}{c^2}V + \frac{2}{c^4}V^2 + \frac{8}{c^6}\left(\hat{X} + \cdots\right)\right] \\
&\quad + \frac{4}{c^4}\hat{W}_{ij} + \frac{16}{c^6}\left(\hat{Z}_{ij} + \cdots\right) + \mathcal{O}\left(\frac{1}{c^8}\right)
\end{aligned}}$$

The potentials are generated by $T^{\mu\nu}$

$$\boxed{\begin{aligned}
\sigma &= \frac{T^{00} + T^{ii}}{c^2} \\
\sigma_i &= \frac{T^{0i}}{c} \\
\sigma_{ij} &= T^{ij}
\end{aligned}}$$

$\sigma = \rho + \mathcal{O}\left(\frac{1}{c^2}\right)$
where ρ is source's Newtonian density

V and V_i represent some retarded versions of the Newtonian and "gravitomagnetic" potentials

$$\boxed{\begin{aligned} V &= \Box^{-1}_{Ret}\left(-4\pi G\sigma\right) \\ V_i &= \Box^{-1}_{Ret}\left(-4\pi G\sigma_i\right) \end{aligned}}$$

$\hat{W}_{ij}$ is generated by matter + gravitational "stresses"

$$\boxed{\hat{W}_{ij} = \Box^{-1}_{Ret}\left[-4\pi\left(\sigma_{ij} - \delta_{ij}\,\sigma_{kk}\right) - \underbrace{\partial_i V \partial_j V}_{\text{quadratic non-linearity}}\right]}$$

$\hat{X}$, $\hat{R}_i$, $\hat{Z}_{ij}$, $\hat{T}$ are higer-order PN potentials

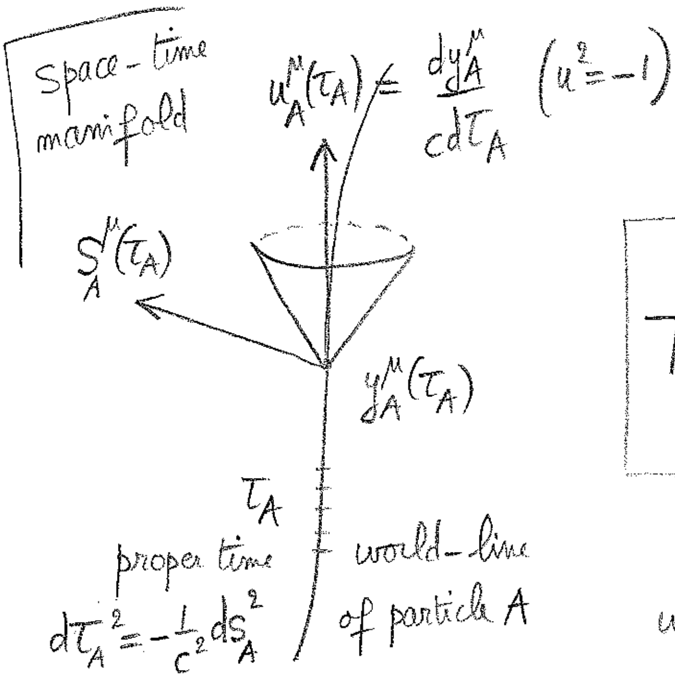
$$\hat{X} = \Box^{-1}_{Ret}\left[-4\pi G V \sigma_{ii} + \underbrace{\hat{W}_{ij}\,\partial_{ij} V}_{\text{cubic term}} + \cdots\right]$$

$$\hat{T} = \Box^{-1}_{Ret}\left[-4\pi G\left(\frac{1}{4}\sigma_{ij}\hat{W}_{ij} + \cdots\right) + \hat{Z}_{ij}\,\partial_{ij} V + \cdots\right]$$

and so on. The 3PN metric parametrized by these potentials is very useful in practice (permits to separate out different problems associated with quadratic, cubic, etc... non-linearities). At Newtonian order

$V = U + O\left(\frac{1}{c^2}\right)$ where $\underbrace{U = \Delta^{-1}\left(-4\pi G\rho\right)}_{\text{Newtonian potential}}$ is the usual

STRESS-ENERGY TENSOR OF POINT PARTICLES



$$\boxed{T^{\mu\nu}(x) = \sum_A \int_{-\infty}^{+\infty} d\tau_A \, p_A^{(\mu} u_A^{\nu)} \frac{\delta_4(x-y_A)}{\sqrt{-g_A}}}$$

where $p^\mu_A = m\, u^\mu_A$ (without spin)

In PN calculations we "split" space & time

$$y^\mu_A = (ct, \vec{y}_A) \qquad v^\mu_A = (c, \vec{v}_A) \qquad \text{where} \quad \boxed{v^i_A = \frac{dy^i_A}{dt} = c\,\frac{u^i_A}{u^0_A}}$$

ordinary (coordinate) velocity

$$\boxed{T^{\mu\nu}(\vec{x},t) = \sum_A \frac{m_A\, v^\mu_A v^\nu_A}{\sqrt{-g^A_{\rho\sigma} v^\rho_A v^\sigma_A}} \frac{\delta(\vec{x}-\vec{y}_A)}{\sqrt{-g_A}}}$$

$\delta(\vec{x}-\vec{y}_A)$ is Dirac's 3-dim function

For spinning particles we can add

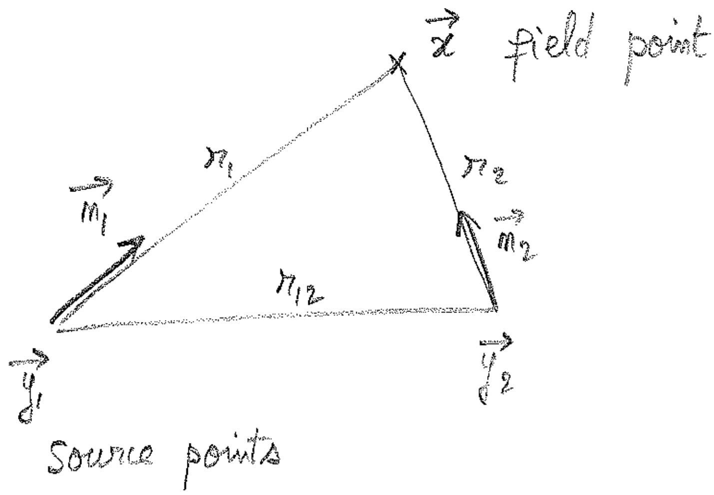
$$\boxed{T^{\mu\nu}_{\text{spin}}(x) = -\sum_A \nabla_\rho \left[\int_{-\infty}^{+\infty} d\tau_A \, S_A^{\rho(\mu} u_A^{\nu)} \frac{\delta_4(x-y_A)}{\sqrt{-g_A}} \right]}$$

(Dixon 1970, Bailey & Israel 1980)

where $S^{\mu\nu}_A$ is the spin anti-symmetric tensor

PROBLEM OF POINT PARTICLES

Two (say) point-like particles (masses m_1 and m_2)



$$r_A = |\vec{x} - \vec{y}_A| \qquad \vec{n}_A = \frac{\vec{x} - \vec{y}_A}{r_A}$$

$$r_{12} = |\vec{y}_1 - \vec{y}_2|$$

Newtonian potential U generated by the point masses

$$\Delta U = -4\pi G \rho = -4\pi G \left[m_1 \delta(\vec{x} - \vec{y}_1) + m_2 \delta(\vec{x} - \vec{y}_2) \right]$$

Using $\Delta \frac{1}{r} = -4\pi\,\delta(\vec{x})$ $\qquad U(\vec{x}) = \frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}$

$$\frac{d\vec{v}_1}{dt} = (\vec{\nabla} U)(\vec{y}_1) = \Bigg(\underbrace{-\frac{Gm_1}{r_1^2}\vec{n}_1}_{\substack{\text{self-force on the} \\ \text{point-particle is divergent}}} - \frac{Gm_2}{r_2^2}\vec{n}_2 \Bigg)(\vec{y}_1)$$

Problem 1 $\quad$ If $F(\vec{x})$ is divergent at $\vec{y}_1$ (say, with a power-like singular expansion around $\vec{y}_1$) what is the meaning of $F(\vec{y}_1)$?

Stress-energy tensor of point-particles

$$T^{\mu\nu} = \sum_A m_A \int_{-\infty}^{+\infty} d\tau_A\, u_A^\mu u_A^\nu \frac{\delta_4(x-y_A)}{\sqrt{-g}} = \sum_A \frac{m_A v_A^\mu v_A^\nu}{\sqrt{-g^A_{\rho\sigma} v_A^\rho v_A^\sigma}} \frac{\delta(\vec{x}-\vec{y}_A)}{\sqrt{-g}}$$

But $g \simeq -1 + \frac{U}{c^2} + \cdots$ where $U(\vec{x})$ is singular at $\vec{x} = \vec{y}_A$

Problem 2 What is the meaning of $F(\vec{x})\,\delta(\vec{x}-\vec{y}_1)$?

Non-linear source of Einstein-field eqs

$$\Lambda_2^{00} \simeq h^{ij}\partial_i\partial_j h^{00} + \partial_i h^{00}\partial_j h^{00} + \cdots$$

with $h^{00} \simeq \frac{U}{c^2}$ Need to differentiate U

Problem 3 How to differentiate singular functions

$$\partial_i\partial_j F \,?$$

For instance should we use standard distribution theory

$$\partial_i\partial_j \frac{1}{r_1} = \frac{3n_1^i n_1^j - \delta^{ij}}{r_1^3} - \underbrace{\frac{4\pi}{3}\delta^{ij}\,\delta(\vec{x}-\vec{y}_1)}_{\text{distributional term}} \quad ?$$

Problem 4 What is the meaning of the divergent integral

$$\int d^3\vec{x}\, F(\vec{x}) \;?$$

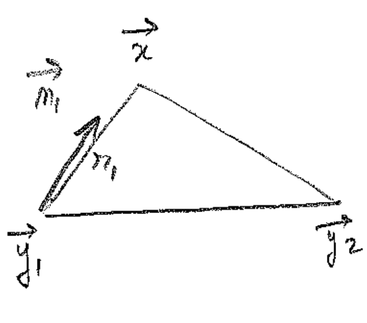
We must suplement the calculation of point particles by some <u>self-field regularization</u> to remove the formally infinite "self-field" of point particles.

- Hadamard's regularization (Hadamard 1932, Schwartz 1957) which is very efficient in practical calculations but yields some <u>ambiguity parameters</u> (coefficients which cannot be computed) at high PN orders (≥ 3PN)

- Dimensional regularization ('t Hooft and Veltman 1972), extremely powerful and <u>free of ambiguities</u> but cannot be implemented at present for general d (only $d = 3+\varepsilon$ where $\varepsilon \to 0$)

<u>HADAMARD SELF-FIELD REGULARIZATION</u>

$F(\vec{x})$ is smooth except at $\vec{y}_1$ and $\vec{y}_2$. When $r_1 \to 0$

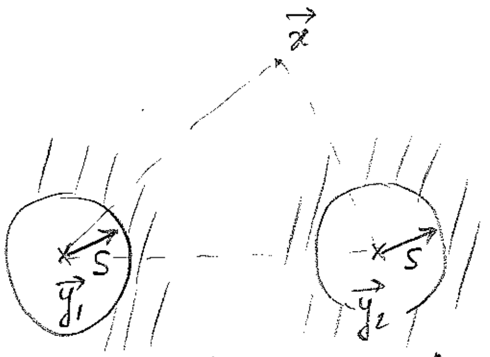


$$F(\vec{x}) = \sum_{a_0 \leq a \leq N} r_1^a \, \underset{1}{f_a}(\vec{n}_1) + o(r_1^N) \qquad a_0 \in \mathbb{Z}$$

Hadamard's partie finie of F at singular point $\vec{y}_1$

$$\boxed{(F)_1 \equiv \int \frac{d\Omega_1}{4\pi} \, \underset{1}{f_0}(\vec{n}_1)}$$

Hadamard's partie finie (Pf) of the divergent integral $\int d^3\vec{x}\, F(\vec{x})$



Two balls (radius s) are excised

$$\boxed{\mathrm{Pf}\int d^3x\, F(\vec{x}) = \lim_{s\to 0}\left\{ \int_{\substack{r_1>s\\ r_2>s}} d^3x\, F(\vec{x}) + \sum_{a+3<0} \frac{s^{a+3}}{a+3}\int d\Omega_1\, \underset{1}{f_a} + \ln\left(\frac{s}{s_1}\right)\int d\Omega_1\, \underset{1}{f_{-3}} + 1 \leftrightarrow 2 \right\}}$$

These terms cancel out the divergencies of the integral over the "exterior"

Note the log terms depending on <u>two arbitrary constants</u> s_1 s_2 (one for each particle)

Hadamard Pf is equivalent to an analytic continuation

$$\boxed{\underset{s_1 s_2}{\mathrm{Pf}}\int d^3x\, F = \underbrace{\underset{\alpha\to 0}{\mathrm{FP}}\ \underset{\beta\to 0}{\mathrm{FP}}}_{\text{operations in whatever order}} \int d^3x \left(\frac{r_1}{s_1}\right)^{\alpha}\left(\frac{r_2}{s_2}\right)^{\beta} F}$$

Note the integral of a gradient is not zero (because of the singularities)

$$\boxed{\mathrm{Pf}\int d^3x\, \partial_i F = -4\pi\left(n_1^i r_1^2 F\right)_1 - 4\pi\left(n_2^i r_2^2 F\right)_2}$$

Partie finie pseudo-functions (Blanchet & Faye 2000)

$\mathcal{F}$ is the set of all such $F(\vec{x})$

$\mathrm{Pf}\,F$ is a linear form on $\mathcal{F}$

$$\forall G \in \mathcal{F} \qquad \boxed{\underbrace{\langle \mathrm{Pf}\,F, G \rangle}_{\text{result of action of Pf F on } G\in\mathcal{F}} = \underbrace{\mathrm{Pf}\int d^3x\, FG}_{\text{a real number given by Hadamard's Pf}}}$$

Define the partie finie δ-function $\mathrm{Pf}\,\delta_1$ (where $\delta_1 = \delta(\vec{x}-\vec{y}_1)$)

$$\forall F \qquad \langle \mathrm{Pf}\,\delta_1, F \rangle = (F)_1$$

We can give also a meaning to the product of F with δ_1

$$\forall G \qquad \langle \mathrm{Pf}(F\delta_1), G \rangle = (FG)_1$$

Derivative of a pseudo-function is defined by the "rule of integration by parts"

$$\boxed{\forall F, G \qquad \langle \partial_i(\mathrm{Pf}\,F), G \rangle = -\langle \partial_i(\mathrm{Pf}\,G), F \rangle}$$

This constitutes a generalization of Schwartz distributional derivative.

This "extended Hadamard regularization" gives a set of rules to compute all terms at say 3PN. However because of the constants s_1 s_2 it leaves finally some unknown "ambiguity parameters" at 3PN order (Jaranowski & Schäfer 1999).

Hadamard's regularization works well up to 2PN but fails to provide a complete answer at 3PN. One reason is that from the definition of $(F)_1$ we have

$$(FG)_1 \neq (F)_1 (G)_1 \quad \text{in general.}$$

Hence basic symmetries of GR such as diffeomorphism invariance at not respected (at PN orders $\geq$ 3PN)

DIMENSIONAL SELF-FIELD REGULARIZATION

Work in a space with d dimensions (so space-time has $D = d+1$ dimensions).

Idea of the regularization is to apply complex analytic continuation in the dimension $d \in \mathbb{C}$.

Volume element $\boxed{d^d x = r^{d-1} dr \, d\Omega_{d-1}}$ $\quad r = |\vec{x}|$

Volume of $(d-1)$ dimensional sphere $\Omega_{d-1} = \int d\Omega_{d-1}$

From the Gaussian integral $\int d^d x \, e^{-r^2} = \left(\int dx \, e^{-x^2} \right)^d = \pi^{d/2}$

$$= \Omega_{d-1} \int_0^\infty dr \, r^{d-1} e^{-r^2} = \frac{\Omega_{d-1}}{2} \Gamma\left(\frac{d}{2}\right)$$

$$\boxed{\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}}$$

For instance $\Omega_2 = 4\pi$ and $\Omega_1 = 2\pi$
and $\Omega_0 = 2$ (sphere with 0 dimension is made of 2 points!)

Green's function of Laplace operator:

$$\boxed{\begin{array}{ll} \Delta u = -4\pi\, \delta^{(d)}(\vec{x}) & \text{d-dimensional Dirac function} \\ u = \tilde{k}\, r^{2-d} \quad \text{where } \tilde{k} = \dfrac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-1}{2}}} & \end{array}}$$

Riesz (1949) Euclidean kernels (generalize $\delta^{(d)}$ and u)

$$\boxed{\begin{array}{l} \delta_\alpha^{(d)}(\vec{x}) = K_\alpha\, r^{\alpha-d} \\ \quad \text{where } K_\alpha = \dfrac{\Gamma\left(\frac{d-\alpha}{2}\right)}{2^\alpha \pi^{d/2}\, \Gamma(d/2)} \\ \text{are such that } \Delta\, \delta_{\alpha+2}^{(d)} = -\delta_\alpha^{(d)} \\ \text{and } \delta_\alpha^{(d)} * \delta_\beta^{(d)} = \delta_{\alpha+\beta}^{(d)} \end{array}}$$

hence $\delta^{(d)} = \delta_0^{(d)}$
and $u = 4\pi\, \delta_2^{(d)}$

this beautiful convolution property is an elegant formulation of Riesz's formula in d dimensions

$$\boxed{\int d^d x\; r_1^\alpha\, r_2^\beta = \pi^{d/2}\, \frac{\Gamma\left(\frac{\alpha+d}{2}\right)\Gamma\left(\frac{\beta+d}{2}\right)\Gamma\left(-\frac{\alpha+\beta+d}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(-\frac{\beta}{2}\right)\Gamma\left(\frac{\alpha+\beta+2d}{2}\right)}\; r_{12}^{\alpha+\beta+d}}$$

For instance $\displaystyle \int \frac{d^3x}{r_1^2 r_2^2} = \frac{\pi^3}{r_{12}}$

Einstein field equations in $D = d+1$ dimensions

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{8\pi G}{c^4} T^{\mu\nu} \Leftrightarrow R^{\mu\nu} = \frac{8\pi G}{c^4}\left(T^{\mu\nu} - \underbrace{\frac{1}{d-1} g^{\mu\nu} T}_{\text{dimension appears explicitly here}}\right)$$

still we have
$$\begin{cases} \Box h^{\mu\nu} = \dfrac{16\pi G}{c^4} \tau^{\mu\nu} \text{ with } \partial_\nu h^{\mu\nu} = 0 \\ \tau^{\mu\nu} = |g| \, T^{\mu\nu} + \dfrac{c^4}{16\pi G} \Lambda^{\mu\nu} \end{cases}$$

$$\Lambda^{\mu\nu} = - h^{\rho\sigma} \partial_{\rho\sigma} h^{\mu\nu} + \partial_\rho h^{\mu\sigma} \partial h^{\nu\rho}_{\sigma} + \dots + \frac{1}{d-1} g \cdots g \, \partial h \, \partial h$$

$$\boxed{G = \ell_0^{d-3} G_N}$$

usual Newtonian gravitational constant

DIFFERENCE BETWEEN HADAMARD AND DIMENSIONAL REGULARIZATIONS

Iterating the field equations in PN form we have to solve Poisson equations $\Delta P = F$ with some source term $F(\vec{x})$ which is singular at $\vec{y}_1$ and $\vec{y}_2$ $(F \in \mathcal{F})$. Then we need to compute the value of P at $\vec{y}_1$ and $\vec{y}_2$.

In Had. reg. we use the Partie finie of a Poisson integral

$$P(\vec{x}') = -\frac{1}{4\pi} \underbrace{\mathrm{Pf}_{s_1 s_2}}_{\text{depends on constants } s_1\, s_2} \int \frac{d^3x}{|\vec{x} - \vec{x}'|} F(\vec{x})$$

To compute the value when $\vec{x}' \longrightarrow \vec{y}_1$ one applies the Partie finie of a singular function.

$$\underline{P}(\vec{x}') = \sum_{p \leq N} r_1'^{\,p} \Big[\underset{1}{g_p}(\vec{n}_1') + \underbrace{\underset{1p}{h}(\vec{n}_1') \ln r_1'}_{\text{appearance of } \ln r_1' \text{ terms in the Poisson integral}} \Big] + o(r_1'^{\,N})$$

$$\boxed{(\underline{P})_1 = \int \frac{d\Omega_1'}{4\pi} \Big[\underset{1}{g_0} + \underset{10}{h} \ln r_1' \Big]}$$

here $\ln r_1'$ is considered as a "constant" (though it is really infinite $\ln 0 = -\infty$)

Explicit calulation shows

$$\boxed{(\underline{P})_1 = -\frac{1}{4\pi} \underset{r_1', s_2}{\mathrm{Pf}} \int \frac{d^3x}{r_1} F(x) - (r_1^2 F)_1}$$

depends on r_1' and s_2

(similarly $(\underline{P})_2$ depends on r_2' and s_1)

In dim. reg. things are simpler:

$$\underline{P}^{(d)}(\vec{x}') = -\frac{\tilde{k}}{4\pi} \int \frac{d^dx}{|\vec{x}' - \vec{x}|^{d-2}} F^{(d)}(\vec{x})$$

and value at $\vec{x}' = \vec{y}_1$ is obtained by replacing $\vec{x}' \longrightarrow \vec{y}_1$

$$\boxed{\underline{P}^{(d)}(\vec{y}_1) = -\frac{\tilde{k}}{4\pi} \int \frac{d^dx}{r_1^{d-2}} F^{(d)}(\vec{x})}$$

Point is that the difference between the two regularization depends on the <u>vicinity of singularities</u> only

$$\boxed{\mathcal{D}P(1) \equiv P^{(d)}(\vec{y}_1) - (P)_1}$$

When $r_1 \longrightarrow 0$ (near $\vec{y}_1$)

$$F(\vec{x}) = \sum_p r_1^p \, {}_1f_p(\vec{n}_1) + o(r_1^N)$$

while the analogue in d dimensions, $F^{(d)}(\vec{x})$ (defined by the same PN iteration of field equations but in d dim) admits

$$F^{(d)}(\vec{x}) = \sum_{p,q} r_1^{p+q\varepsilon} \, {}_1f^{(\varepsilon)}_{p,q}(\vec{n}_1) + o(r_1^N)$$

where $\varepsilon = d-3$.

$$\boxed{\begin{aligned}\mathcal{D}P(1) = &-\frac{1}{\varepsilon(1+\varepsilon)}\sum_q \left(\frac{1}{q} + \varepsilon\left[\ln r_1' - 1\right]\right)\int \frac{d\Omega_1}{4\pi}\, {}_1f^{(\varepsilon)}_{-2,q}(\vec{n}_1) \\ &-\frac{1}{\varepsilon(1+\varepsilon)}\sum_q \left(\frac{1}{q+1} + \varepsilon \ln s_2\right) \\ &\quad\times \sum_{\ell=0}^{\infty}\frac{(-)^\ell}{\ell!}\,\partial_L\left(\frac{1}{r_{12}^{1+\varepsilon}}\right)\int\frac{d\Omega_2}{4\pi}\, n_2^L\, {}_2f^{(\varepsilon)}_{-\ell-3,q}(\vec{n}_2) \\ &+\mathcal{O}(\varepsilon)\end{aligned}}$$

can be computed from the knowledge of the expansions of $F^{(d)}$ when $r_1 \to 0$, $r_2 \to 0$

<u>Conclusions</u> The difference between Had reg and Dim reg is made of the <u>contribution of poles</u>

$$\boxed{(\text{Dim reg}) - (\text{Had reg}) = \frac{a_{-1}}{\varepsilon} + a_0 + \mathcal{O}(\varepsilon)}$$

$$\varepsilon = d-3$$

This difference can be computed <u>locally</u>, i.e. depends only on the expansions of $F^{(d)}$ around the singularities ($r_1 \longrightarrow 0$ and $r_2 \longrightarrow 0$)

The two regs. agree in the absence of poles. Since no poles occur up to 2PN order (poles in ε correspond to logarithmic divergences in $d=3$) Had reg can be employed without problem up to 2PN.

At 3PN order poles in ε occur and as a result Had reg is not able to give a complete answer, and becomes "ambiguous" with the appearance of unknown "ambiguity parameters" (λ, ξ, κ and ς) which cannot be computed.

Technically one of the reasons for the problems with Had reg is the "<u>non-distributivity</u>" <u>of the partie finie</u>

$$(F\,G)_1 \neq (F)_1\,(G)_1 \qquad \text{in general}$$

(because of the angular integration in the definition of the p.f.)

However Had. reg. is extremely convenient in practical calculations and permits to compute unambiguously all the terms but a few (those corresponding to poles in ε)

By contrast Dim. reg. cannot be implemented (for the moment) for general d but only in the limit $d \to 3$

<u>Strategy</u>

(1) Compute all the terms using Had reg (in $d=3$)

(2) Obtain the Dim reg result by

$$(\text{Dim reg}) = (\text{Had reg}) + \underbrace{\frac{a_{-1}}{\varepsilon} + a_0 + \mathcal{O}(\varepsilon)}_{\text{computed locally } r_{1,2} \to 0}$$

SOME EXAMPLES OF COMPUTATION IN $d=3$

In a PN expansion the metric is

$$\begin{cases} g_{00} = -1 + \dfrac{2U}{c^2} + \cdots + \dfrac{\hat{X}}{c^6} + \cdots \\[2ex] g_{0i} = \dfrac{4V_i}{c^3} + \cdots \\[2ex] g_{ij} = \delta_{ij}\left(1 + \dfrac{2U}{c^2} + \cdots\right) + \dfrac{1}{c^4}\hat{W}_{ij} \end{cases}$$

U = Newtonian potential $\quad \Delta U = -4\pi G\rho$

$\hat{X}$ = some higher potential

V_i = gravitomagnetic potential $\quad \Delta V_i = -4\pi G\rho v^i$

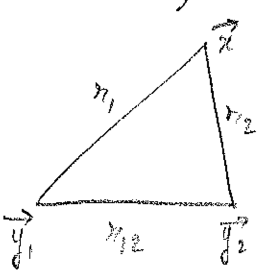
$\hat{W}_{ij}$ = potential generated by gravitational stresses

$$\Delta \hat{W}_{ij} = \partial_i U \partial_j U + \cdots$$

For 2 particles

$$\rho = m_1\delta_1 + m_2\delta_2 \Rightarrow U = \frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}$$

$$V_i = \frac{Gm_1 v_1^i}{r_1} + \frac{Gm_2 v_2^i}{r_2}$$



$$\Delta W_{ij} = \partial_i\left(\frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}\right)\partial_j\left(\frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}\right)$$

$$= \partial_i\left(\frac{Gm_1}{r_1}\right)\partial_j\left(\frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}\right) + 1 \longleftrightarrow 2$$

$$= G^2 m_1^2 \frac{n_1^i n_1^j}{r^4} + G^2 m_1 m_2 \frac{\partial^2}{\partial y_1^{(i}\partial y_2^{j)}}\left(\frac{1}{r_1 r_2}\right) + 1 \longleftrightarrow 2$$

Can be integrated using

$$\boxed{\begin{aligned} &g = \ln S \qquad S = r_1 + r_2 + r_{12} \\ &\Delta g = \frac{1}{r_1 r_2} \end{aligned}}$$

extremely useful function which permits the 3PN calculation in closed-analytic form

$$\boxed{W_{ij} = \frac{G^2 m_1^2}{8}\left(\partial_{ij} \ln r_1 + \frac{\delta^{ij}}{r^2}\right) + G^2 m_1 m_2 \frac{\partial^2 g}{\partial y_1^{(i} \partial y_2^{j)}} + 1 \longleftrightarrow 2}$$

At higher PN order needs to compute solutions of eqs like

$$\Delta X = W_{ij} \partial_{ij} U \qquad \text{where}$$

The closed-form solution can be found using the elementary solutions

$$\boxed{\begin{aligned} \Delta K_1 &= 2\, \partial_{ij} \frac{1}{r_2} \partial_{ij} \ln r_1 \\ \Delta H_1 &= 2\, \partial_{ij} \frac{1}{r_1} \frac{\partial^2 g}{\partial y_1^i \partial y_2^j} \end{aligned}}$$

which are known in closed form
$$K_1 = \left(\frac{1}{2}\Delta - \Delta_1 - \Delta_2\right)\left(\frac{\ln r_1}{r_2}\right) + \cdots$$
$$H_1 = \frac{1}{2}\Delta_1\left(\frac{g}{r_1}\right) + \cdots$$

These results permit to derive the metric $g_{\mu\nu}$ at 2PN hence we can deduce the EOM at 2PN (by replacing $g_{\mu\nu}$ into the geodesic equation and applying the regularization)

However at 3PN one cannot derive the metric $g_{\mu\nu}^{3PN}$ in closed form for any field point $\vec{x}$ in the NZ. Only the limit $\vec{x} \to \vec{y}_1$ can be computed (using the regularization) so the 3PN EOM can be obtained (after long and tedious calculations) (Blanchet & Faye 2000)

For the computation of the multipole moments $\underbrace{I_L \; J_L}$:

source-type moments whose general expression is known

At Newtonian order (quadrupole formula)

$$I_{ij} = \int d^3x \, \rho \, \hat{x}_{ij} = m_1 \, y_1^{<i} y_1^{j>} + m_2 \, y_2^{<i} y_2^{j>} + \dots$$
$$+ \dots$$

At higher PN order we have <u>non-compact support</u> terms such as

$$I_{ij}^{(NC)} = \underset{B \to 0}{\text{F.P.}} \int d^3x \, |\vec{x}|^B \, \hat{x}_{ij} \, \partial_k U \, \partial_k U$$

$$= \text{FP} \int d^3x \, |x|^B \, \hat{x}_{ij} \left\{ \underbrace{\frac{G^2 m_1^2}{r_1^4}}_{\substack{\text{gives zero with} \\ \text{Had reg}}} + G^2 m_1 m_2 \frac{\partial^2}{\partial y_1^k \partial y_2^k} \left(\frac{1}{r_1 r_2} \right) + 1 \longleftrightarrow 2 \right\}$$

Computation (to this order) is reduced to the computation of

$$\boxed{\begin{aligned} Y_L(\vec{y}_1, \vec{y}_2) &= -\frac{1}{2\pi} \, \text{F.P.} \int d^3x \, |x|^B \frac{\hat{x}_L}{r_1 r_2} \\ Y_L(y_1, y_2) &= \frac{r_{12}}{l+1} \sum_{p=0}^{l} y_1^{<L-P} \, y_2^{P>} \end{aligned}}$$

To higher PN order more complicated integrals appear
(Blanchet, Iyer & Joguet 2002)

Ambiguity parameter λ in 3PN Had. reg. EOM

There are 4 constants which appear (inside logs)

$r'_1 \quad r'_2$ (come from reg. of the potentials)

$s_1 \quad s_2$ (come from reg. of the EOM)

However two of these constants can be removed by a coordinate transformation. It remains only the 2 "constants"

$$\ln\left(\frac{r'_1}{s_1}\right) \quad \text{and} \quad \ln\left(\frac{r'_2}{s_2}\right)$$

We find (Blanchet & Faye 2000) these constants have the form

$$\ln\left(\frac{r'_1}{s_1}\right) = \frac{159}{308} + \lambda \frac{m}{m_1} \qquad (m = m_1 + m_2)$$

and $1 \longleftrightarrow 2$

λ is equivalent to ω_{static} introduced by Jaranowski & Schäfer (1999)

Ambiguity parameters ξ, κ, ζ in 3PN quad. moment

(Blanchet, Iyer & Joguet 2002)

$$\ln\left(\frac{r'_1}{u_1}\right) = \xi + \kappa \frac{m_2}{m_1}$$

(ambiguities in the relation between Had. reg. constants u_1, u_2 similar to $s_1 s_2$ and the EOM-related constants $r'_1\, r'_2$).

In addition ζ reflects the Poincaré invariance of the field (not necessarily satisfied by Had. reg.)

There is complete agreement between all these works (whenever this can be compared) up to 3.5PN.

Final values for the ambiguity parameters are

$$\lambda = -\frac{1987}{3080} \qquad \text{(3PN equations of motion)}$$

$$\begin{cases} \xi = -\dfrac{9871}{9240} \\ \kappa = 0 \\ \zeta = -\dfrac{7}{33} \end{cases} \qquad \text{(3PN radiation field)}$$

All these parameters have been checked by methods <u>independent of the regularization</u>

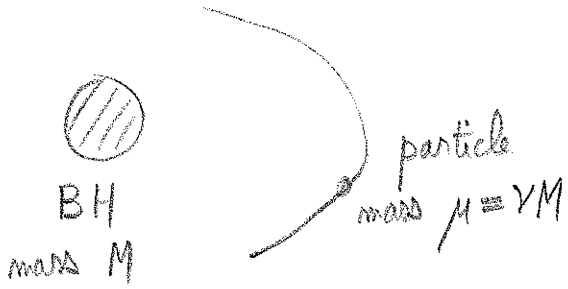
λ by surface-integral method (Itoh & Futamase 2004)

$\xi + \kappa$ by requiring that the binary's mass dipole agrees with the center-of-mass deduced from EOM (BI04)

κ from argument based on space-time diagrams (BDEI05)

ζ from a computation of the multipole moments of a boosted Schwarzschild solution (BDI04)

All results are in agreement with black-hole perturbation theory in the limit $\nu \to 0$



Sasaki (1994)
Tagoshi & Sasaki (1994) } 4PN

Tanaka, Tagoshi & Sasaki (1996) } 5.5PN

Spin effects have been added

Barker & O'Connell (1975, 1979)
Kidder, Will & Wiseman (1993)
Kidder (1995)
Porto (2006)
} spin-orbit (1.5PN) and spin-spin (2PN)

Tagoshi, Ohashi & Owen (2000)
Faye, Blanchet & Buonanno (2006)
Blanchet, Buonanno & Faye (2006)
} spin-orbit (2.5PN)

Templates for inspiralling compact binaries (ICBs) are known up to

3.5PN for the phase
2.5PN for the waveform

With spins they are known up to 2.5PN for the phase.

HISTORY OF PN EOM AND RADIATION OF COMPACT BINARIES

PN equations of motion

Lorentz & Droste 1917
Einstein, Infeld & Hoffmann 1938
← surface integral approach

1PN

Damour & Deruelle (1982, 1983) Harm. coord.
Damour & Schäfer (1985) ADM coord.
Kopeikin & Grischuk (1985) extended body approach
Blanchet, Faye & Ponsot (1998) point-particles computation of EOM and metric
Itoh, Futamase & Asada (2001) surface-integral

2.5PN

Jaranowski & Schäfer (1998, 1999) Hadamard reg. in ADM coord. Two ambiguity parameters ω_s, ω_k
Blanchet & Faye (2000, 2001) Had. reg. in harmonic coord. One ambiguity parameter $\lambda \Leftrightarrow \omega_s$
Damour, Jaranowski & Schäfer (2001) Dimensional reg. computation of ω_s
Blanchet, Damour & Esposito-Farèse (2004) Dim reg. computation of $\lambda \Leftrightarrow \omega_s$
Itoh & Futamase (2004) surface-integral method free of ambiguity parameters

3PN

Iyer & Will (1993, 1995) balance equation for computing rad. reaction
Pati & Will (2001) harm. coord.
Königsdörffer, Faye & Schäfer (2003) ADM coord.
Nissanke & Blanchet (2005) harm. coord.
} 3.5 PN

PN radiation field

Landau & Lifchitz (1941)
Peters & Mathews (1963)
} Newtonian (quadrupole order)

Wagoner & Will (1976) using Epstein–Wagoner–Thorne moments
Blanchet & Schäfer (1989) using BD moments
} 1 PN

Poisson (1993) perturbative limit $\nu \to 0$
Wiseman (1993)
Blanchet & Schäfer (1993)
} 1.5 PN (tail)

Blanchet, Damour, Iyer, Will & Wiseman (1995)
Blanchet, Iyer, Will & Wiseman (1996) waveform
Blanchet (1996) 2.5 PN tail
Arun, Blanchet, Iyer & Qusailah (2004) 2.5 PN waveform
} 2 PN + 2.5 PN

Blanchet (1998) 3 PN tail-of-tail
Blanchet, Iyer & Joguet (2001) Hadamard reg. 3 ambiguity parameters ξ, κ, ζ
Blanchet & Iyer (2004) Had. reg., general orbits
Blanchet, Damour, Esposito-Farèse & Iyer (2005) Dim. reg. computation of ξ, κ, ζ
} 3 PN + 3.5 PN