PART 5

EQUATIONS OF MOTION AND RADIATION OF INSPIRALLING COMPACT BINARIES

5.1

9 Newtonian-like Equations of Motion

9.1 The 3PN acceleration and energy

We present the acceleration of one of the particles, say the particle 1, at the 3PN order, as well as the 3PN energy of the binary, which is conserved in the absence of radiation reaction. To get this result we used essentially a "direct" post-Newtonian method (issued from Ref. [36]), which consists of reducing the 3PN metric of an extended regular source, worked out in Eqs. (115), to the case where the matter tensor is made of delta functions, and then curing the self-field divergences by means of the Hadamard regularization technique. The equations of motion are simply the geodesic equations associated with the regularized metric (see Ref. [34] for a proof). The Hadamard ambiguity parameter λ is computed from dimensional regularization in Section 8.3. We also add the 3.5PN terms which are known from Refs. [133, 134, 138, 171, 145, 161].

Though the successive post-Newtonian approximations are really a consequence of general relativity, the final equations of motion must be interpreted in a Newtonian-like fashion. That is, once a convenient general-relativistic (Cartesian) coordinate system is chosen, we should express the results in terms of the *coordinate* positions, velocities, and accelerations of the bodies, and view the trajectories of the particles as taking place in the absolute Euclidean space of Newton. But because the equations of motion are actually relativistic, they must

(i) stay manifestly invariant – at least in harmonic coordinates – when we perform a global post-Newtonian-expanded Lorentz transformation,

(ii) possess the correct "perturbative" limit, given by the geodesics of the (post-Newtonian-expanded) Schwarzschild metric, when one of the masses tends to zero, and

(iii) be conservative, *i.e.* to admit a Lagrangian or Hamiltonian formulation, when the gravitational radiation reaction is turned off.

We denote by $r_{12} = |\mathbf{y}_1(t) - \mathbf{y}_2(t)|$ the harmonic-coordinate distance between the two particles, with $\mathbf{y}_1 = (y_1^i)$ and $\mathbf{y}_2 = (y_2^i)$, by $n_{12}^i = (y_1^i - y_2^i)/r_{12}$ the corresponding unit direction, and by $v_1^i = dy_1^i/dt$ and $a_1^i = dv_1^i/dt$ the coordinate velocity and acceleration of the particle 1 (and *idem* for 2). Sometimes we pose $v_{12}^i = v_1^i - v_2^i$ for the relative velocity. The usual Euclidean scalar product of vectors is denoted with parentheses, *e.g.*, $(n_{12}v_1) = \mathbf{n}_{12} \cdot \mathbf{v}_1$ and $(v_1v_2) = \mathbf{v}_1 \cdot \mathbf{v}_2$. The equations of the body 2 are obtained by exchanging all the particle labels $1 \leftrightarrow 2$ (remembering that n_{12}^i and v_{12}^i change sign in this operation):

$$a_1^i = -\frac{Gm_2 n_{12}^i}{r_{12}^2}$$

1 PN (EIH acceleration)

$$+\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3} + \frac{4G^2m_2^2}{r_{12}^3} + \frac{Gm_2}{r_{12}^2}\left(\frac{3}{2}(n_{12}v_2)^2 - v_1^2 + 4(v_1v_2) - 2v_2^2\right)\right]n_{12}^i + \frac{Gm_2}{r_{12}^2}\left(4(n_{12}v_1) - 3(n_{12}v_2)\right)v_{12}^i\right\}$$

$$+\frac{1}{c^4}\left\{\left[-\frac{57G^3m_1^2m_2}{4r_{12}^4} - \frac{69G^3m_1m_2^2}{2r_{12}^4} - \frac{9G^3m_2^3}{r_{12}^4}\right.\right.$$
$$+\frac{Gm_2}{r_{12}^2}\left(-\frac{15}{8}(n_{12}v_2)^4 + \frac{3}{2}(n_{12}v_2)^2v_1^2 - 6(n_{12}v_2)^2(v_1v_2) - 2(v_1v_2)^2 + \frac{9}{2}(n_{12}v_2)^2v_2^2\right.$$
$$\left.+ 4(v_1v_2)v_2^2 - 2v_2^4\right)$$

2PN

$$+\frac{G^2m_1m_2}{r_{12}^3}\left(\frac{39}{2}(n_{12}v_1)^2-39(n_{12}v_1)(n_{12}v_2)+\frac{17}{2}(n_{12}v_2)^2-\frac{15}{4}v_1^2-\frac{5}{2}(v_1v_2)+\frac{5}{4}v_2^2\right)$$

$$+\frac{G^2m_2^2}{r_{12}^3}\left(2(n_{12}v_1)^2-4(n_{12}v_1)(n_{12}v_2)-6(n_{12}v_2)^2-8(v_1v_2)+4v_2^2\right)\Bigg]n_{12}^i$$

$$+\Bigg[\frac{G^2m_2^2}{r_{12}^3}\left(-2(n_{12}v_1)-2(n_{12}v_2)\right)+\frac{G^2m_1m_2}{r_{12}^3}\left(-\frac{63}{4}(n_{12}v_1)+\frac{55}{4}(n_{12}v_2)\right)$$

$$+\frac{Gm_2}{r_{12}^2}\Big(-6(n_{12}v_1)(n_{12}v_2)^2+\frac{9}{2}(n_{12}v_2)^3+(n_{12}v_2)v_1^2-4(n_{12}v_1)(v_1v_2)$$

$$+4(n_{12}v_2)(v_1v_2)+4(n_{12}v_1)v_2^2-5(n_{12}v_2)v_2^2\Big)\Bigg]v_{12}^i\Bigg\}$$

2.5PN
(radiation reaction)

$$+\frac{1}{c^5}\Bigg\{\left[\frac{208G^3m_1m_2^2}{15r_{12}^4}(n_{12}v_{12})-\frac{24G^3m_1^2m_2}{5r_{12}^4}(n_{12}v_{12})+\frac{12G^2m_1m_2}{5r_{12}^3}(n_{12}v_{12})v_{12}^2\right]n_{12}^i$$

$$+\left[\frac{8G^3m_1^2m_2}{5r_{12}^4}-\frac{32G^3m_1m_2^2}{5r_{12}^4}-\frac{4G^2m_1m_2}{5r_{12}^3}v_{12}^2\right]v_{12}^i\Bigg\}$$

3PN

$$+\frac{1}{c^6}\Bigg\{\Bigg[\frac{Gm_2}{r_{12}^2}\Big(\frac{35}{16}(n_{12}v_2)^6-\frac{15}{8}(n_{12}v_2)^4v_1^2+\frac{15}{2}(n_{12}v_2)^4(v_1v_2)+3(n_{12}v_2)^2(v_1v_2)^2$$

$$-\frac{15}{2}(n_{12}v_2)^4v_2^2+\frac{3}{2}(n_{12}v_2)^2v_1^2v_2^2-12(n_{12}v_2)^2(v_1v_2)v_2^2-2(v_1v_2)^2v_2^2$$

$$+\frac{15}{2}(n_{12}v_2)^2v_2^4+4(v_1v_2)v_2^4-2v_2^6\Big)$$

$$+\frac{G^2m_1m_2}{r_{12}^3}\Big(-\frac{171}{8}(n_{12}v_1)^4+\frac{171}{2}(n_{12}v_1)^3(n_{12}v_2)-\frac{723}{4}(n_{12}v_1)^2(n_{12}v_2)^2$$

$$+\frac{383}{2}(n_{12}v_1)(n_{12}v_2)^3-\frac{455}{8}(n_{12}v_2)^4+\frac{229}{4}(n_{12}v_1)^2v_1^2$$

$$-\frac{205}{2}(n_{12}v_1)(n_{12}v_2)v_1^2+\frac{191}{4}(n_{12}v_2)^2v_1^2-\frac{91}{8}v_1^4-\frac{229}{2}(n_{12}v_1)^2(v_1v_2)$$

$$+244(n_{12}v_1)(n_{12}v_2)(v_1v_2)-\frac{225}{2}(n_{12}v_2)^2(v_1v_2)+\frac{91}{2}v_1^2(v_1v_2)$$

$$-\frac{177}{4}(v_1v_2)^2+\frac{229}{4}(n_{12}v_1)^2v_2^2-\frac{283}{2}(n_{12}v_1)(n_{12}v_2)v_2^2$$

$$+\frac{259}{4}(n_{12}v_2)^2v_2^2-\frac{91}{4}v_1^2v_2^2+43(v_1v_2)v_2^2-\frac{81}{8}v_2^4\Big)$$

$$+\frac{G^2m_2^2}{r_{12}^3}\Big(-6(n_{12}v_1)^2(n_{12}v_2)^2+12(n_{12}v_1)(n_{12}v_2)^3+6(n_{12}v_2)^4$$

$$+4(n_{12}v_1)(n_{12}v_2)(v_1v_2)+12(n_{12}v_2)^2(v_1v_2)+4(v_1v_2)^2$$

$$-4(n_{12}v_1)(n_{12}v_2)v_2^2-12(n_{12}v_2)^2v_2^2-8(v_1v_2)v_2^2+4v_2^4\Big)$$

$$+\frac{G^3m_2^3}{r_{12}^4}\left(-(n_{12}v_1)^2+2(n_{12}v_1)(n_{12}v_2)+\frac{43}{2}(n_{12}v_2)^2+18(v_1v_2)-9v_2^2\right)$$

$$+\frac{G^3m_1m_2^2}{r_{12}^4}\Big(\frac{415}{8}(n_{12}v_1)^2-\frac{375}{4}(n_{12}v_1)(n_{12}v_2)+\frac{1113}{8}(n_{12}v_2)^2-\frac{615}{64}(n_{12}v_{12})^2\pi^2$$

$$+18v_1^2+\frac{123}{64}\pi^2v_{12}^2+33(v_1v_2)-\frac{33}{2}v_2^2\Big)$$

5.3

The logs can be removed
by coord. transformation
(r_1', r_2' = "gauge" constants — unphysical)

only coefficient
which is ambiguous
in Had. reg.
$\lambda = -\dfrac{1987}{3080}$
has been
replaced

$$
\begin{aligned}
&+\frac{G^3m_1^2m_2}{r_{12}^4}\Big(-\frac{45887}{168}(n_{12}v_1)^2+\frac{24025}{42}(n_{12}v_1)(n_{12}v_2)-\frac{10469}{42}(n_{12}v_2)^2+\frac{48197}{840}v_1^2\\
&\qquad -\frac{36227}{420}(v_1v_2)+\frac{36227}{840}v_2^2+110(n_{12}v_{12})^2\ln\left(\frac{r_{12}}{r_1'}\right)-22v_{12}^2\ln\left(\frac{r_{12}}{r_1'}\right)\Big)\\
&+\frac{16G^4m_2^4}{r_{12}^5}+\frac{G^4m_1^2m_2^2}{r_{12}^5}\left(175-\frac{41}{16}\pi^2\right)+\frac{G^4m_1^3m_2}{r_{12}^5}\left(-\frac{3187}{1260}+\frac{44}{3}\ln\left(\frac{r_{12}}{r_1'}\right)\right)\\
&+\frac{G^4m_1m_2^3}{r_{12}^5}\left(\frac{110741}{630}-\frac{41}{16}\pi^2-\frac{44}{3}\ln\left(\frac{r_{12}}{r_2'}\right)\right)\Big]n_{12}^i\\
&+\Big[\frac{Gm_2}{r_{12}^2}\Big(\frac{15}{2}(n_{12}v_1)(n_{12}v_2)^4-\frac{45}{8}(n_{12}v_2)^5-\frac{3}{2}(n_{12}v_2)^3v_1^2+6(n_{12}v_1)(n_{12}v_2)^2(v_1v_2)\\
&\qquad -6(n_{12}v_2)^3(v_1v_2)-2(n_{12}v_2)(v_1v_2)^2-12(n_{12}v_1)(n_{12}v_2)^2v_2^2+12(n_{12}v_2)^3v_2^2\\
&\qquad +(n_{12}v_2)v_1^2v_2^2-4(n_{12}v_1)(v_1v_2)v_2^2+8(n_{12}v_2)(v_1v_2)v_2^2+4(n_{12}v_1)v_2^4\\
&\qquad -7(n_{12}v_2)v_2^4\Big)\\
&+\frac{G^2m_2^2}{r_{12}^3}\Big(-2(n_{12}v_1)^2(n_{12}v_2)+8(n_{12}v_1)(n_{12}v_2)^2+2(n_{12}v_2)^3+2(n_{12}v_1)(v_1v_2)\\
&\qquad +4(n_{12}v_2)(v_1v_2)-2(n_{12}v_1)v_2^2-4(n_{12}v_2)v_2^2\Big)\\
&+\frac{G^2m_1m_2}{r_{12}^3}\Big(-\frac{243}{4}(n_{12}v_1)^3+\frac{565}{4}(n_{12}v_1)^2(n_{12}v_2)-\frac{269}{4}(n_{12}v_1)(n_{12}v_2)^2\\
&\qquad -\frac{95}{12}(n_{12}v_2)^3+\frac{207}{8}(n_{12}v_1)v_1^2-\frac{137}{8}(n_{12}v_2)v_1^2-36(n_{12}v_1)(v_1v_2)\\
&\qquad +\frac{27}{4}(n_{12}v_2)(v_1v_2)+\frac{81}{8}(n_{12}v_1)v_2^2+\frac{83}{8}(n_{12}v_2)v_2^2\Big)\\
&+\frac{G^3m_2^3}{r_{12}^4}\Big(4(n_{12}v_1)+5(n_{12}v_2)\Big)\\
&+\frac{G^3m_1m_2^2}{r_{12}^4}\Big(-\frac{307}{8}(n_{12}v_1)+\frac{479}{8}(n_{12}v_2)+\frac{123}{32}(n_{12}v_{12})\pi^2\Big)\\
&+\frac{G^3m_1^2m_2}{r_{12}^4}\left(\frac{31397}{420}(n_{12}v_1)-\frac{36227}{420}(n_{12}v_2)-44(n_{12}v_{12})\ln\left(\frac{r_{12}}{r_1'}\right)\right)\Big]v_{12}^i\Big\}
\end{aligned}
$$

3.5 PN
(or 1PN
rad. reaction)

$$
\begin{aligned}
&+\frac{1}{c^7}\Big\{\Big[\frac{G^4m_1^3m_2}{r_{12}^5}\Big(\frac{3992}{105}(n_{12}v_1)-\frac{4328}{105}(n_{12}v_2)\Big)\\
&+\frac{G^4m_1^2m_2^2}{r_{12}^6}\Big(-\frac{13576}{105}(n_{12}v_1)+\frac{2872}{21}(n_{12}v_2)\Big)-\frac{3172}{21}\frac{G^4m_1m_2^3}{r_{12}^6}(n_{12}v_{12})\\
&+\frac{G^3m_1^2m_2}{r_{12}^4}\Big(48(n_{12}v_1)^3-\frac{696}{5}(n_{12}v_1)^2(n_{12}v_2)+\frac{744}{5}(n_{12}v_1)(n_{12}v_2)^2\\
&\qquad -\frac{288}{5}(n_{12}v_2)^3-\frac{4888}{105}(n_{12}v_1)v_1^2+\frac{5056}{105}(n_{12}v_2)v_1^2\\
&\qquad +\frac{2056}{21}(n_{12}v_1)(v_1v_2)-\frac{2224}{21}(n_{12}v_2)(v_1v_2)\\
&\qquad -\frac{1028}{21}(n_{12}v_1)v_2^2+\frac{5812}{105}(n_{12}v_2)v_2^2\Big)
\end{aligned}
$$

5.4

$$
\begin{aligned}
&+\frac{G^3 m_1 m_2^2}{r_{12}^4}\Big(-\frac{582}{5}(n_{12}v_1)^3+\frac{1746}{5}(n_{12}v_1)^2(n_{12}v_2)-\frac{1954}{5}(n_{12}v_1)(n_{12}v_2)^2\\
&\quad+158(n_{12}v_2)^3+\frac{3568}{105}(n_{12}v_{12})v_1^2-\frac{2864}{35}(n_{12}v_1)(v_1v_2)\\
&\quad+\frac{10048}{105}(n_{12}v_2)(v_1v_2)+\frac{1432}{35}(n_{12}v_1)v_2^2-\frac{5752}{105}(n_{12}v_2)v_2^2\Big)\\
&+\frac{G^2 m_1 m_2}{r_{12}^3}\Big(-56(n_{12}v_{12})^5+60(n_{12}v_1)^3v_{12}^2-180(n_{12}v_1)^2(n_{12}v_2)v_{12}^2\\
&\quad+174(n_{12}v_1)(n_{12}v_2)^2v_{12}^2-54(n_{12}v_2)^3v_{12}^2\\
&\quad-\frac{246}{35}(n_{12}v_{12})v_1^4+\frac{1068}{35}(n_{12}v_1)v_1^2(v_1v_2)\\
&\quad-\frac{984}{35}(n_{12}v_2)v_1^2(v_1v_2)-\frac{1068}{35}(n_{12}v_1)(v_1v_2)^2+\frac{180}{7}(n_{12}v_2)(v_1v_2)^2\\
&\quad-\frac{534}{35}(n_{12}v_1)v_1^2v_2^2+\frac{90}{7}(n_{12}v_2)v_1^2v_2^2+\frac{984}{35}(n_{12}v_1)(v_1v_2)v_2^2\\
&\quad-\frac{732}{35}(n_{12}v_2)(v_1v_2)v_2^2-\frac{204}{35}(n_{12}v_1)v_2^4+\frac{24}{7}(n_{12}v_2)v_2^4\Big)\Big]n_{12}^i\\
&+\Big[-\frac{184}{21}\frac{G^4m_1^3m_2}{r_{12}^5}+\frac{6224}{105}\frac{G^4m_1^2m_2^2}{r_{12}^6}+\frac{6388}{105}\frac{G^4m_1m_2^3}{r_{12}^6}\\
&\quad+\frac{G^3m_1^2m_2}{r_{12}^4}\Big(\frac{52}{15}(n_{12}v_1)^2-\frac{56}{15}(n_{12}v_1)(n_{12}v_2)-\frac{44}{15}(n_{12}v_2)^2-\frac{132}{35}v_1^2\\
&\quad+\frac{152}{35}(v_1v_2)-\frac{48}{35}v_2^2\Big)\\
&\quad+\frac{G^3m_1m_2^2}{r_{12}^4}\Big(\frac{454}{15}(n_{12}v_1)^2-\frac{372}{5}(n_{12}v_1)(n_{12}v_2)+\frac{854}{15}(n_{12}v_2)^2-\frac{152}{21}v_1^2\\
&\quad+\frac{2864}{105}(v_1v_2)-\frac{1768}{105}v_2^2\Big)\\
&\quad+\frac{G^2m_1m_2}{r_{12}^3}\Big(60(n_{12}v_{12})^4-\frac{348}{5}(n_{12}v_1)^2v_{12}^2+\frac{684}{5}(n_{12}v_1)(n_{12}v_2)v_{12}^2\\
&\quad-66(n_{12}v_2)^2v_{12}^2+\frac{334}{35}v_1^4-\frac{1336}{35}v_1^2(v_1v_2)+\frac{1308}{35}(v_1v_2)^2+\frac{654}{35}v_1^2v_2^2\\
&\quad-\frac{1252}{35}(v_1v_2)v_2^2+\frac{292}{35}v_2^4\Big)\Big]v_{12}^i\Big\}
\end{aligned}
$$

neglect 4PN $\longrightarrow$ $+\mathcal{O}\left(\frac{1}{c^8}\right).$ (168)

The 2.5PN and 3.5PN terms are associated with gravitational radiation reaction. The 3PN harmonic-coordinates equations of motion depend on two arbitrary length scales r'_1 and r'_2 associated with the logarithms present at the 3PN order[31]. It has been proved in Ref. [33] that r'_1 and r'_2 are merely linked with the choice of coordinates – we can refer to r'_1 and r'_2 as "gauge constants". In our approach [32, 33], the harmonic coordinate system is not uniquely fixed by the coordinate condition $\partial_\mu h^{\alpha\mu}=0$. In fact there are infinitely many harmonic coordinate systems

[31]Notice also the dependence upon π^2. Technically, the π^2 terms arise from non-linear interactions involving some integrals such as

$$\frac{1}{\pi}\int\frac{d^3\mathbf{x}}{r_1^2r_2^2}=\frac{\pi^2}{r_{12}}.$$

5.5

where the Newtonian trace-free quadrupole moment is $\mathrm{Q}_{ij} = m_1(y_1^i y_1^j - \frac{1}{3}\delta^{ij}\mathrm{y}_1^2) + 1 \leftrightarrow 2$. We refer to Iyer and Will [133, 134] for the discussion of the energy balance equation at the next 3.5PN order. As we can see, the 3.5PN equations of motion (168) are highly relativistic when describing the *motion*, but concerning the *radiation* they are in fact 1PN, because they contain merely the radiation reaction force at the 2.5PN+3.5PN orders.

9.2 Lagrangian and Hamiltonian formulations

The conservative part of the equations of motion in harmonic coordinates (168) is derivable from a *generalized* Lagrangian, depending not only on the positions and velocities of the bodies, but also on their accelerations: $a_1^i = dv_1^i/dt$ and $a_2^i = dv_2^i/dt$. As shown by Damour and Deruelle [86], the accelerations in the harmonic-coordinates Lagrangian occur already from the 2PN order. This fact is in accordance with a general result of Martin and Sanz [155] that N-body equations of motion cannot be derived from an ordinary Lagrangian beyond the 1PN level, provided that the gauge conditions preserve the Lorentz invariance. Note that we can always arrange for the dependence of the Lagrangian upon the accelerations to be *linear*, at the price of adding some so-called "multi-zero" terms to the Lagrangian, which do not modify the equations of motion (see, *e.g.*, Ref. [96]). At the 3PN level, we find that the Lagrangian also depends on accelerations. It is notable that these accelerations are sufficient – there is no need to include derivatives of accelerations. Note also that the Lagrangian is not unique because we can always add to it a total time derivative dF/dt, where F depends on the positions and velocities, without changing the dynamics. We find [3]

1PN

2PN

Note dependence on accelerations (in harm. coord.)

(DD 1982)

3PN

$$
\begin{aligned}
L^{\mathrm{harm}} = {} & \frac{G m_1 m_2}{2 r_{12}} + \frac{m_1 v_1^2}{2} \\
& + \frac{1}{c^2}\left\{ -\frac{G^2 m_1^2 m_2}{2 r_{12}^2} + \frac{m_1 v_1^4}{8} + \frac{G m_1 m_2}{r_{12}}\left(-\frac{1}{4}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2}v_1^2 - \frac{7}{4}(v_1 v_2)\right)\right\} \\
& + \frac{1}{c^4}\left\{ \frac{G^3 m_1^3 m_2}{2 r_{12}^3} + \frac{19 G^3 m_1^2 m_2^2}{8 r_{12}^3} \right. \\
& \qquad + \frac{G^2 m_1^2 m_2}{r_{12}^2}\left(\frac{7}{2}(n_{12}v_1)^2 - \frac{7}{2}(n_{12}v_1)(n_{12}v_2) + \frac{1}{2}(n_{12}v_2)^2 + \frac{1}{4}v_1^2 - \frac{7}{4}(v_1 v_2) + \frac{7}{4}v_2^2 \right) \\
& \qquad + \frac{G m_1 m_2}{r_{12}}\left(\frac{3}{16}(n_{12}v_1)^2(n_{12}v_2)^2 - \frac{7}{8}(n_{12}v_2)^2 v_1^2 + \frac{7}{8}v_1^4 + \frac{3}{4}(n_{12}v_1)(n_{12}v_2)(v_1 v_2) \right. \\
& \qquad\qquad \left. - 2 v_1^2 (v_1 v_2) + \frac{1}{8}(v_1 v_2)^2 + \frac{15}{16}v_1^2 v_2^2 \right) + \frac{m_1 v_1^6}{16} \\
& \qquad \left. + G m_1 m_2 \left(-\frac{7}{4}(a_1 v_2)(n_{12}v_2) - \frac{1}{8}(n_{12}a_1)(n_{12}v_2)^2 + \frac{7}{8}(n_{12}a_1)v_2^2 \right) \right\} \\
& + \frac{1}{c^6}\left\{ \frac{G^2 m_1^2 m_2}{r_{12}^2}\left(\frac{13}{18}(n_{12}v_1)^4 + \frac{83}{18}(n_{12}v_1)^3(n_{12}v_2) - \frac{35}{6}(n_{12}v_1)^2(n_{12}v_2)^2 - \frac{245}{24}(n_{12}v_1)^2 v_1^2 \right. \right. \\
& \qquad\qquad + \frac{179}{12}(n_{12}v_1)(n_{12}v_2)v_1^2 - \frac{235}{24}(n_{12}v_2)^2 v_1^2 + \frac{373}{48}v_1^4 + \frac{529}{24}(n_{12}v_1)^2(v_1 v_2) \\
& \qquad\qquad - \frac{97}{6}(n_{12}v_1)(n_{12}v_2)(v_1 v_2) - \frac{719}{24}v_1^2(v_1 v_2) + \frac{463}{24}(v_1 v_2)^2 - \frac{7}{24}(n_{12}v_1)^2 v_2^2 \\
& \qquad\qquad \left. - \frac{1}{2}(n_{12}v_1)(n_{12}v_2)v_2^2 + \frac{1}{4}(n_{12}v_2)^2 v_2^2 + \frac{463}{48}v_1^2 v_2^2 - \frac{19}{2}(v_1 v_2)v_2^2 + \frac{45}{16}v_2^4 \right) \\
& \qquad + \frac{5 m_1 v_1^8}{128}
\end{aligned}
$$

$$
\begin{aligned}
&+ Gm_1m_2\Bigg(\frac{3}{8}(a_1v_2)(n_{12}v_1)(n_{12}v_2)^2 + \frac{5}{12}(a_1v_2)(n_{12}v_2)^3 + \frac{1}{8}(n_{12}a_1)(n_{12}v_1)(n_{12}v_2)^3\\
&\qquad +\frac{1}{16}(n_{12}a_1)(n_{12}v_2)^4 + \frac{11}{4}(a_1v_1)(n_{12}v_2)v_1^2 - (a_1v_2)(n_{12}v_2)v_1^2\\
&\qquad -2(a_1v_1)(n_{12}v_2)(v_1v_2) + \frac{1}{4}(a_1v_2)(n_{12}v_2)(v_1v_2)\\
&\qquad +\frac{3}{8}(n_{12}a_1)(n_{12}v_2)^2(v_1v_2) - \frac{5}{8}(n_{12}a_1)(n_{12}v_1)^2v_2^2 + \frac{15}{8}(a_1v_1)(n_{12}v_2)v_2^2\\
&\qquad -\frac{15}{8}(a_1v_2)(n_{12}v_2)v_2^2 - \frac{1}{2}(n_{12}a_1)(n_{12}v_1)(n_{12}v_2)v_2^2\\
&\qquad -\frac{5}{16}(n_{12}a_1)(n_{12}v_2)^2v_2^2\Bigg)\\
&+ \frac{G^2m_1^2m_2}{r_{12}}\Bigg(-\frac{235}{24}(a_2v_1)(n_{12}v_1) - \frac{29}{24}(n_{12}a_2)(n_{12}v_1)^2 - \frac{235}{24}(a_1v_2)(n_{12}v_2)\\
&\qquad -\frac{17}{6}(n_{12}a_1)(n_{12}v_2)^2 + \frac{185}{16}(n_{12}a_1)v_1^2 - \frac{235}{48}(n_{12}a_2)v_1^2\\
&\qquad -\frac{185}{8}(n_{12}a_1)(v_1v_2) + \frac{20}{3}(n_{12}a_1)v_2^2\Bigg)\\
&+ \frac{Gm_1m_2}{r_{12}}\Bigg(-\frac{5}{32}(n_{12}v_1)^3(n_{12}v_2)^3 + \frac{1}{8}(n_{12}v_1)(n_{12}v_2)^3v_1^2 + \frac{5}{8}(n_{12}v_2)^4v_1^2\\
&\qquad -\frac{11}{16}(n_{12}v_1)(n_{12}v_2)v_1^4 + \frac{1}{4}(n_{12}v_2)^2v_1^4 + \frac{11}{16}v_1^6\\
&\qquad -\frac{15}{32}(n_{12}v_1)^2(n_{12}v_2)^2(v_1v_2) + (n_{12}v_1)(n_{12}v_2)v_1^2(v_1v_2)\\
&\qquad +\frac{3}{8}(n_{12}v_2)^2v_1^2(v_1v_2) - \frac{13}{16}v_1^4(v_1v_2) + \frac{5}{16}(n_{12}v_1)(n_{12}v_2)(v_1v_2)^2\\
&\qquad +\frac{1}{16}(v_1v_2)^3 - \frac{5}{8}(n_{12}v_1)^2v_1^2v_2^2 - \frac{23}{32}(n_{12}v_1)(n_{12}v_2)v_1^2v_2^2 + \frac{1}{16}v_1^4v_2^2\\
&\qquad -\frac{1}{32}v_1^2(v_1v_2)v_2^2\Bigg)\\
&- \frac{3G^4m_1^4m_2}{8r_{12}^4} + \frac{G^4m_1^3m_2^2}{r_{12}^4}\left(-\frac{9707}{420} + \frac{22}{3}\ln\left(\frac{r_{12}}{r_1'}\right)\right)\\
&+ \frac{G^3m_1^2m_2^2}{r_{12}^3}\Bigg(\frac{383}{24}(n_{12}v_1)^2 - \frac{889}{48}(n_{12}v_1)(n_{12}v_2) - \frac{123}{64}(n_{12}v_1)(n_{12}v_{12})\pi^2 - \frac{305}{72}v_1^2\\
&\qquad +\frac{41}{64}\pi^2(v_1v_{12}) + \frac{439}{144}(v_1v_2)\Bigg)\\
&+ \frac{G^3m_1^3m_2}{r_{12}^3}\Bigg(-\frac{8243}{210}(n_{12}v_1)^2 + \frac{15541}{420}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2}(n_{12}v_2)^2 + \frac{15611}{1260}v_1^2\\
&\qquad -\frac{17501}{1260}(v_1v_2) + \frac{5}{4}v_2^2 + 22(n_{12}v_1)(n_{12}v_{12})\ln\left(\frac{r_{12}}{r_1'}\right)\\
&\qquad -\frac{22}{3}(v_1v_{12})\ln\left(\frac{r_{12}}{r_1'}\right)\Bigg)\Bigg\}\\
&+1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^7}\right). \qquad (174)
\end{aligned}
$$

5.7

We have

$$E = -\frac{\mu c^2 \gamma}{2}\left\{1 + \left(-\frac{7}{4} + \frac{1}{4}\nu\right)\gamma + \left(-\frac{7}{8} + \frac{49}{8}\nu + \frac{1}{8}\nu^2\right)\gamma^2 + \left(-\frac{235}{64} + \left[\frac{46031}{2240} - \frac{123}{64}\pi^2 + \frac{22}{3}\ln\left(\frac{r}{r'_0}\right)\right]\nu + \frac{27}{32}\nu^2 + \frac{5}{64}\nu^3\right)\gamma^3\right\} + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{189}$$

This expression is that of a physical observable E; however, it depends on the choice of a coordinate system, as it involves the post-Newtonian parameter γ defined from the harmonic-coordinate separation r_{12}. But the *numerical* value of E should not depend on the choice of a coordinate system, so E must admit a frame-invariant expression, the same in all coordinate systems. To find it we re-express E with the help of a frequency-related parameter x instead of the post-Newtonian parameter γ. Posing

$$x \equiv \left(\frac{G\,m\,\omega}{c^3}\right)^{2/3} = \mathcal{O}\left(\frac{1}{c^2}\right), \tag{190}$$

we readily obtain from Eq. (188) the expression of γ in terms of x at 3PN order,

$$\gamma = x\left\{1 + \left(1 - \frac{\nu}{3}\right)x + \left(1 - \frac{65}{12}\nu\right)x^2 + \left(1 + \left[-\frac{2203}{2520} - \frac{41}{192}\pi^2 - \frac{22}{3}\ln\left(\frac{r}{r'_0}\right)\right]\nu + \frac{229}{36}\nu^2 + \frac{1}{81}\nu^3\right)x^3 + \mathcal{O}\left(\frac{1}{c^8}\right)\right\}, \tag{191}$$

that we substitute back into Eq. (189), making all appropriate post-Newtonian re-expansions. As a result, we gladly discover that the logarithms together with their associated gauge constant r'_0 have cancelled out. Therefore, our result is

Energy E is expressed in terms of orbital frequ ω

$x = \left(\frac{Gm\omega}{c^3}\right)^{2/3}$

$$E = -\frac{\mu c^2 x}{2}\left\{1 + \left(-\frac{3}{4} - \frac{1}{12}\nu\right)x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{1}{24}\nu^2\right)x^2 + \left(-\frac{675}{64} + \left[\frac{34445}{576} - \frac{205}{96}\pi^2\right]\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\right)x^3\right\} + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{192}$$

For circular orbits one can check that there are no terms of order $x^{7/2}$ in Eq. (192), so our result for E is actually valid up to the 3.5PN order.

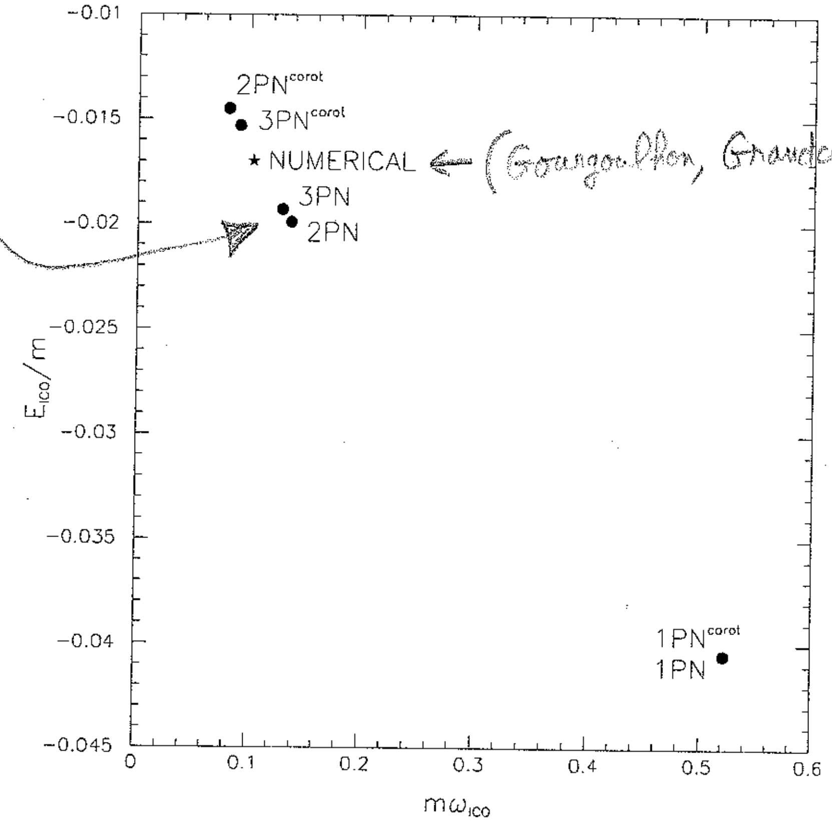
9.5 The innermost circular orbit (ICO)

Having in hand the circular-orbit energy, we define the innermost circular orbit (ICO) as the minimum, when it exists, of the energy function $E(x)$. Notice that we do not define the ICO as a point of dynamical general-relativistic unstability. Hence, we prefer to call this point the ICO rather than, strictly speaking, an innermost stable circular orbit or ISCO. A study of the dynamical stability of circular binary orbits in the post-Newtonian approximation of general relativity can be found in Ref. [50].

The previous definition of the ICO is motivated by our comparison with the results of numerical relativity. Indeed we shall confront the prediction of the standard (Taylor-based) post-Newtonian

I.C.O. = innermost circular orbit defined by the minimum of $E(\omega)$

Note the very good convergence of the PN expansion



← (Gourgoulhon, Grandclément, Bonazzola 2001)

Figure 1: Results for the binding energy $E_{\rm ICO}$ versus $\omega_{\rm ICO}$ in the equal-mass case ($\nu = 1/4$). The asterisk marks the result calculated by numerical relativity. The points indicated by 1PN, 2PN and 3PN are computed from the minimum of Eq. (192), and correspond to irrotational binaries. The points denoted by $1PN^{corot}$, $2PN^{corot}$ and $3PN^{corot}$ come from the minimum of the sum of Eqs. (192) and (201), and describe corotational binaries.

5.9

where $r_1 = |\mathbf{x} - \mathbf{y}_1|$ and $r_2 = |\mathbf{x} - \mathbf{y}_2|$. When $n > -3$ and $p > -3$, this integral is perfectly well-defined (recall that the finite part $\mathcal{FP}$ deals with the bound at infinity). When $n \leq -3$ or $p \leq -3$, our basic ansatz is that we apply the definition of the Hadamard *partie finie* provided by Eq. (124). Two examples of closed-form formulas that we get, which do not necessitate the Hadamard *partie finie*, are (quadrupole case $l = 2$)

$$
\begin{aligned}
\overset{(-1,-1)}{Y_{ij}} &= \frac{r_{12}}{3}\left[y_1^{\langle ij\rangle} + y_1^{\langle i}y_2^{j\rangle} + y_2^{\langle ij\rangle}\right], \\
\overset{(-2,-1)}{Y_{ij}} &= y_1^{\langle ij\rangle}\left[\frac{16}{15}\ln\left(\frac{r_{12}}{r_0}\right) - \frac{188}{225}\right] + y_1^{\langle i}y_2^{j\rangle}\left[\frac{8}{15}\ln\left(\frac{r_{12}}{r_0}\right) - \frac{4}{225}\right] + y_2^{\langle ij\rangle}\left[\frac{2}{5}\ln\left(\frac{r_{12}}{r_0}\right) - \frac{2}{25}\right].
\end{aligned}
\tag{216}
$$

We denote for example $y_1^{\langle ij\rangle} = y_1^{\langle i}y_1^{j\rangle}$ (and $r_{12} = r|\mathbf{y}_1 - \mathbf{y}_2|$); the constant r_0 is the one pertaining to the finite-part process (see Eq. (36)). One example where the integral diverges at the location of the particle 1 is

$$
\overset{(-3,0)}{Y_{ij}} = \left[2\ln\left(\frac{s_1}{r_0}\right) + \frac{16}{15}\right] y_1^{\langle ij\rangle}, \tag{217}
$$

where s_1 is the Hadamard-regularization constant introduced in Eq. (124)[37].

The crucial input of the computation of the flux at the 3PN order is the mass quadrupole moment I_{ij}, since this moment necessitates the full 3PN precision. The result of Ref. [37] for this moment (in the case of circular orbits) is

$$
\mathrm{I}_{ij} = \mu\left(A\, x_{\langle ij\rangle} + B\,\frac{r_{12}^3}{Gm} v_{\langle ij\rangle} + \frac{48}{7}\frac{G^2 m^2 \nu}{c^5 r_{12}} x_{\langle i}v_{j\rangle}\right) + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{218}
$$

where we pose $x_i = x^i \equiv y_{12}^i$ and $v_i = v^i \equiv v_{12}^i$. The third term is the 2.5PN radiation-reaction term, which does not contribute to the energy flux for circular orbits. The two important coefficients are A and B, whose expressions through 3PN order are

3 PN source quadrupole I_{ij}

$$
\begin{aligned}
A &= 1 + \gamma\left(-\frac{1}{42} - \frac{13}{14}\nu\right) + \gamma^2\left(-\frac{461}{1512} - \frac{18395}{1512}\nu - \frac{241}{1512}\nu^2\right) \\
&\quad + \gamma^3\left\{\frac{395899}{13200} - \frac{428}{105}\ln\left(\frac{r_{12}}{r_0}\right) + \left[\frac{139675}{33264} - \frac{44}{3}\xi - \frac{88}{3}\kappa - \frac{44}{3}\ln\left(\frac{r_{12}}{r'_0}\right)\right]\nu \right. \\
&\quad \left. + \frac{162539}{16632}\nu^2 + \frac{2351}{33264}\nu^3\right\}, \\
B &= \gamma\left(\frac{11}{21} - \frac{11}{7}\nu\right) + \gamma^2\left(\frac{1607}{378} - \frac{1681}{378}\nu + \frac{229}{378}\nu^2\right) \\
&\quad + \gamma^3\left(-\frac{357761}{19800} + \frac{428}{105}\ln\left(\frac{r_{12}}{r_0}\right) + \left[-\frac{75091}{5544} + \frac{44}{3}\zeta\right]\nu + \frac{35759}{924}\nu^2 + \frac{457}{5544}\nu^3\right).
\end{aligned}
\tag{219}
$$

These expressions are valid in harmonic coordinates *via* the post-Newtonian parameter γ given by Eq. (186). As we see, there are two types of logarithms in the moment: One type involves the length scale r'_0 related by Eq. (183) to the two gauge constants r'_1 and r'_2 present in the 3PN equations

[37]When computing the gravitational-wave flux in Ref. [37] we preferred to call the Hadamard-regularization constants u_1 and u_2, in order to distinguish them from the constants s_1 and s_2 that were used in our previous computation of the equations of motion in Ref. [33]. Indeed these regularization constants need not neccessarily to be the same when employed in different contexts.

Let us give the two basic technical formulas needed when carrying out this reduction:

$$\begin{aligned}\int_0^{+\infty} d\tau \ln\tau\, e^{-\sigma\tau} &= -\frac{1}{\sigma}(C+\ln\sigma),\\ \int_0^{+\infty} d\tau \ln^2\tau\, e^{-\sigma\tau} &= \frac{1}{\sigma}\left[\frac{\pi^2}{6}+(C+\ln\sigma)^2\right],\end{aligned} \tag{226}$$

where $\sigma \in \mathbb{C}$ and $C = 0.577\cdots$ denotes the Euler constant [122]. The tail integrals are evaluated thanks to these formulas for a *fixed* (non-decaying) circular orbit. Indeed it can be shown [42] that the "remote-past" contribution to the tail integrals is negligible; the errors due to the fact that the orbit actually spirals in by gravitational radiation do not affect the signal before the 4PN order. We then find, for the quadratic tail term *stricto sensu*, the 1.5PN, 2.5PN and 3.5PN amounts[39]

$$\begin{aligned}\mathcal{L}_{\rm tail} = \frac{32c^5}{5G}\gamma^5\nu^2\Bigg\{&4\pi\gamma^{3/2}+\left(-\frac{25663}{672}-\frac{125}{8}\nu\right)\pi\gamma^{5/2}+\left(\frac{90205}{576}+\frac{505747}{1512}\nu+\frac{12809}{756}\nu^2\right)\pi\gamma^{7/2}\\ &+\mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}.\end{aligned} \tag{227}$$

For the sum of squared tails and cubic tails of tails at 3PN, we get

$$\begin{aligned}\mathcal{L}_{({\rm tail})^2+{\rm tail(tail)}} = \frac{32c^5}{5G}\gamma^5\nu^2\Bigg\{&\left(-\frac{116761}{3675}+\frac{16}{3}\pi^2-\frac{1712}{105}C+\frac{1712}{105}\ln\left(\frac{r_{12}}{r_0}\right)-\frac{856}{105}\ln(16\gamma)\right)\gamma^3\\ &+\mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}.\end{aligned} \tag{228}$$

By comparing Eqs. (223) and (228) we observe that the constants r_0 cleanly cancel out. Adding together all these contributions we obtain

$$\begin{aligned}\mathcal{L} = \frac{32c^5}{5G}\gamma^5\nu^2\Bigg\{&1+\left(-\frac{2927}{336}-\frac{5}{4}\nu\right)\gamma+4\pi\gamma^{3/2}+\left(\frac{293383}{9072}+\frac{380}{9}\nu\right)\gamma^2+\left(-\frac{25663}{672}-\frac{125}{8}\nu\right)\pi\gamma^{5/2}\\ &+\Bigg[\frac{129386791}{7761600}+\frac{16\pi^2}{3}-\frac{1712}{105}C-\frac{856}{105}\ln(16\gamma)\\ &\quad+\left(-\frac{50625}{112}+\frac{110}{3}\ln\left(\frac{r_{12}}{r'_0}\right)+\frac{123\pi^2}{64}\right)\nu-\frac{383}{9}\nu^2\Bigg]\gamma^3\\ &+\left(\frac{90205}{576}+\frac{505747}{1512}\nu+\frac{12809}{756}\nu^2\right)\pi\gamma^{7/2}+\mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}.\end{aligned} \tag{229}$$

The gauge constant r'_0 has not yet disappeared because the post-Newtonian expansion is still parametrized by γ instead of the frequency-related parameter x defined by Eq. (190) – just as for E when it was given by Eq. (189). After substituting the expression $\gamma(x)$ given by Eq. (191), we find that r'_0 does cancel as well. Because the relation $\gamma(x)$ is issued from the equations of motion, the latter cancellation represents an interesting test of the consistency of the two computations, in harmonic coordinates, of the 3PN multipole moments and the 3PN equations of motion. At long last we obtain our end result:

Final result for the 3.5PN flux (expressed in term of x)

$$\mathcal{L} = \frac{32c^5}{5G}\nu^2x^5\Bigg\{1+\left(-\frac{1247}{336}-\frac{35}{12}\nu\right)x+4\pi x^{3/2}+\left(-\frac{44711}{9072}+\frac{9271}{504}\nu+\frac{65}{18}\nu^2\right)x^2$$

[39] All formulas incorporate the changes in some equations following the published Errata (2005) to the works [17, 20, 37, 35, 5].

$$+\left(-\frac{8191}{672}-\frac{583}{24}\nu\right)\pi x^{5/2}$$
$$+\left[\frac{6643739519}{69854400}+\frac{16}{3}\pi^2-\frac{1712}{105}C-\frac{856}{105}\ln(16\,x)\right.$$
$$\left.+\left(-\frac{134543}{7776}+\frac{41}{48}\pi^2\right)\nu-\frac{94403}{3024}\nu^2-\frac{775}{324}\nu^3\right]x^3$$
$$+\left(-\frac{16285}{504}+\frac{214745}{1728}\nu+\frac{193385}{3024}\nu^2\right)\pi x^{7/2}+\mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}. \tag{230}$$

In the test-mass limit $\nu \to 0$ for one of the bodies, we recover exactly the result following from linear black-hole perturbations obtained by Tagoshi and Sasaki [202]. In particular, the rational fraction 6643739519/69854400 comes out exactly the same as in black-hole perturbations. On the other hand, the ambiguity parameters λ and θ are part of the rational fraction $-134543/7776$, belonging to the coefficient of the term at 3PN order proportional to ν (hence this coefficient cannot be computed by linear black hole perturbations)[40].

10.3 Orbital phase evolution

We shall now deduce the laws of variation with time of the orbital frequency and phase of an inspiralling compact binary from the energy balance equation (214). The center-of-mass energy E is given by Eq. (192) and the total flux $\mathcal{L}$ by Eq. (230). For convenience we adopt the dimensionless time variable[41]

$$\Theta \equiv \frac{\nu c^3}{5Gm}(t_c - t), \tag{231}$$

where t_c denotes the instant of coalescence, at which the frequency tends to infinity (evidently, the post-Newtonian method breaks down well before this point). We transform the balance equation into an ordinary differential equation for the parameter x, which is immediately integrated with the result

$$x=\frac{1}{4}\Theta^{-1/4}\Bigg\{1+\left(\frac{743}{4032}+\frac{11}{48}\nu\right)\Theta^{-1/4}-\frac{1}{5}\pi\Theta^{-3/8}+\left(\frac{19583}{254016}+\frac{24401}{193536}\nu+\frac{31}{288}\nu^2\right)\Theta^{-1/2}$$
$$+\left(-\frac{11891}{53760}+\frac{109}{1920}\nu\right)\pi\Theta^{-5/8}$$
$$+\left[-\frac{10052469856691}{6008596070400}+\frac{1}{6}\pi^2+\frac{107}{420}C-\frac{107}{3360}\ln\left(\frac{\Theta}{256}\right)\right.$$
$$\left.+\left(\frac{3147553127}{780337152}-\frac{451}{3072}\pi^2\right)\nu-\frac{15211}{442368}\nu^2+\frac{25565}{331776}\nu^3\right]\Theta^{-3/4}$$
$$+\left(-\frac{113868647}{433520640}-\frac{31821}{143360}\nu+\frac{294941}{3870720}\nu^2\right)\pi\Theta^{-7/8}+\mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}. \tag{232}$$

The orbital phase is defined as the angle ϕ, oriented in the sense of the motion, between the separation of the two bodies and the direction of the ascending node $\mathcal{N}$ within the plane of the sky, namely the point on the orbit at which the bodies cross the plane of the sky moving toward

[40] Generalizing the flux formula (230) to point masses moving on *quasi elliptic* orbits dates back from the work of Peters and Mathews [175] at Newtonian order. The result was obtained in [214, 41] at 1PN order, and then further extended by Gopakumar and Iyer [119] up to 2PN order using an explicit quasi-Keplerian representation of the motion [97, 194]. No complete result at 3PN order is yet available.

[41] Notice the "strange" post-Newtonian order of this time variable: $\Theta = \mathcal{O}(c^{+8})$.

5.12

the detector. We have $d\phi/dt = \omega$, which translates, with our notation, into $d\phi/d\Theta = -5/\nu \cdot x^{3/2}$, from which we determine

$$
\begin{aligned}
\phi = -\frac{1}{\nu}\Theta^{5/8}\Bigg\{ & 1+\left(\frac{3715}{8064}+\frac{55}{96}\nu\right)\Theta^{-1/4}-\frac{3}{4}\pi\Theta^{-3/8}+\left(\frac{9275495}{14450688}+\frac{284875}{258048}\nu+\frac{1855}{2048}\nu^2\right)\Theta^{-1/2} \\
& +\left(-\frac{38645}{172032}+\frac{65}{2048}\nu\right)\pi\Theta^{-5/8}\ln\left(\frac{\Theta}{\Theta_0}\right) \\
& +\left[\frac{831032450749357}{57682522275840}-\frac{53}{40}\pi^2-\frac{107}{56}C+\frac{107}{448}\ln\left(\frac{\Theta}{256}\right)\right. \\
& \quad +\left(-\frac{126510089885}{4161798144}+\frac{2255}{2048}\pi^2\right)\nu \\
& \quad \left.+\frac{154565}{1835008}\nu^2-\frac{1179625}{1769472}\nu^3\right]\Theta^{-3/4} \\
& +\left(\frac{188516689}{173408256}+\frac{488825}{516096}\nu-\frac{141769}{516096}\nu^2\right)\pi\Theta^{-7/8}+\mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\},
\end{aligned}
\tag{233}
$$

where Θ_0 is a constant of integration that can be fixed by the initial conditions when the wave frequency enters the detector's bandwidth. Finally we want also to dispose of the important expression of the phase in terms of the frequency x. For this we get

3.5PN orbital phase $\phi(x)$

$$
\begin{aligned}
\phi = -\frac{x^{-5/2}}{32\nu}\Bigg\{ & 1+\left(\frac{3715}{1008}+\frac{55}{12}\nu\right)x-10\pi x^{3/2}+\left(\frac{15293365}{1016064}+\frac{27145}{1008}\nu+\frac{3085}{144}\nu^2\right)x^2 \\
& +\left(\frac{38645}{1344}-\frac{65}{16}\nu\right)\pi x^{5/2}\ln\left(\frac{x}{x_0}\right) \\
& +\left[\frac{12348611926451}{18776862720}-\frac{160}{3}\pi^2-\frac{1712}{21}C-\frac{856}{21}\ln(16\,x)\right. \\
& \quad \left.+\left(-\frac{15737765635}{12192768}+\frac{2255}{48}\pi^2\right)\nu+\frac{76055}{6912}\nu^2-\frac{127825}{5184}\nu^3\right]x^3 \\
& +\left(\frac{77096675}{2032128}+\frac{378515}{12096}\nu-\frac{74045}{6048}\nu^2\right)\pi x^{7/2}+\mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\},
\end{aligned}
\tag{234}
$$

where x_0 is another constant of integration. With the formula (234) the orbital phase is complete up to the 3.5PN order. The effects due to the spins of the particles, *i.e.* the spin-orbit (SO) coupling arising at the 1.5PN order for maximally rotating compact bodies and the spin-spin (SS) coupling at the 2PN order, can be added if necessary; they are known up to the 2.5PN order included [143, 141, 165, 201, 109, 24]. On the other hand, the contribution of the quadrupole moments of the compact objects, which are induced by tidal effects, is expected to come only at the 5PN order [see Eq. (8)].

As a rough estimate of the relative importance of the various post-Newtonian terms, let us give in Table 2 their contributions to the accumulated number of gravitational-wave cycles $\mathcal{N}$ in the bandwidth of the LIGO and VIRGO detectors (see also Table I in Ref. [30] for the contributions of the SO and SS effects). Note that such an estimate is only indicative, because a full treatment would require the knowledge of the detector's power spectral density of noise, and a complete simulation of the parameter estimation using matched filtering [80, 181, 149]. We define $\mathcal{N}$ by

$$
\mathcal{N} = \frac{1}{\pi}\left[\phi_{\mathrm{ISCO}} - \phi_{\mathrm{seismic}}\right]. \tag{235}
$$

The frequency of the signal at the entrance of the bandwidth is the seismic cut-off frequency f_{seismic} of ground-based detectors; the terminal frequency f_{ISCO} is assumed for simplicity's sake to be given

	$2\times 1.4M_\odot$	$10M_\odot + 1.4M_\odot$	$2\times 10M_\odot$
Newtonian order	16031	3576	602
1PN	441	213	59
1.5PN (dominant tail)	-211	-181	-51
2PN	9.9	9.8	4.1
2.5PN	-11.7	-20.0	-7.1
3PN	2.6	2.3	2.2
3.5PN	-0.9	-1.8	-0.8

Table 2: Contributions of post-Newtonian orders to the accumulated number of gravitational-wave cycles $\mathcal{N}$ [defined by (235)] in the bandwidth of VIRGO and LIGO detectors. Neutron stars have mass $1.4M_\odot$, and black holes $10\,M_\odot$. The entry frequency is $f_{\text{seismic}} = 10$ Hz, and the terminal frequency is $f_{\text{ISCO}} = c^3/(6^{3/2}\pi\, G\, m)$.

by the Schwarzschild innermost stable circular orbit. Here $f = \frac{\omega}{\pi} = \frac{2}{P}$ is the signal frequency at the dominant harmonics (twice the orbital frequency). As we see in Table 2, with the 3PN or 3.5PN approximations we reach an acceptable level of, say, a few cycles, that roughly corresponds to the demand which was made by data-analysists in the case of neutron-star binaries [78, 79, 80, 180, 59, 60]. Indeed, the above estimation suggests that the neglected 4PN terms will yield some systematic errors that are, at most, of the same order of magnitude, *i.e.* a few cycles, and perhaps much less (see also the discussion in Section 9.6).

10.4 The two polarization wave-forms

The theoretical templates of the compact binary inspiral follow from insertion of the previous solutions for the 3.5PN-accurate orbital frequency and phase into the binary's two polarization wave-forms h_+ and $h_\times$. We shall include in h_+ and $h_\times$ all the harmonics, besides the dominant one at twice the orbital frequency, up to the 2.5PN order, as they have been calculated in Refs. [38, 5]. The polarization wave-forms are defined with respect to two polarization vectors $\mathbf{p} = (p_i)$ and $\mathbf{q} = (q_i)$:

$$\begin{aligned} h_+ &= \frac{1}{2}(p_i p_j - q_i q_j) h_{ij}^{\mathrm{TT}}, \\ h_\times &= \frac{1}{2}(p_i q_j + p_j q_i) h_{ij}^{\mathrm{TT}}, \end{aligned} \tag{236}$$

where $\mathbf{p}$ and $\mathbf{q}$ are chosen to lie along the major and minor axis, respectively, of the projection onto the plane of the sky of the circular orbit, with $\mathbf{p}$ oriented toward the ascending node $\mathcal{N}$. To the 2PN order we have

$$h_{+,\times} = \frac{2G\mu x}{c^2 R}\left\{ H^{(0)}_{+,\times} + x^{1/2} H^{(1/2)}_{+,\times} + x H^{(1)}_{+,\times} + x^{3/2} H^{(3/2)}_{+,\times} + x^2 H^{(2)}_{+,\times} + x^{5/2} H^{(5/2)}_{+,\times} + \mathcal{O}\left(\frac{1}{c^6}\right)\right\}. \tag{237}$$

The post-Newtonian terms are ordered by means of the frequency-related variable x. They depend on the binary's 3.5PN-accurate phase ϕ through the auxiliary phase variable

$$\psi = \phi - \frac{2GM\omega}{c^3}\ln\left(\frac{\omega}{\omega_0}\right), \tag{238}$$

5.14

where $\mathrm{M} = m\left[1 - \nu\gamma/2 + \mathcal{O}\left(1/c^4\right)\right]$ is the ADM mass [*cf.* Eq. (225)], and where ω_0 is a constant frequency that can conveniently be chosen to be the entry frequency of a laser-interferometric detector (say $\omega_0/\pi = 10\,\mathrm{Hz}$). For the plus polarization we have[42]

2PN waveform

$$
\begin{aligned}
H_+^{(0)} &= -(1+c_i^2)\cos 2\psi\,,\\
H_+^{(1/2)} &= -\frac{s_i}{8}\frac{\delta m}{m}\left[(5+c_i^2)\cos\psi - 9(1+c_i^2)\cos 3\psi\right],\\
H_+^{(1)} &= \frac{1}{6}\left[19 + 9c_i^2 - 2c_i^4 - \nu(19 - 11c_i^2 - 6c_i^4)\right]\cos 2\psi - \frac{4}{3}s_i^2(1+c_i^2)(1-3\nu)\cos 4\psi\,,\\
H_+^{(3/2)} &= \frac{s_i}{192}\frac{\delta m}{m}\left\{\left[57 + 60c_i^2 - c_i^4 - 2\nu(49 - 12c_i^2 - c_i^4)\right]\cos\psi\right.\\
&\qquad - \frac{27}{2}\left[73 + 40c_i^2 - 9c_i^4 - 2\nu(25 - 8c_i^2 - 9c_i^4)\right]\cos 3\psi\\
&\qquad \left. + \frac{625}{2}(1-2\nu)s_i^2(1+c_i^2)\cos 5\psi\right\} - 2\pi(1+c_i^2)\cos 2\psi\,,\\
H_+^{(2)} &= \frac{1}{120}\left[22 + 396c_i^2 + 145c_i^4 - 5c_i^6 + \frac{5}{3}\nu(706 - 216c_i^2 - 251c_i^4 + 15c_i^6)\right.\\
&\qquad \left. -5\nu^2(98 - 108c_i^2 + 7c_i^4 + 5c_i^6)\right]\cos 2\psi\\
&\quad + \frac{2}{15}s_i^2\left[59 + 35c_i^2 - 8c_i^4 - \frac{5}{3}\nu(131 + 59c_i^2 - 24c_i^4) + 5\nu^2(21 - 3c_i^2 - 8c_i^4)\right]\cos 4\psi\\
&\quad - \frac{81}{40}(1 - 5\nu + 5\nu^2)s_i^4(1+c_i^2)\cos 6\psi\\
&\quad + \frac{s_i}{40}\frac{\delta m}{m}\left\{\left[11 + 7c_i^2 + 10(5+c_i^2)\ln 2\right]\sin\psi - 5\pi(5+c_i^2)\cos\psi\right.\\
&\qquad \left. - 27\left[7 - 10\ln(3/2)\right](1+c_i^2)\sin 3\psi + 135\pi(1+c_i^2)\cos 3\psi\right\}.
\end{aligned}
\tag{239}
$$

For the cross polarization,

$$
\begin{aligned}
H_\times^{(0)} &= -2c_i\sin 2\psi\,,\\
H_\times^{(1/2)} &= -\frac{3}{4}s_i c_i\frac{\delta m}{m}\left[\sin\psi - 3\sin 3\psi\right],\\
H_\times^{(1)} &= \frac{c_i}{3}\left[17 - 4c_i^2 - \nu(13 - 12c_i^2)\right]\sin 2\psi - \frac{8}{3}(1-3\nu)c_i s_i^2\sin 4\psi\,,
\end{aligned}
$$

[42] We neglect the non-linear memory (DC) term present in the Newtonian plus polarization $H_+^{(0)}$. See Wiseman and Will [219] and Arun *et al.* [5] for the computation of this term.