AGR 2.0

2

General Relativistic N-body Problem:

Multi-chart approach, BD multipoles, Equations of motion

Relevant background references:

[BK88] V.A. Brumberg, S.M. Kopeikin, Nuovo. Cim. B 103, 63 (1988)

[DSX1] T. Damour, M. Soffel, C. Xu, 'General relativistic celestial mechanics I. Method and definition of reference systems' Phys. Rev. D 43, 3273 (1991)

[DSX2] T. Damour, M. Soffel, C. Xu 'General relativistic celestial mechanics II. Translational equations of motion', Phys. Rev. D 45, 1017 (1992)

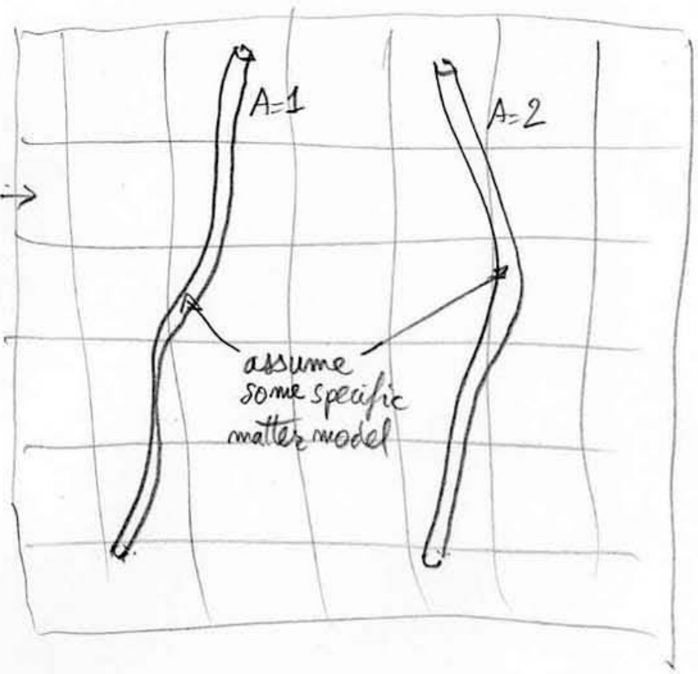
Review of problem of motion

[D87] T. Damour, 'The problem of motion in Newtonian and Einsteinian gravity', in "300 Years of Gravitation", ed. S.W. Hawking and W. Israel, Cambridge U. Press, 1987, pp 128-198.

$\boxed{2.1}$ Traditional (one chart) approach to the N-body problem

Traditionally

ONE
GLOBAL
COORDINATE
CHART
x^μ



Usual strategy: Try to solve
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$
$$\Downarrow$$
together with
$$\nabla_\nu T^{\mu\nu} = 0$$

assuming

specific
$$T^{\mu\nu} \overset{e.g}{=} (\varepsilon + p) u^\mu u^\nu + p g^{\mu\nu}$$

and global expansion:
$$g_{\mu\nu}(x^\lambda) = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + \dots$$

together, may be, with global PN approximation ansätze

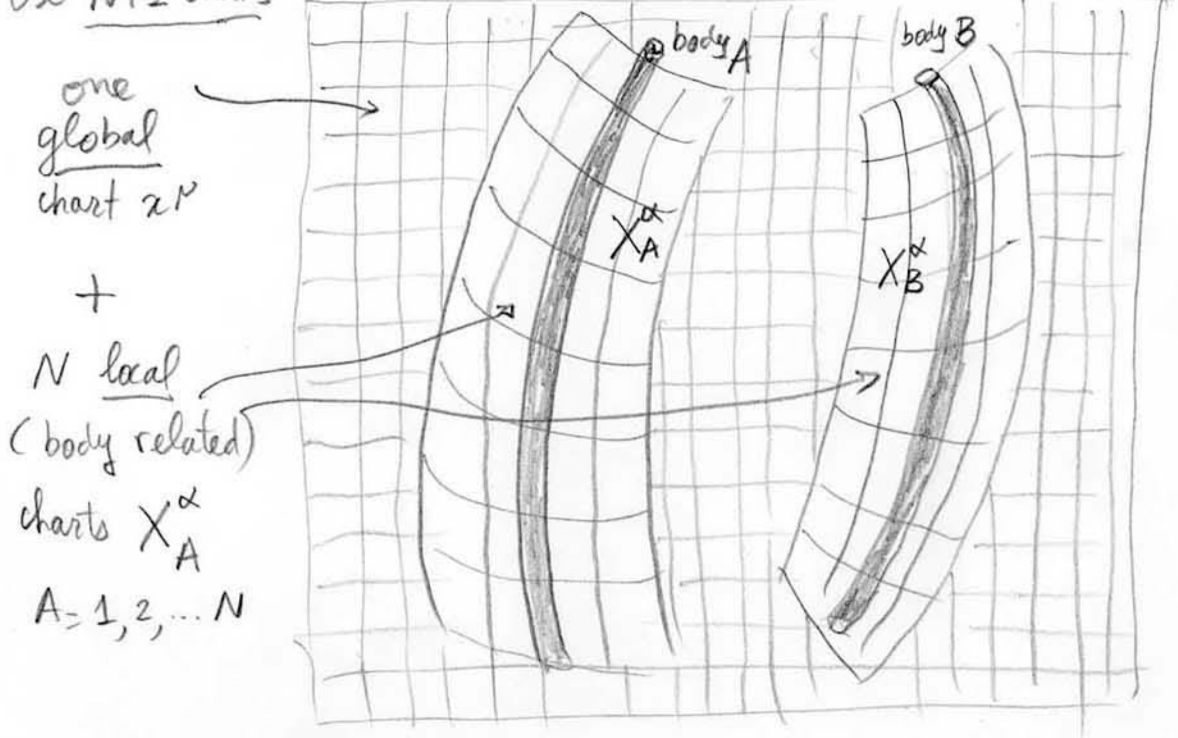
$$\partial_0 h_{\mu\nu} = \frac{1}{c} \partial_t h_{\mu\nu} \ll \partial_i h_{\mu\nu}$$
$$v \ll c$$
$$p \ll \varepsilon$$

Inconvenient aspects of global, one-chart approach:

- technically : e.g. a body which is (essentially) spherically symmetric w.r.t. its own rest frame, will appear, in global x^μ, as some deformed ellipsoid (with time-dependent deformation). Deformation important w.r.t. accuracy of modern techniques: VLBI, laser tracking, ...
- the gravitational field, and multipole moments, of body A in common x^μ chart are not directly related to the relevant observable quantities
- conceptually : PN approximation tends to reintroduce Newtonian way of thinking (absolute space and time) which can lead to errors or confusions.

$\boxed{2.2}$ Multi-chart approach to the N-body problem

Use N+1 charts



one global chart x^μ

+

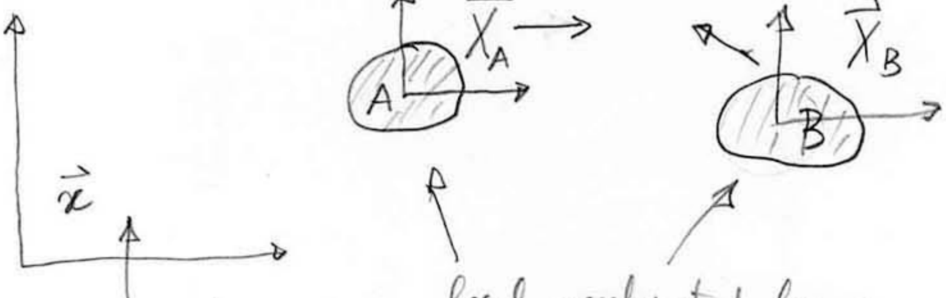
N local (body related) charts X^α_A

$A = 1, 2, \dots N$

Basic ideas:

- to describe the metric $ds^2 = G^A_{\alpha\beta}(X^\gamma_A)\, dX^\alpha_A\, dX^\beta_A$ in each local chart X^α_A in terms mainly of quantities observed in this system ($T^{\alpha\beta}_A(X_A)$, multipole moments measured by satellites of A, ...) , plus some extra 'tidal-like' terms coming from the influence of far away bodies
- to obtain the equations of motion (in common x^μ chart) of each body as generalizations of the 'd'Alembert approach', i.e. reducing dynamics in x^μ to an 'equilibrium' problem in the local frame X^α_A
- to end up by expressing the global eqs of motion in terms of the locally measured multipole moments of each body
- framework is rather simple and fully explicit at the 1PN approximation (weakly self-gravitating bodies and v^2/c^2 beyond Newton), but the general idea is usefully extended to more general cases (notably strongly self-gravitating bodies)
- NB: We are considering general deformable bodies, with unspecified equation of state.

2.3 Reminder: d'Alembert approach to the Newtonian N-body problem



inertial frame

local accelerated frames associated with each body

in inertial frame

$i = 1,2,3$

$$\boxed{\begin{aligned} &\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x^i}(\rho v^i) = 0 \\ &\frac{\partial}{\partial t}(\rho v^i) + \frac{\partial}{\partial x^j}\left[\rho v^i v^j + \overset{\text{material stresses}}{t^{ij}}\right] = \rho \frac{\partial U}{\partial x^i} \\ &\Delta_x U(\vec{x},t) = -4\pi G \rho(\vec{x},t) \end{aligned}}$$

formal solution $U(\vec{x},t) = \sum_{A=1}^{N} U^A(\vec{x},t)$

with $U^A(\vec{x},t) \equiv G \int_A d^3x' \, \frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}$

yields <u>non local</u> (and non linear) evolution system for $\rho(\vec{x},t)$, $\vec{v}(\vec{x},t)$, which cannot be solved exactly.

Introduce transformation to some accelerated reference frames associated with each body

$$\boxed{x^i = z^i_A(t) + X^i_a}$$

assuming for simplicity no time-dependent rotation

arbitrary accelerated motion to be determined later

in accelerated A-frame

$$\rho_A \equiv \rho(\vec{X}_A, t) = \rho(\vec{z}_A + \vec{X}_A, t)$$

$$V^i_A \equiv v^i - \frac{dz^i_A}{dt}$$

$$\boxed{\begin{aligned} &\frac{\partial \rho_A}{\partial t} + \frac{\partial}{\partial X^i_A}(\rho_A V^i_A) = 0 \\ &\frac{\partial}{\partial t}(\rho_A V^i_A) + \frac{\partial}{\partial X^j_A}(\rho_A V^i_A V^j_A + t^{ij}_A) = \rho_A \frac{\partial U^{eff}_A}{\partial X^i_A} \end{aligned}}$$

effective gravitational potential (modified by inertial forces)

$$\boxed{U^{eff}_A(\vec{X}_A) = U(\vec{z}_A + \vec{X}_A) - C(t) - \frac{d^2\vec{z}_A}{dt^2} \cdot \vec{X}_A}$$

can be decomposed as

$$\boxed{U^{eff}_A = U^A + \bar{U}^A}$$

locally generated

$$\boxed{U^A(t, \vec{X}_A) = G \int d^3X'_A \frac{\rho_A(\vec{X}'_A, t)}{|\vec{X}_A - \vec{X}'_A|}}$$

externally generated + inertia effects

$$\boxed{\bar{U}^A = \sum_{B \neq A} U^B(\vec{z}_A + \vec{X}_A) - C(t) - \frac{d^2\vec{z}_A}{dt^2} \cdot \vec{X}_A}$$

Introduce (mass) multipole moments of body A wrt local A frame

$$\boxed{m^A_L(t) \equiv \int_A d^3X_A \; X_A^{\langle L\rangle} \rho_A} \qquad l = 0, 1, 2, \dots$$

+ local spin vector

$$\boxed{s^A_i(t) \equiv \int_A d^3X_A \; \varepsilon_{iab} \, X_A^a \, \rho \, V_A^b}$$

Using Action and Reaction principle, one has

ONLY $\bar{U}^A$ enters here

$$\boxed{\begin{aligned} &\frac{d}{dt} m^A(t) = 0 \\ &\frac{d^2}{dt^2} m^A_i(t) = \int_A d^3X_A \, \rho_A \frac{\partial \bar{U}^A}{\partial X_A^i} \\ &\frac{d}{dt} s_i^A(t) = \varepsilon_{iab} \int d^3X_A \, \rho_A \, X_A^a \frac{\partial \bar{U}^A}{\partial X_A^b} \end{aligned}}$$

Expand $\bar{U}^A(\vec{X}_A)$ in <u>tidal series</u>

$$\boxed{\bar{U}^A(\vec{X}_A, t) = g^A(t) + g^A_i(t) X_A^i + \frac{1}{2!} g^A_{ij}(t) X_A^{\langle i} X_A^{j\rangle} + \dots + \frac{1}{l!} g^A_L(t) X_A^{\langle L\rangle} + \dots}$$

$$g^A(t) = \sum_{B \neq A} U^B(\vec{z}_A(t)) - C(t)$$

$$g^A_i(t) = \sum_{B \neq A} \partial_i U^B(\vec{z}_A(t)) - \frac{d^2 z_A^i}{dt^2}$$

crucial inertial contribution at <u>dipole</u> level

$$g^A_L(t) = \sum_{B \neq A} \partial_L U^B(\vec{z}_A(t)) \quad \text{for } l \geq 2$$

- One can choose arbitrary $C(t)$ to set $g^A(t) = 0$; $l=0$ term
- One could also think that it is a good idea to choose the 'so far arbitrary $\frac{d^2\vec{z}_A}{dt^2}$ to set tidal dipole

  $$g^A_i(t) \overset{?}{=} 0$$

  In fact, NO!

- Insert tidal expansion of $\bar{U}^A$ in eqs for m^A, m^A_i, s^A_i

$$\boxed{\begin{aligned}
&\frac{d\,m^A(t)}{dt} = 0\\
&\frac{d^2 m^A_i(t)}{dt^2} = m^A g^A_i + m^A_j\, g^A_{ij} + \frac{1}{2!} m^A_{jk}\, g^A_{ijk} + \cdots + \frac{1}{l!} m^A_L\, g^A_{iL} + \cdots\\
&\frac{d s^A_i(t)}{dt} = \varepsilon_{iab}\, m^A_a\, g^A_b + \varepsilon_{iab}\, m^A_{aj}\, g^A_{bj} + \cdots + \frac{1}{l!} \varepsilon_{iab}\, m^A_{aL}\, g^A_{bL} + \cdots
\end{aligned}}$$

- Let us now fix the motion of the local A frame by imposing that it follows, for all times, the overall motion of body A by setting

$$\boxed{0 = m^A_i(t) = \int_A d^3X_A\, \rho_A\, X^i_A = \int_A d^3x\, \rho(\vec{x},t)\,[x^i - z^i_A(t)]}$$

i.e. saying that the local A frame stays centered at the center of mass of body A

This condition of 'equilibrium' in A frame yields

$$\frac{d^2 m^A_i}{dt^2} = 0 = \underset{\swarrow}{m^A g^A_i} + \underset{\overset{\|}{0}}{m^A_j g^A_{ij}} + \frac{1}{2!} m^A_{jk} g^A_{ijk} + \cdots$$

$$\boxed{0 = m^A \Big(\sum_{B \neq A} \partial_i U^B(\vec{z}_A) - \frac{d^2 z^i_A}{dt^2} \Big) + \frac{1}{2!} m^A_{jk} g^A_{ijk} + \cdots + \frac{1}{\ell!} m^A_L g^A_{iL} + \cdots}$$

'd'Alembert'

But we have fixed $z^i_A \equiv z^i_{cm\,A}$, hence we get the following (global) inertial-frame translational eqs of motion for body A

$$\boxed{m^A \frac{d^2 z^i_{cm.A}}{dt^2} = \sum_{B \neq A} \Big\{ m^A \partial_i U^B(\vec{z}_{cmA}) + \frac{1}{2!} m^A_{jk} \partial_{ijk} U^B(z_{cmA}) + \cdots + \frac{1}{\ell!} m^A_L \partial_{iL} U^B(z_{cmA}) \Big\}}$$

Replacing U^B by its multipole expansion

$$U^B(\vec{x},t) = \frac{G m^B}{|\vec{x} - \vec{z}_B|} - \partial_i \Big(\frac{G m^B_i}{|\vec{x} - \vec{z}_B|} \Big) + \frac{1}{2!} \partial_{ij} \Big(\frac{G m^B_{ij}}{|\vec{x} - \vec{z}_B|} \Big) \cdots + \frac{(-)^\ell}{\ell!} \partial_L \frac{G m^B_L}{|x - z_B|} + \cdots$$

Finally we get a double series in the multipoles of all bodies

$$\boxed{m^A \frac{d^2 z^i_{cmA}}{dt^2} = G \sum_{B \neq A} \sum_{\ell \geq 0} \sum_{k \geq 0} \frac{(-)^k}{\ell!\,k!} m^A_L m^B_K \partial^A_{iLK} \Big(\frac{1}{|\vec{z}_{cmA} - \vec{z}_{cmB}|} \Big)}$$

Useful facts in Newtonian N-body problem:

- Eq. for U was linear $\Rightarrow$ allowed linear decomposition $U = \sum_A U^A$
- inertial forces in local frame $\Rightarrow$ linear addition to U
- $\exists$ multipole expansion for each U^A
- $\exists$ tidal expansion for U_A^{eff}, including inertial effects, in A frame

The 1PN generalrelativistic N-body problem is NON LINEAR

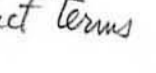
and the transformation $x^\mu = f^\mu(X_A^\alpha)$ is also NON LINEAR

BUT the problem can be reformulated (at 1PN) in a quasi-linear way which allows one to physically parallel the Newtonian treatment

One makes use of a hidden linearity in Einstein's equations at 1PN level. Essentially, after using at 1PN:

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$; $h_{ij} \simeq h_{00}\delta_{ij}$, $h_{0i} \ll h_{00}$, $\partial_0 \ll \partial_i$, the cubic piece in Einstein's action is

$$\mathcal{L}^{Ein} \sim (\partial_i h_{00})^2 + \overbrace{\alpha\, h_{00}(\partial_i h_{00})^2}^{\text{cubic vertex}} + \text{contact terms}$$



$$h_{00} = \varphi \quad : \quad (\partial\varphi)^2 + \alpha\varphi(\partial\varphi)^2 = (\partial\psi)^2$$

$$\text{if} \quad d\psi = \sqrt{1+\alpha\varphi}\, d\varphi \Rightarrow \boxed{\psi \simeq \varphi + \frac{1}{4}\alpha\varphi^2 = h_{00} + \frac{1}{4}\alpha h_{00}^2}$$

i.e. change of variable : $h_{00} \to \psi \Rightarrow \simeq$ linear theory $\mathcal{L} \sim (\partial\psi)^2$

2.4 Quasi-linear formulation of 1PN relativity

In *each* of the N+1 coordinate charts one can write 1PN-accurate metric in terms of

a *scalar potential* $w \propto \overset{\text{non-linear function of } g_{00}}{\ln(-g_{00})}$ cf 24 above

a *vector potential* $w_i \propto g_{0i}$

with

$g_{\mu\nu}(x^\lambda)$ in global chart

$$\begin{aligned} g_{00} &= -\exp\left(-\frac{2}{c^2}w\right), \\ g_{0i} &= -\frac{4}{c^3}w_i, \\ g_{ij} &= \delta_{ij}\exp\left(+\frac{2}{c^2}w\right) + \overset{= \mathcal{O}(\frac{1}{c^4})}{\mathcal{O}(4)} \end{aligned}$$

$G^A_{\alpha\beta}(X^\gamma_A)$ in local A chart

$$\begin{aligned} G^A_{00} &= -\exp\left(-\frac{2}{c^2}W^A\right), \\ G^A_{0a} &= -\frac{4}{c^3}W^A_a, \\ G^A_{ab} &= \delta_{ab}\exp\left(+\frac{2}{c^2}W^A\right) + \mathcal{O}(4) \end{aligned}$$

In each frame, the 'scalar' and 'vector' potentials satisfy <u>linear</u> ('Maxwell-like') equations

$$\boxed{\begin{aligned} &\Delta_x w + \frac{3}{c^2}\partial_t^2 w + \frac{4}{c^2}\partial_t \partial_i w_i = -4\pi G\sigma + O(4) \\ &\Delta_x w_i - \partial_{ij} w_j - \partial_t \partial_i w = -4\pi G \sigma^i + O(2) \end{aligned}}$$

where

$$\boxed{\begin{aligned} \sigma(\vec{x},t) &= \frac{T^{00} + T^{ss}}{c^2} \\ \sigma^i(\vec{x},t) &= \frac{T^{0i}}{c} \end{aligned}}$$

Similarly for W^A, W_a^A: $\Delta_{X_A} W^A + \frac{3}{c^2}\partial_{T_A}^2 W^A + \cdots = -4\pi G \Sigma_A$

$$\Sigma_A(\vec{X},T) = \frac{T_A^{00}(X) + T_A^{ss}(X)}{c^2}$$

in X_A-chart

∃ Maxwell-like <u>gauge invariance</u> of 1PN field eqs

$$\begin{cases} w' = w - \frac{1}{c^2}\partial_t \lambda \\ w_i' = w_i + \frac{1}{4}\partial_i \lambda \end{cases}$$

which corresponds to a shift of the time variable

$$\delta t = \frac{1}{c^4}\lambda(\vec{x},t)$$

This residual gauge invariance (after fixing the spatial gauge by the form of the metric) for the

'4 vector' $a_\mu = (c w, -4 w_i)$

suggests to introduce the gauge-invariant object

$$b_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$$

i.e.

'gravito electric field' $\qquad e_i[w] \equiv \partial_i w + \frac{4}{c^2} \partial_t w_i$

'gravito magnetic field' $\qquad b_{ij}[w) \equiv \varepsilon_{ijk} b_k \equiv -4[\partial_i w_j - \partial_j w_i]$

They satisfy Maxwell-like eqs

$$\boxed{\begin{aligned} \vec{\nabla} \cdot \vec{b} &= 0 \\ \vec{\nabla} \times \vec{e} &= -\tfrac{1}{c^2} \partial_t \vec{b} \\ \vec{\nabla} \cdot \vec{e} &= -\tfrac{3}{c^2} \partial_t^2 w - 4\pi G \sigma + \mathcal{O}(4) \\ \vec{\nabla} \times \vec{b} &= +4\, \partial_t \vec{e} - 16\pi G \vec{\sigma} + \mathcal{O}(2) \end{aligned}}$$

If the time coordinate x^0 is harmonic:

$$0 = \Box_g x^0 = -\frac{4}{c^3} \left(\partial_t w + \partial_i w_i\right) + \mathcal{O}(5)$$

we have the "Lorenz"-like gauge $\quad \boxed{\partial_t w + \partial_i w_i = 0}$

and

$$\Delta - \frac{1}{c^2}\partial_t^2 \longrightarrow \boxed{\begin{aligned} \Box w &= -4\pi G \sigma + \mathcal{O}(4) \\ \Delta w_i &= -4\pi G \sigma^i + \mathcal{O}(2) \end{aligned}}$$

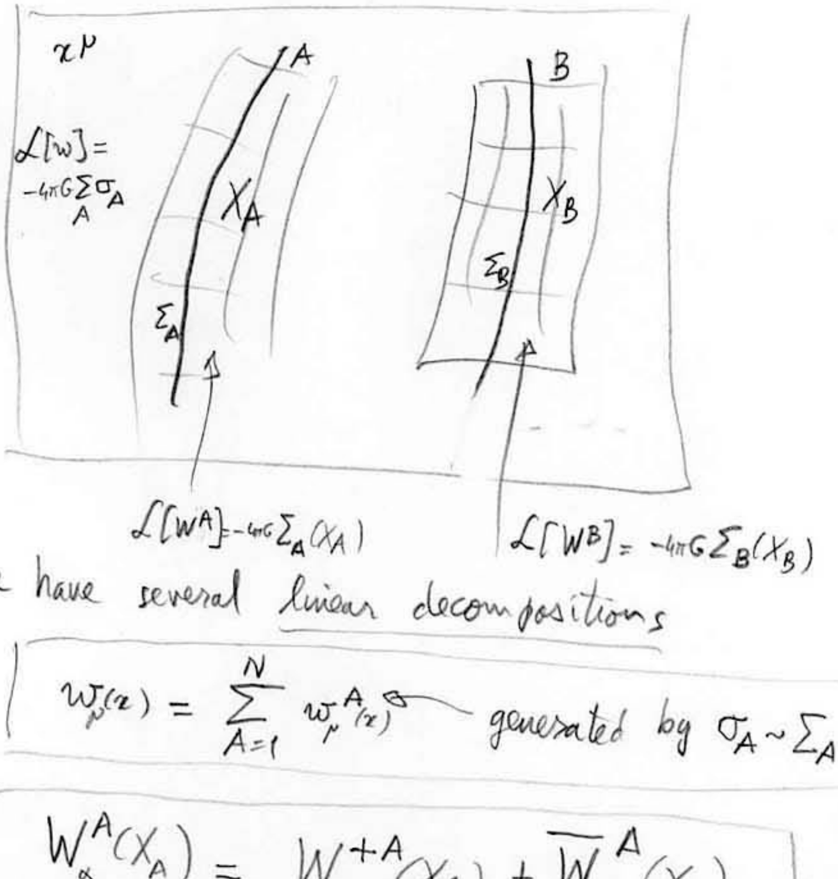
2.5 Structure of PN metric in various charts

In each frame $w_\mu \equiv (w, w_i)$ or $W^A_\alpha \equiv (W^A, W^A_a)$
satisfy some unhomogeneous linear equations

in global frame $\mathcal{L}^\mu[w_\nu] = -4\pi G \sum_{A=1}^{N} \sigma_A{}^\mu(x)$

in A-local frame $\mathcal{L}^\alpha[W^A_\beta] = -4\pi G \; \Sigma^\alpha_A(X_A)$

only one source term present in X_A frame.



Therefore we have several linear decompositions

$$\boxed{w_\mu(x) = \sum_{A=1}^{N} w_\mu^A(x)}$$ generated by $\sigma_A \sim \Sigma_A$

$$\boxed{W^A_\alpha(X_A) = W^{+A}_\alpha(X_A) + \overline{W}^A_\alpha(X_A)}$$

locally generated by Σ_A

unknown homogeneous solution : $\mathcal{L}[\overline{W}^A] = 0$

$\boxed{2.6}$ Relation between the various linear decompositions

One proves that the transformation $x^\mu = f^\mu(X^\alpha)$ between global x^μ and any local X^α (index A suppressed here) is

$$x^\mu = z^\mu(X^0) + e^\mu_a(X^0)\left[X^a + \frac{1}{c^2}\left(\frac{1}{2}A_a \vec{X}^2 - X^a(\vec{A}\cdot\vec{X})\right)\right] + \eta^\mu$$

some worldline in x^μ chart representing the moving origin of X-frame

some triad of spacetime vectors which depends only on choice of some slowly changing rotation matrix $R^i{}_a(T)$

$$\eta^0 = \frac{1}{c^3}\eta(X^0, X^a) = \mathcal{O}(\vec{X}^2)$$

$$\eta^i = \mathcal{O}\left(\frac{1}{c^4}\right)$$

(Minkowski acceleration)

$$A_a \equiv \eta_{\mu\nu} e^\mu_a \frac{d^2 z^\nu}{d\tau^2}$$

with

$$d\tau^2 = -\eta_{\mu\nu} dz^\mu dz^\nu$$

Effect of coordinate transformation $x^\mu = f^\mu(X^\alpha)$:

$$g^{\mu\nu}(x) = \frac{\partial x^\mu(X)}{\partial X^\alpha}\frac{\partial x^\nu(X)}{\partial X^\beta} G^{\alpha\beta}(X)$$

non linear in terms of $w_\mu(x)$

non linear coordinate transf

non linear in terms of $W_\alpha(X)$

where $w_\mu = \sum_{A=1}^{N} w_\mu^A$

where $W = W^+ + \overline{W}$

locally generated

(homogeneous rest)

After non trivial analysis, simple links

locally generated pieces

$$\boxed{w^A_\mu(x) = \mathcal{A}^A_{\mu\alpha}(X^0_A)\, W^{+A}_\alpha(X)}$$

'externally generated' pieces

$$\boxed{\sum_{B\neq A} w^B_\mu(x) = \mathcal{A}^A_{\mu\alpha}(X^0_A)\, \overline{W}^A_\alpha(X_A) + \mathcal{B}^A_\mu(X_A)}$$

explicitly computed in terms of Z^μ_A, $e_A{}^\mu_a$

can be rewritten as

$$\boxed{\overline{W}^A_\alpha = \mathcal{A}^{A(-1)}_{\alpha\mu}\left[\sum_{B\neq A} w^B_\mu - \mathcal{B}^A_\mu(X)\right]}$$

'tidal grav. field in A frame'

generated by far away bodies

additional 'inertial contribution'

closely analogous to Newtonian result:

$$\boxed{\overline{U}^A = \sum_{B\neq A} U^B - C(t) - \frac{d^2\vec{Z}_A}{dt^2}\cdot\vec{X}_A}$$

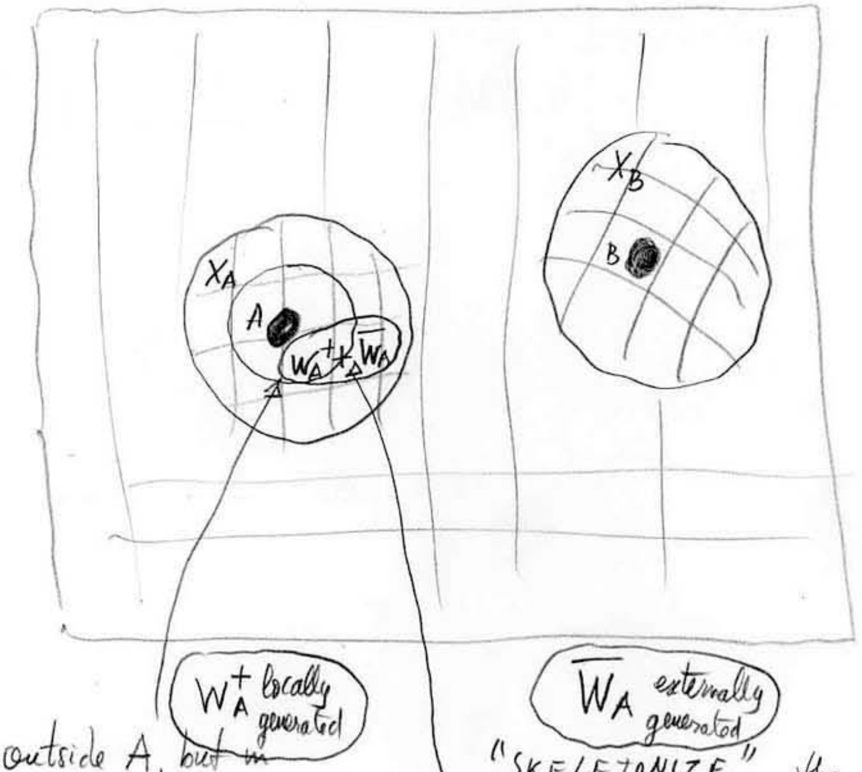
tidal potential

externally generated

inertia effects of accelerated frame

$\boxed{2.7}$ Introduction of relativistic multipole and tidal moments

Basic idea



outside A, ~~but in~~ local A frame

SKELETONIZE A

by 1PN-accurate multipoles

M_L^A

S_L^A

describing

W_A^+

"SKELETONIZE" the effect of other bodies + inertial effects in local A frame

by expanding

$\overline{W}_A$

in Taylor series in $\vec{X}_A$

relativistic multipole moments (1PN) : BD moments [BD89]

$$M^A_{a_1a_2\cdots a_l}(T) \equiv \int_A d^3X\, X^{\langle a_1}\cdots X^{a_l\rangle}\, \frac{T^{00}_A + T^{ss}_A}{c^2}$$

everything in the A-frame

$$+\frac{1}{2(2l+3)}\frac{1}{c^2}\frac{d^2}{dT^2}\int_A d^3X\, \vec{X}^2 X^{\langle a_1}\cdots X^{a_l\rangle}\,\frac{T^{00}_A}{c^2}$$

$$-\frac{4(2l+1)}{(l+1)(2l+3)c^2}\frac{d}{dT}\int_A d^3X\, X^{\langle b}X^{a_1}\cdots X^{a_l\rangle}\,\frac{T^{0b}_A}{c}$$

$$S^A_{a_1\cdots a_l}(T) \equiv \int_A d^3X\, \varepsilon_{bc\langle a_l} X_{a_1}\cdots X_{a_{l-1}\rangle} X_b\, \frac{T^{0c}_A}{c}$$

relativistic tidal moments (1PN)

gauge-invariant gravito-electric and -magnetic "tidal fields"

$$\bar{E}^A_a(X_A) = \partial_a \bar{W}^A + \frac{4}{c^2}\partial_T \bar{W}^A_a$$

$$\bar{B}^A_a(X_A) = \varepsilon_{abc}\,\partial_b\left(-4\bar{W}^A_c\right)$$

externally generated, as seen in X_A frame

'electric-' and 'magnetic-type' relativistic tidal moments

$$G^A_{a_1\cdots a_l}(T) \equiv \partial_{\langle a_1 a_2\cdots a_{l-1}}\bar{E}^A_{a_l\rangle}\Big|_{\vec{X}_A=0}$$

$$H^A_{a_1\cdots a_l}(T) \equiv \partial_{\langle a_1\cdots a_{l-1}}\bar{B}^A_{a_l\rangle}\Big|_{\vec{X}_A=0}$$

- Expression of local gravitational field in terms of multipole and tidal moments

$$\overset{+}{W}{}^{A}\overset{\text{outside}\,A}{(T,\vec{X})} = G\sum_{l\geq 0}\frac{(-)^l}{l!}\partial_L\left(\frac{M_L^A(T\overset{\text{half-sum}}{\mp}R/c)}{R}\right)+\frac{1}{c^2}\partial_T(\Lambda^A-\lambda)+O(4)$$

$$\overset{+}{W}{}^{A}_{a}(T,\vec{X}) = -G\sum_{l\geq 1}\frac{(-)^l}{l!}\left[\partial_{L-1}\left(\frac{\dot{M}^A_{aL-1}}{R}\right)+\frac{l}{l+1}\varepsilon_{abc}\partial_{bL-1}\left(\frac{S^A_{cL-1}}{R}\right)\right]$$
$$-\frac{1}{4}\partial_a(\Lambda^A-\lambda)+O(2)$$

$$\overline{W}{}^{A}(T,\vec{X}) = \sum_l\frac{1}{l!}\left[\hat{X}^L G_L^A(T)+\frac{1}{2(2l+3)c^2}\vec{X}^2\hat{X}^L\ddot{G}_L^A(T)\right]+\frac{1}{c^2}\partial_T\overline{\Lambda}^A+O(4)$$

$$\overline{W}{}^{A}_{a}(T,\vec{X}) = \sum_l\frac{1}{l!}\left[-\frac{2l+1}{(l+1)(2l+3)}\hat{X}^{aL}\dot{G}_L^A+\frac{l}{4(l+1)}\varepsilon_{abc}\hat{X}^{bL-1}H^A_{cL-1}\right]$$
$$-\frac{1}{4}\partial_a\overline{\Lambda}^A+O(2)$$

2.8 Relativistic tidally expanded equations of motion

d'Alembert-type approach

Local evolution equation for matter variables

$$0 = \nabla_\beta T^{\alpha\beta} = \frac{\partial}{\partial x^\beta} T^{\alpha\beta} + \Gamma^\alpha_{\sigma\beta} T^{\sigma\beta} + \Gamma^\beta_{\sigma\beta} T^{\alpha\sigma}$$

$$\boxed{\begin{aligned} &\partial_T \Sigma + \partial_{X^a} \Sigma^a = \frac{1}{c^2} \partial_T T^{ss} - \frac{1}{c^2} \Sigma \, \partial_T W + O(4) \\ &\partial_T \left[\left(1 + \frac{4W}{c^2}\right) \Sigma^a \right] + \partial_{X^b} \left[\left(1 + \frac{4W}{c^2}\right) T^{ab} \right] = \mathcal{F}^a_{(T,\vec{X})} + O(4) \end{aligned}}$$

$$\boxed{\mathcal{F}^a(T,\vec{X}) = \Sigma E_a + \frac{1}{c^2} B_{ab} \Sigma^b = \left(\Sigma \vec{E} + \frac{1}{c^2} \vec{\Sigma} \times \vec{B} \right)_a}$$

TOTAL
$\vec{E}$ and $\vec{B}$ fields
$E^+ + \overline{E}$ ...

remarkable Lorentz form of relativistic force density

Note:

LINEAR in W

$$\boxed{\mathcal{F}^a = \mathcal{F}^a[W_\alpha^{tot}] = \mathcal{F}^a[\overset{+}{W}] + \mathcal{F}^a[\overline{W}]}$$

'self-force density' $\overset{+}{\mathcal{F}}{}^a$

'external force density' $\overline{\mathcal{F}}{}^a$

AGR 2.20

Consider evolution of $l=0$ and $l=1$ BD multipoles

$$\left\{ \begin{aligned} \frac{d}{dT} \overset{l=0}{M^A}(T) &= F_0^A[W] \equiv -\frac{1}{c^2}\int_A d^3X\,\Sigma\,\partial_T W - \frac{1}{c^2}\frac{d}{dT}\int_A d^3X\,\Sigma\, X^b \partial_b W \\ \frac{d^2}{dT^2} M_a^A(T) &= F_a^A[W] = \int_A d^3X\, \mathcal{F}^a[W] - \frac{1}{c^2}\frac{d}{dT}\int d^3X\,(4W\Sigma^a + X^a \Sigma\partial_T W) \\ &\quad - \frac{1}{c^2}\frac{d^2}{dT^2}\int_A d^3X\,(X^a X^b - \tfrac{1}{2}\vec{X}^2\delta^{ab})\Sigma\partial_b W \end{aligned} \right.$$

again RHS's are linear in W

$$\Rightarrow \quad F_\alpha^A[W^A] = \overset{\overset{+}{W}}{\overset{+}{F}{}_\alpha^A} + \overset{\overline{W}}{\overline{F}{}_\alpha^A}$$

Theorem ('1PN Action and Reaction'): $\boxed{\overset{+}{F}{}_\alpha^A = 0 \mod \mathcal{O}(4)}$

$$\Rightarrow \quad F_\alpha^A[W] = F_\alpha^A[\underset{\text{insert tidal expansion}}{\overline{W}}]$$

complicated intermediate calculations
but nice simplifications allows one to express
the final result only in terms of <u>multipole</u> and
<u>tidal</u> moments

$$\boxed{\begin{aligned} \frac{d M^A}{d T_A} &= \overline{F}^A_0 \left[M^A_L, G^A_L \right] + O(4) \\ \frac{d^2 M^A_a}{d T_A^2} &= \overline{F}^A_a \left[M^A_L, S^A_L, G^A_L, H^A_L \right] + O(4) \end{aligned}}$$

where, for instance,

$$\boxed{\overline{F}_a = \sum_\ell \frac{1}{\ell!} \Big\{ \underset{\uparrow}{M_L G_{aL}} + \frac{1}{c^2} \underset{\uparrow}{\frac{\ell}{\ell+1}} S_L H_{aL} + \text{seven other } {}^1\!/_{c^2} \text{ terms} \Big\}}$$

"Newtonian-like" term

explicit 1PN corrections

NB : $\overline{F}_0 = -\frac{1}{c^2} \sum_\ell \frac{1}{\ell!} \left\{ (\ell+1) M_L \dot{G}_L + \ell \dot{M}_L G_L \right\}$ is $\neq 0$ in general

$\boxed{2.9}$ Translational equations of motion à la d'Alembert

① Attach the spatial origin of A-frame to body A by requiring $\boxed{M^A_a(T) = 0}$ ← specific way of defining a relativistic center of mass for A

② Write the consequence of center of mass definition

$$\frac{d^2 M^A_a}{dT^2} = 0 = \overline{F}_a \left[M^A_L , S^A_L , G^A_L , H^A_L \right]$$

i.e. $$0 = M^A G^A_a + \frac{1}{2!} M^A_{bc} G^A_{abc} + \dots + \frac{1}{l!} M^A_L G^A_{aL}$$

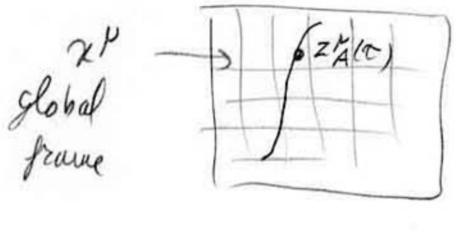
$$+ \frac{1}{c^2} \sum_l \frac{1}{l!} \frac{l}{l+1} S_L H_{aL} + \text{seven other } \frac{1}{c^2} f(MSGH)$$

'tidal dipole' $$\boxed{G^A_a = \sum_{B \neq A} G^{B/A}_a - A^A_a}$$

contributions from other bodies

$$\boxed{A^A_a = \eta_{\mu\nu} \, e^\mu_{Aa} \frac{d^2 z^\nu_A}{d\tau^2}}$$

Minkowski acceleration of z^μ_A worldline



$\Rightarrow$ yields translational equations of motion for body A

$$\boxed{\begin{aligned} M^A A^A_a = & \sum_{B \neq A} \left(M^A G^{B/A}_a \left[M^B_K , S^B_K \right] + \sum_{l \geq 2} \frac{1}{l!} M^A_L G^{B/A}_{aL} \left[M^B_K , S^B_K \right] \right) \\ & + \frac{1}{c^2} F^{(1PN)}_a \left(M^A_L , S^A_L ; G^A_L , \left[M^B_K , S^B_K \right] , H^A_L , \left[M^B_K , S^B_K \right] \right) + O(4) \end{aligned}}$$

RHS can be fully expressed in terms of <u>multipole</u> moments

$\boxed{2.10}$ Rotational equations of motion

Similarly, with extra work (DSX3: PRD 47, 3124 (1993)), one can derive a 1PN-accurate law of evolution for a certain 1PN-accurate 'spin vector' of body A.

$$\boxed{\frac{d}{dT_A} S_a^{A(1PN)} = \Gamma_a^A \left[M_L^A, S_L^A, G_L^A, H_L^A \right] + \mathcal{O}(4)}$$

$$\uparrow \qquad\qquad \uparrow$$

$$S_a^{\overset{(1PN)}{A}} = S_a^{\substack{BD\\ \text{usual}}}[\Sigma_A^\alpha] + \frac{1}{c^2}\bar{S}_a[\underset{\substack{\uparrow\\ \text{tidal fields}}}{\bar{E}, \bar{B}}]$$

$$\sum_l \frac{1}{l!}\left[\underset{\substack{\uparrow\\ \text{Newtonian torque}}}{\varepsilon_{abc} M_{bL}^A G_{cL}^A} + \underset{\substack{\uparrow\\ \frac{1}{c^2}\text{ corrections}}}{\frac{1}{c^2}\frac{l+1}{l+2}\varepsilon_{abc} S_{bL}^A H_{cL}^A} \right]$$

2.11 Applications of DSX formalism

- Allows one to control / correct the application of GR in, e.g., the relativistic description of the Earth environment: tests of GR on Earth, satellite motion around the Earth (DSX4: PRD 49, 618 (1994))

- Allows one to describe, with relativistic accuracy, the use of modern technologies in solar-system: VLBI, laser tracking of Moon etc ...

- Allows one to derive 1PN accurate equations of motion for solar-system, including all effects of higher multipoles (notably spin and quadrupole, which are the dominant corrections)

2.12 Application to a derivation of the Lorentz-Droste-Einstein-Infeld-Hoffmann equations of motion for N monopolar bodies

truncation to a 'monopolar model' for each body:

i.e. $\forall A = 1, \dots, N$: $\quad l \geq 1 \Rightarrow M^A_L = 0 = S^A_L$

i.e. keep only mass monopole $l = 0$ $\quad \boxed{M^A \neq 0}$

- The general law
$$\frac{dM^A}{dT} = -\frac{1}{c^2} \sum_l \frac{1}{l!} \left\{ (l+1) M_L \dot{G}_L + l \dot{M}_L G_L \right\}$$
$$= -\frac{1}{c^2} M^A \dot{G}^A$$

actually 'monopole tidal moment'
$$\overset{A}{G}(T) = \overline{W}_A(T, \underset{\swarrow \; \vec{X}_A = 0}{\vec{0}})$$

Fix freedom in definition of local time T_A by requiring

$$\overline{W}_A(T, \vec{0}) = 0 \quad \rightarrow \quad G^A(T) = 0$$

$$\Rightarrow \qquad \boxed{\frac{dM^A}{dT} = 0}$$

Fix origin of A frame to be the 1PN 'center of mass'

i.e. $$\boxed{M_a^A(T) = 0}$$

$$\Rightarrow 0 = \frac{d^2 M_a^A}{dT^2} = M^A G_a^A + \underbrace{\frac{1}{2!} M_{bc}^A G_{abc}^A}_{0} + \cdots + \underbrace{\frac{1}{c^2} \sum_l \frac{1}{l!} \frac{l}{l+1} S_L H_{aL} + \cdots}_{0}$$

$$\Rightarrow \qquad \boxed{G_a^A = 0}$$

i.e. $$\boxed{\bar{E}_a^A \Big|_{\vec{X}_A = 0} = 0}$$

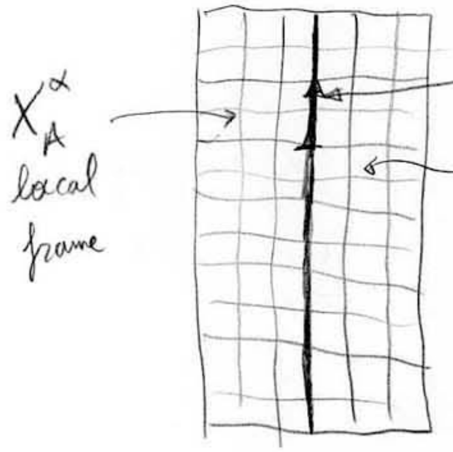
Let us define the following metric around body A

$$\boxed{\begin{aligned} \bar{G}_{00}^A(T,\vec{X}) &\equiv -e^{-\frac{2}{c^2}\bar{W}^A} \\ \bar{G}_{0a}^A(T,\vec{X}) &\equiv -\frac{4}{c^3}\bar{W}_a^A \\ \bar{G}_{ab}^A(T,\vec{X}) &\equiv \delta_{ab}\, e^{+\frac{2}{c^2}\bar{W}^A} \end{aligned}} \neq \left\{ \begin{aligned} G_{00} &= -e^{-\frac{2}{c^2}W} \\ G_{0a} &= -\frac{4}{c^3} W_a \\ G_{ab} &= \delta_{ab}\, e^{\frac{2}{c^2}W} \end{aligned} \right.$$

↑ defines an 'external metric' obtained by discarding the self-terms $\overset{+}{W}{}^A$ in the exponential parametrization

↑ real metric, with $W = \overset{+}{W}{}^A + \bar{W}^A$

Then consider the A worldline within metric $\bar{G}_{\alpha\beta}$



A worldline $X^a_A = 0$

$d\bar{s}^2 = \bar{G}_{\alpha\beta}(X)\,dX^\alpha dX^\beta$

unit tangent vector

$$\boxed{\bar{u}^\alpha_A = e^{\bar{W}/c^2}\frac{\partial}{\partial T} = \left(e^{\bar{W}/c^2}, 0, 0, 0\right)}$$

$$\bar{G}^A_{\alpha\beta}\,\bar{u}^\alpha_A\,u^\beta_A = -1$$

coordinate vectorial basis: $\epsilon_0 \equiv \frac{1\partial}{c\partial T_A} \leftrightarrow \bar{u}_A = c\,e^{\bar{W}/c^2}\epsilon_0$

$\epsilon_a \equiv \frac{\partial}{\partial X^a_A}$

simple calculation

$$\boxed{\bar{G}_A\left(\epsilon_a, \bar{\nabla}_{\bar{u}_A}\bar{u}_A\right) \equiv \bar{G}^A_{\alpha\beta}\;\epsilon^\alpha_a\;\;\bar{u}^\sigma_A\,\overset{\text{with }\bar{\Gamma}_A}{\bar{\nabla}_\sigma}\,\bar{u}^\beta_A = -\,\bar{E}^A_a\Big|_{\vec{X}=0}}$$

Therefore

$$G^A_a = 0 \to \bar{E}^A_a = 0 \to \boxed{\bar{\nabla}_{\bar{u}_A}\,\bar{u}_A = 0}$$

i.e. the A worldline is a <u>geodesic</u> of the A-external metric $\boxed{d\bar{s}^2_A = \bar{G}^A_{\alpha\beta}(X)\,dX^\alpha dX^\beta}$

by transforming to global coordinates $X_A^\alpha \to x^\mu$ one concludes that the A worldline is a geodesic of the following global-coords A-external metric

$d\bar{s}^2 = \bar{g}^A_{\mu\nu}(x^\lambda)\, dx^\mu dx^\nu$ where

$$\begin{cases} \bar{g}^A_{00}(x) = -\exp\left(-\frac{2}{c^2} \sum_{B \neq A} w^B(x)\right) \\ \bar{g}^A_{0i}(x) = -\frac{4}{c^3} \sum_{B \neq A} w_i^B(x) \\ \bar{g}^A_{ij}(x) = \delta_{ij} \exp\left(\frac{2}{c^2} \sum_{B \neq A} w^B(x)\right) \end{cases}$$

with some extra work (using $w^A_\mu = \mathcal{A}^A_{\mu\alpha} \overset{+}{W}{}^A_\alpha$)

one can express $w^B(x)$ in terms of M_B and x-coordinates.

Finally one checks that the 1PN eqs of motion $\frac{d^2 z_A^i}{dt^2} = A^i_{A0} + \frac{1}{c^2} A^i_{A2}$ can be derived from the Lagrangian (Lorentz Droste '17; EIH '38)

$$\boxed{\begin{aligned} L(\vec{z}_A, \vec{v}_A) = &\sum_A \frac{1}{2} M_A \vec{v}_A^{\,2} + \frac{1}{2} \sum_{A \neq B} \frac{G M_A M_B}{r_{AB}} + \frac{1}{8c^2} \sum M_A \vec{v}_A^{\,4} \\ &+ \frac{3}{2c^2} \sum_{A \neq B} \frac{G M_A M_B \vec{v}_A^{\,2}}{r_{AB}} - \frac{1}{4c^2} \sum_{A \neq B} \frac{G M_A M_B}{r_{AB}} \left[7 v_A \cdot v_B + (\vec{n}_{AB} \cdot \vec{v}_A)(\vec{n}_{AB} \cdot \vec{v}_B) \right] \\ &- \sum_A \sum_{B \neq A} \sum_{C \neq A} \frac{1}{2c^2} \frac{G^2 M_A M_B M_C}{r_{AB}\, r_{AC}} \end{aligned}}$$