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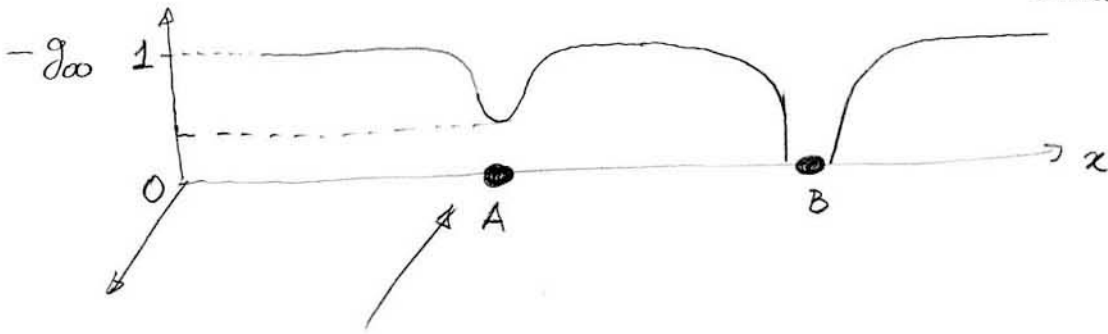
MOTION OF STRONGLY SELF-GRAVITATING
BODIES

Some relevant references

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3.1 Matched Multi-chart approach

Problem with one-chart approach for strong self-gravity bodies



on surface of A : $-g_{00} \approx 1 - \frac{2GM_A}{c^2 R_A}$

$\frac{GM_\odot}{c^2} \approx 1.5 \text{ km}$

for Neutron Star

$$\frac{2GM_A}{c^2 R_A} \approx \frac{2 \times 1.5 \times 1.5 \text{ km}}{10 \text{ km}} \approx 0.45$$

$\nearrow \approx 1.5 M_\odot$
 $\uparrow \approx 10 \text{ km}$

! comparable to 1

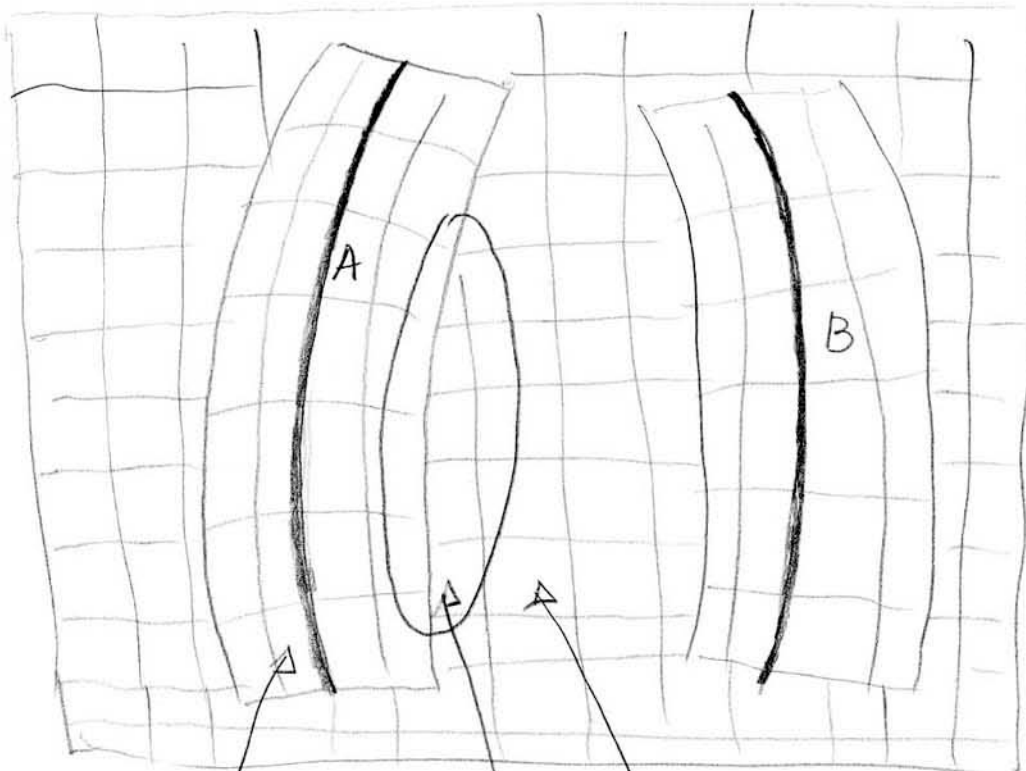
for Black Hole : by definition $\frac{2GM_A}{c^2 R_A} = 1$

Not (a priori) justified to use perturbation theory

$$g_{\mu\nu}(z) = \eta_{\mu\nu} + h_{\mu\nu}, \quad h_{\mu\nu} \ll 1 \quad \text{in one chart } x^\mu$$

⇒ Need to use

- multi-chart
 - multiple expansions
 - relate the various expansions



LOCAL CHART OF BODY A

$$x_A^\alpha$$

perturbation of the metric of an isolated body

$$G_{\alpha\beta}(x_A^\gamma) = G_{\alpha\beta}^{(0)}(x_A^\gamma; m_A) + H_{\alpha\beta}^{(1)}(x_A^\gamma; m_A, m_B) + \dots$$

exact isolated A body

perturbation due to far away bodies + inertial effects

INTERNAL PERTURBATION SCHEME

GLOBAL CHART

$$x^\mu$$

usual (PN or PM)

weak-field expansion

$$g_{\mu\nu}(z) = \eta_{\mu\nu} +$$

$$h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)} + h_{\mu\nu}^{(3)} + \dots$$

EXTERNAL PERTURBATION SCHEME

'MATCHING' OF TWO EXPANSIONS IN INTERMEDIATE DOMAIN

$$\frac{GM_A}{c^2} \lesssim r_A \ll |x-z_A| \ll d$$

3.2 Matching

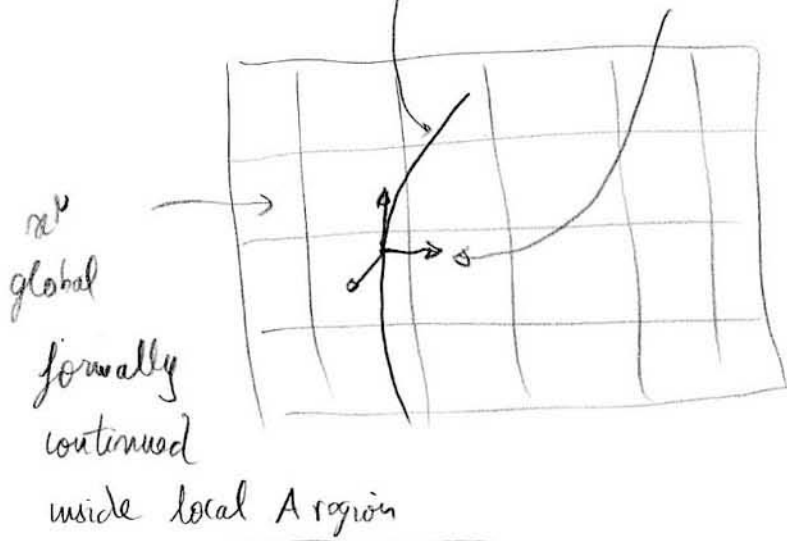
Like in 1PN multi-chart approach, relate the coordinate systems by

$$x^\mu(x_A^\alpha) = z_A^\mu(\tau_A) + e_a^\mu(\tau_A) X_A^a + \frac{1}{2} f_{ab}^\mu(\tau_A) X_A^a X_A^b + \dots$$

defines a worldline in x^μ chart

defines a spatial frame in x^μ chart

Taylor expansion in $(\frac{|\vec{X}_A|}{d})^n$
 d distance to other bodies



MATCHING CONDITION

$$g^{\mu\nu}(x(x_A)) = \frac{\partial x^\mu}{\partial X_A^\alpha} \frac{\partial x^\nu}{\partial X_A^\beta} G^{\alpha\beta}(X_A)$$

external expansion in intermediate domain

expansion $z + eX + \dots$

internal expansion

$$R_A \ll |\vec{x} - \vec{z}_A| \ll d$$

3.3 Linear perturbation scheme

For simplicity, consider case where $A = \text{Block Hole}$ NON-ROTATING

$$G_{\alpha\beta}(\vec{X}, T) = G_{\alpha\beta}^{(0)}(X; m) + H_{\alpha\beta}^{(1)} + \dots$$

\uparrow Schwarzschild \uparrow Schwarzschild mass of isolated body \uparrow linear perturbation of Schwarzschild

Regge-Wheeler-Zerilli formalism

$$H_{\alpha\beta}^{(1)}(\vec{X}, T) = \sum_{l \geq 0} \sum_{m=-l}^{+l} \sum_{I=1}^{10} H_{lm}^{(I, \pi)}(\hat{R}, T) Y_{\alpha\beta}^{lm(I, \pi)}(\theta, \phi)$$

\uparrow radial (+time) functions with $\hat{R} = \frac{R}{(GM/c^2)}$ \uparrow basis of 10 tensorial spherical harmonics (in general gauge, or 6 in special gauge)

PARITY: even or odd

Result of Regge-Wheeler-Zerilli:

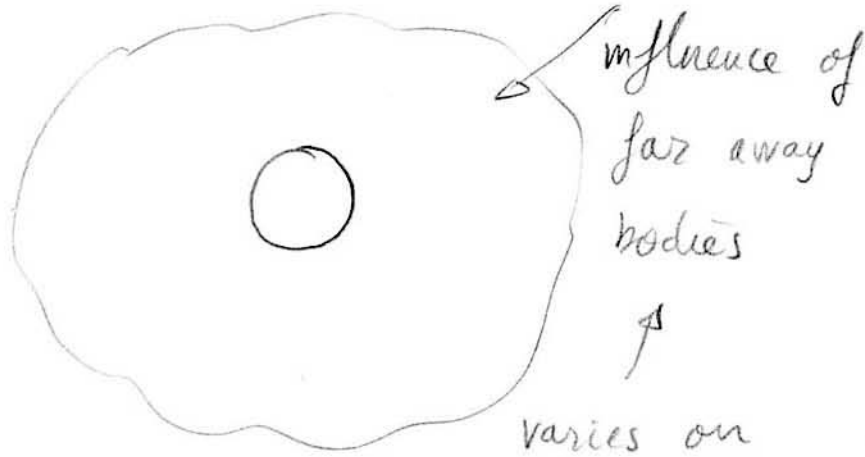
the terms $\{l=0 \text{ or } l=1\}$ can be gauged away

strong-field analog of fixing local frame in DSX formalism by setting to zero the first tidal moments

$$\begin{cases} G(T) = 0 & \leftarrow \text{fixing } T \text{ scale} \\ G_a(T) = 0 & \leftarrow \text{fixing motion of origin of A frame} \\ H_a(T) = 0 & \leftarrow \text{fixing rotation of A frame} \end{cases}$$

Leading perturbation: $l=2$, $\pi = \text{even parity}$

Idea:



SLOW SCALE compared to $\frac{Gm}{c^3}$

Leading effect: neglect $\left[\frac{\partial^2}{\partial T^2} \ll \partial_{x^a} \right]$

\Rightarrow stationary (even) perturbations of Schwarzschild
 Regge-Wheeler '57, Zerilli '70, Edelman Vishveshvara '70
 reduced to a decoupled eq. for one metric component: H

$$\hat{R}(\hat{R}-2) \frac{d^2}{d\hat{R}^2} \left(\frac{H}{\hat{R}(\hat{R}-2)} \right) + 3(2\hat{R}-2) \frac{d}{d\hat{R}} \left(\frac{H}{\hat{R}(\hat{R}-2)} \right)$$

$$- (l-2)(l+3) \frac{H}{\hat{R}(\hat{R}-2)} = 0$$

\uparrow
 simplifies for $l=2$

General solution for $l=2$

$$H_{lm} = D_{lm} \left[\hat{R}(\hat{R}-2) + k_2 \hat{R}(\hat{R}-2) \int_{\hat{R}}^{\infty} \frac{5 dx}{x^3 (x-2)^3} \right]$$

\uparrow $l=2$ \uparrow $D(T)$ \uparrow k

two arbitrary constants

$D(T)$

k

slowly varying

fixed by the condition that the perturbation be regular 'within body A'

determined by going to the intermediate zone

$$1 \ll \hat{R} \ll d$$

$$\hat{R} \equiv \frac{R}{G^m/c^2}$$

in Black Hole case regularity at HORIZON $\hat{R}=2$

$$\Rightarrow \boxed{k_{BH} = 0}$$

for a Neutron Star one will have

$$\boxed{k_{NS} = \text{function of eq of state} = O(1)}$$

$$H_{lm} = D_{lm} \left[\hat{R}^2 - 2\hat{R} + \frac{k}{\hat{R}^3} + O\left(\frac{1}{\hat{R}^4}\right) \right]$$

grows like \hat{R}^2 as $\hat{R} \gg 1$

MATCHING

to $l=2$ tide in metric

w) DSK: $\bar{W} = \frac{1}{2} X^{(ij)} G_{ij}^A(T) + \dots$

$l=2$ external tidal multipole

k depends on the internal structure of body A

3.4 'Effacement of internal structure' AGR 3.7

Matching $\rightarrow \left[\frac{D_{lm}^A}{(GM^A/c^2)^2} \sim G_{\langle ij \rangle}^A \approx \partial_{ij} \bar{U} \sim \sum_{B \neq A} \partial_{ij} \left(\frac{GM_B}{c^2 |\vec{z}_A - \vec{z}_B|} \right) \right]$

\Rightarrow The internal perturbation $H_{\alpha\beta}^{(1)}$ contains two types of contributions

(1) one $\propto D_{lm} \hat{R}(\hat{R}-2)$ which does not depend on the internal structure of A (and which is determined by matching)

(2) another one $\propto D_{lm} k_l \hat{R}(\hat{R}-2) \int_{\hat{R}}^{\infty} \frac{5 dx}{x^3(x-2)^3} \sim D_{lm} k_l \frac{1}{\hat{R}^3}$

which depends on the internal structure of A (BH or NS...)

structure dependent $H_{lm} \hat{R} \gg 1 \approx k_l \frac{D_{lm}}{\hat{R}^3} \sim k_l G_{ij}^A \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \frac{1}{\hat{R}} \times \left(\frac{GM^A}{c^2} \right)^2$

after matching to global coordinate this is because of $\hat{R} \equiv \frac{R}{\left(\frac{GM^A}{c^2} \right)}$

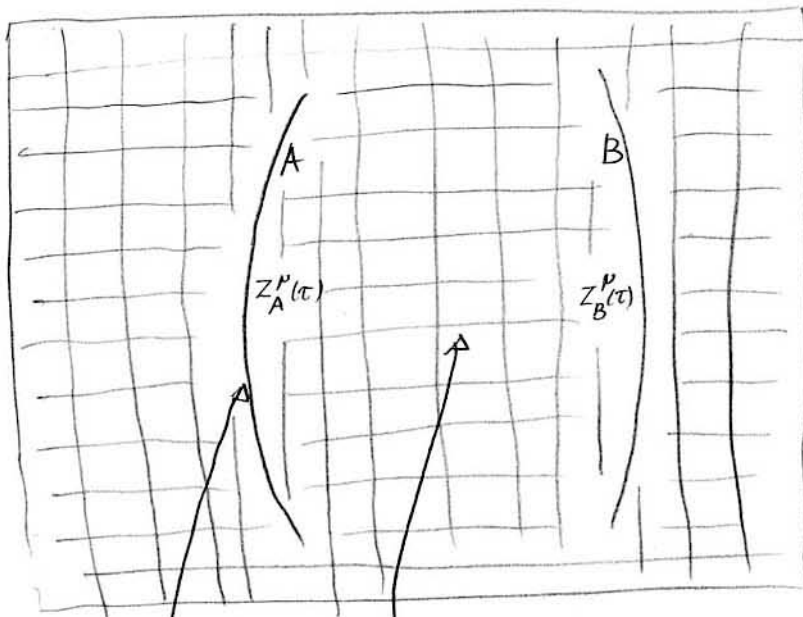
structure dependent $\delta_{\mu\nu} \sim k_2 \partial_{ij} \bar{U} \partial_{ij} \frac{1}{|\vec{r} - \vec{z}_A|} \times \left(\frac{GM^A}{c^2} \right)^5$

EFFACEMENT OF INTERNAL STRUCTURE MODULO $\frac{1}{c^{10}} \sim \left(\frac{v}{c} \right)^{10}$ beyond Newton
5PN

3.5 External perturbation scheme

AGR 3.8

Below 5PN we have



$$R_{\mu\nu}(g) = 0$$

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)} + \dots$$

POST-MINKOWSKIAN EXPANSION in powers of G

near each worldline $|x - z_A| \rightarrow 0$ (ie $|x - z_A| \ll d$) one has some boundary conditions deduced from

$$g^{\mu\nu}(x(X_A)) = \frac{\partial x^\mu}{\partial X_A^\alpha} \frac{\partial x^\nu}{\partial X_A^\beta} G^{\alpha\beta}(X_A)$$

$$= G^{\alpha\beta}_{\text{Schwarzschild}} + \sum_{l \geq 2} O(G_L \hat{x}^L)$$

insert $x^\mu = z_A^\mu(\tau_A) + e_{Aa}^\mu X_A^a + \frac{1}{2} f_{ab}^{\mu\nu} X_A^a X_A^b + \dots$

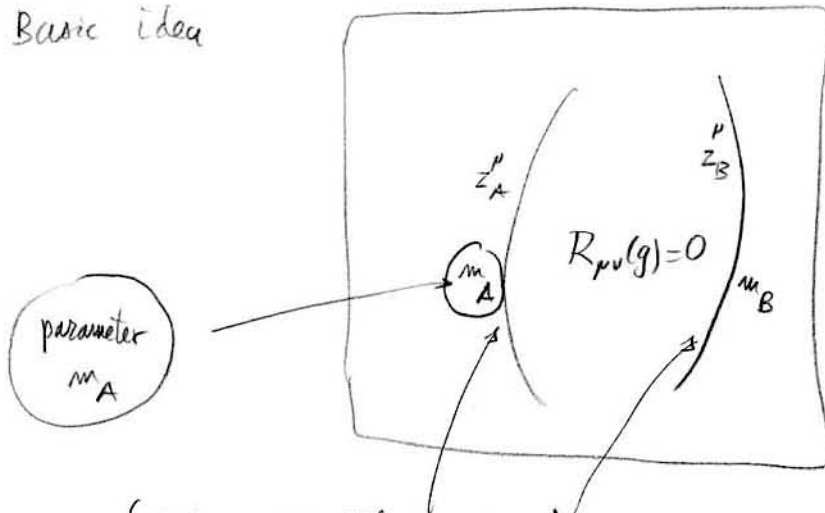
UNIQUELY DETERMINES THE MOST SINGULAR $\sim \frac{m_A^m}{|x - z_A|^m}$ TERMS AT EACH ORDER G^m IN POST-MINKOWSKIAN EXPANSION

3.6

Skeletonization

AGR 3.9

Basic idea



'Dominantly Schwarzschild' boundary conditions near worldlines

⇒ sufficient (at least at 3PM) to prove UNIQUENESS OF
SOLUTION $g_{\mu\nu}(x^\lambda; m_A, m_B, z_A^\mu, z_B^\mu)$

⇒ it is enough to find ONE SPECIFIC METHOD

of finding a solution $g_{\mu\nu}$ satisfying

the required boundary conditions

(including 'no incoming radiation condition')

⇒ construct a solution by using ANALYTIC CONTINUATION
in some complex parameter

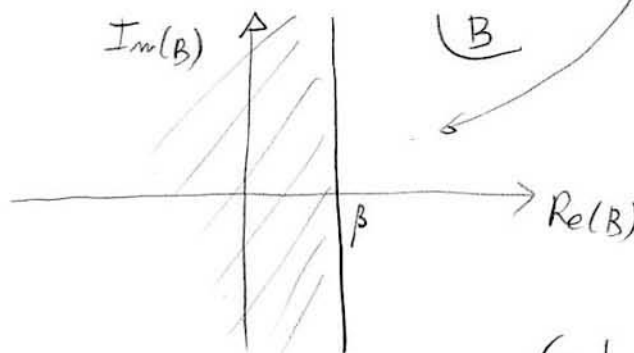
3.7 Analytic continuation

Let $F(z)$ be such that it has at most power-law singularity as $z \rightarrow 0$; i.e. $\exists \beta: z^\beta F(z) = \mathcal{O}(1)$ and decays fast as $z \rightarrow \infty$ $\beta \in \mathbb{R}$ as $z \rightarrow 0$

Consider function

$$B \in \mathbb{C} \longrightarrow I_F(B) \equiv \int_0^{\infty} x^B F(x) dx$$

Complex B plane



integral converges

if

$$\text{Re}(B) > \beta$$

(and possibly $\text{Re}(B) < \gamma$)

$$\frac{d}{dB} I_F(B) = \int_0^{\infty} x^B \log x F(x) dx \text{ also converges}$$

\Rightarrow $I_F(B)$ is an analytic function of $B \in \mathbb{C}$ $\text{Re}(B) > \beta$

Assume: $F(x) = \sum_{\substack{\text{finite} \\ -\alpha_i < \beta}} c_{\alpha_i} x^{\alpha_i} + x^N G_N(x)$

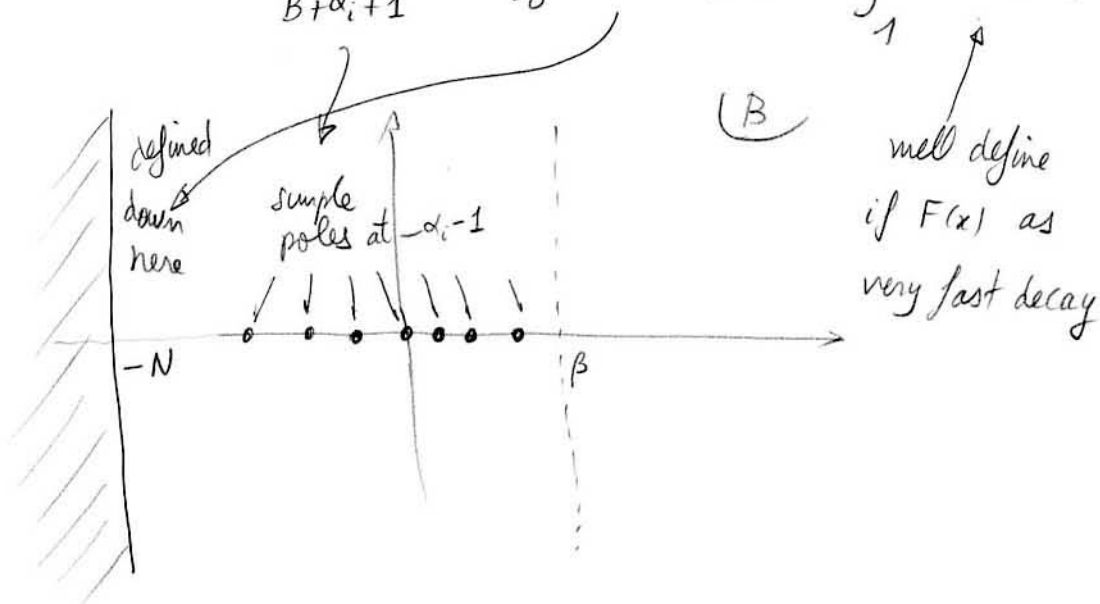
where $G_N(z) = \mathcal{O}(1)$
as $x \rightarrow 0$

for $\text{Re}(B) > \beta$
 $I_F(B) = \int_0^1 dx x^B F(x) + \int_1^\infty dx x^B F(x)$

$$= \int_0^1 dx x^B \left(\sum c_{\alpha_i} x^{\alpha_i} + x^N G_N(x) \right) + \int_1^\infty dx x^B F(x)$$

$$= \sum c_{\alpha_i} \left[\frac{x^{B+\alpha_i+1}}{B+\alpha_i+1} \right]_0^1 + \int_0^1 dx x^{B+N} G_N(x) + \int_1^\infty dx x^B F(x)$$

$$= \sum \frac{c_{\alpha_i}}{B+\alpha_i+1} + \int_0^1 dx x^{B+N} G_N(x) + \int_1^\infty dx x^B F(x)$$



Thm: UNIQUENESS of such an analytic continuation

\Rightarrow Uniquely defines $I_F(B)$ as a meromorphic function (with simple poles at $-\alpha_i - 1$) all over \mathbb{C}

3.8 Useful properties of analytic continuation

$$\bullet \left[\text{AC} \left[\int_0^{\infty} dx x^B (F(x) + G(x)) \right] = \text{AC} \left[\int_0^{\infty} dx x^B F(x) \right] + \text{AC} \left[\int_0^{\infty} dx x^B G(x) \right] \right]$$

• Integrating by parts

in original domain

$$\int_0^{\infty} dx x^B F(x) dx = \left[\frac{x^{B+1}}{B+1} F(x) \right]_0^{\infty} - \frac{1}{B+1} \int_0^{\infty} dx x^{B+1} F'(x) dx$$

↑
vanishes in original domain

⇒ vanishes always! (even when divergent!)

$$\Rightarrow \int_0^{\infty} dx x^B F(x) dx = - \frac{1}{B+1} \int_0^{\infty} dx x^{B+1} F'(x) dx$$

or

$$\int_0^{\infty} dx x^B F'(x) = -B \int_0^{\infty} dx x^{B-1} F(x)$$

more generally: continuation of any (analytic) identities valid for B in some domain.

(3.9) Dimensional regularization (t'Hooft Veltman '72) AGR 3.13

i.e. analytic continuation in the dimension of space d

Consider

$$S = \int \frac{d^{d+1}x}{c} \sqrt{g} \frac{c^4}{16\pi G} R(g) - \sum_A m_A c \int \sqrt{-g_{\mu\nu}(z_A^\lambda)} dz_A^\mu dz_A^\nu$$

or

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} \sum_A T_{\mu\nu}^A(x)$$

POINT MASSES

$$T_{\mu\nu}^A(x) = \int m_A c^2 \frac{ds_A}{\sqrt{g(z_A)}} \frac{dz_A^\mu}{ds_A} \frac{dz_A^\nu}{ds_A} \delta^{d+1}(x^\lambda - z_A^\lambda(s_A))$$

$$ds_A = \sqrt{-g_{\mu\nu}(z_A^\lambda)} dz_A^\mu dz_A^\nu$$

or

$$R_{\mu\nu} = \frac{8\pi G}{c^4} \sum_A \left(T_{\mu\nu}^A(x) - \frac{1}{d-1} g_{\mu\nu} T^A(x) \right)$$

d -independent
in harmonic gauge

$$g^{\mu\nu} \Gamma_{\mu\nu}^\lambda = 0$$

introduces some
explicit d -dependence
in solving Einstein's eqs

$$2 R_{\mu\nu}^{\text{harmonic}} = -g^{\alpha\beta} \partial_{\alpha\beta} g_{\mu\nu} + g'' g'' (\partial g \partial g)$$

3.10 Relativistic gravitational interaction of two point masses AGR 3.14

$$m_A \left(g_{\mu\nu}(z^\lambda) \right) m_B$$

$z_A^\mu \qquad z_B^\nu$

explicit form of Einstein's eqs in harmonic coords

$$-g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} + g^{\alpha\beta} g^{\gamma\delta} (\partial_\alpha g_{\mu\gamma} \partial_\delta g_{\nu\beta} - \partial_\alpha g_{\nu\gamma} \partial_\delta g_{\mu\beta} + \partial_\alpha g_{\mu\delta} \partial_\beta g_{\nu\gamma} - \partial_\alpha g_{\nu\delta} \partial_\beta g_{\mu\gamma} - \frac{1}{2} \partial_\alpha g_{\mu\nu} \partial^\alpha g^{\mu\nu})$$

$$= \frac{16\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{d-1} g_{\mu\nu} T^\lambda{}_\lambda \right)$$

$$T_{\mu\nu}(z) = \sum_A m_A c^2 \int ds_A \frac{g_{\mu\alpha}(z_A) g_{\nu\beta}(z_A) \frac{dz_A^\alpha}{ds_A} \frac{dz_A^\beta}{ds_A} \delta^{(d+1)}(z^\lambda - z_A^\lambda(z_A))}{\sqrt{g(z_A)}}$$

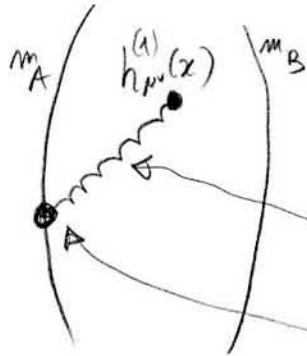
$$g_{\mu\nu}(z^\lambda) = \eta_{\mu\nu} + h_{\mu\nu}^{(1)}(z) + h_{\mu\nu}^{(2)}(z) + \dots$$

$$\square h_{\mu\nu}^{(1)}(z) = -\frac{16\pi G}{c^4} \left(T_{\mu\nu}^{(1)} - \frac{1}{d-1} \eta_{\mu\nu} T^{(1)} \right)$$

$$T_{\mu\nu}^{(1)} = \sum_A m_A c^2 \eta_{\mu\alpha} \eta_{\nu\beta} \int ds_A^{(0)} \frac{dz_A^\alpha}{ds_A^{(0)}} \frac{dz_A^\beta}{ds_A^{(0)}} \delta^{(d+1)}(z - z_A)$$

$$ds_A^{(0)} = \sqrt{-\eta_{\mu\nu} dz_A^\mu dz_A^\nu}$$

3.11 Diagrams



Inverse of \square , i.e. Green function $G_0(x,y)$

$\square G_0(x-y) = \delta^{(d+1)}(x-y)$

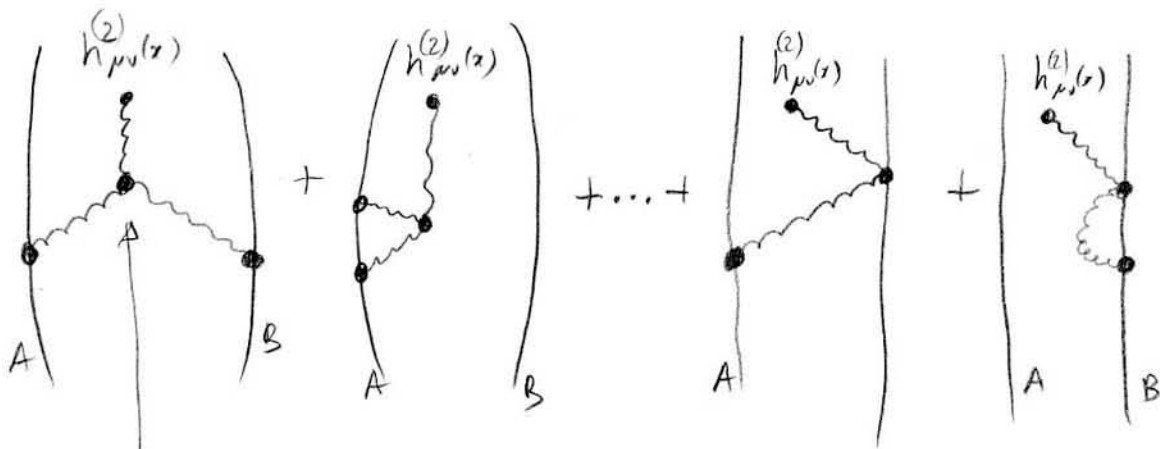
$$h^{(1)}_{\mu\nu}(x) = -\frac{16\pi G}{c^4} \left(\eta_{\mu\alpha} \eta_{\nu\beta} - \frac{1}{d-1} \eta_{\mu\nu} \eta_{\alpha\beta} \right) \square^{-1} T_A^{\alpha\beta}$$

PROPAGATOR

SOURCE

Next approximation

$$\square h^{(2)} \sim h^{(1)} \partial^2 h^{(1)} + \partial h^{(1)} \partial h^{(1)} + h^{(1)} T^{(1)}$$



'CUBIC VERTEX'

i.e. non linearities

$$\sim h \partial^2 h + \partial h \partial h$$

in RHS

3.12 Reduced ('Fokker') action for interacting particles

$$S[z_A^\lambda; g_{\mu\nu}(x)] = -\sum_A m_A c \int \sqrt{-g_{\mu\nu} dz_A^\mu dz_A^\nu} + \int \frac{d^{d+1}}{c} \sqrt{g} \frac{c^4}{16\pi G} R(g)$$

+ surface term
+ gauge-fixing term

replace $g_{\mu\nu}(x)$ by gauge-fixed solution $\rightarrow \propto (\text{gauge condition})^2$

$$g_{\mu\nu}[x; z_A^\lambda] = \eta_{\mu\nu} + h_{\mu\nu}^{(1)}[z_A^\lambda] + \dots$$

\Rightarrow Reduced (Fokker) action for particles $\left(\begin{array}{l} \text{with, say,} \\ G_{\text{sym}} \equiv \\ \frac{1}{2}(G_{\text{ret}} + G_{\text{adv}}) \end{array} \right)$

$$S^{\text{reduced}}[z_A^\lambda] = S[z_A^\lambda; g_{\mu\nu}(x; z_B^\lambda)]$$

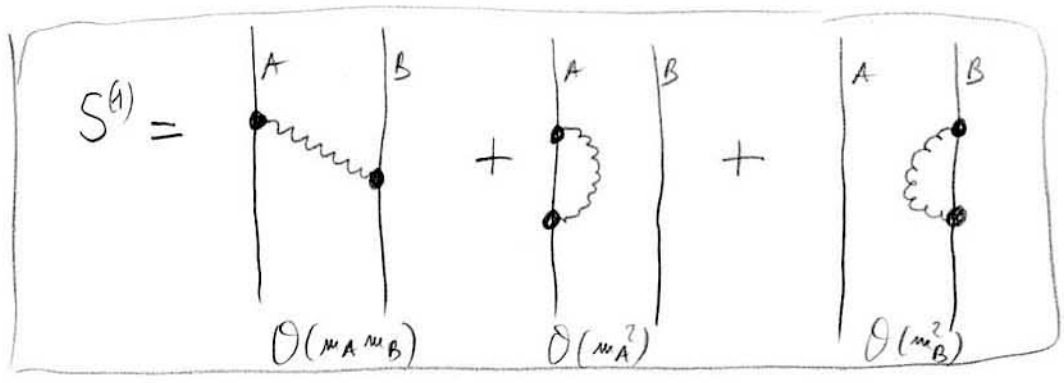
$$= S_{\text{FREE}}^{(0)}[z_A^\lambda] + S^{(1)}[z_A] + \dots$$

$$S^{(0)} = -\sum_A m_A c \int \sqrt{-\eta_{\mu\nu} dz_A^\mu dz_A^\nu} = -\int dt \sum_A m_A c^2 \sqrt{1 - \frac{\dot{\mathbf{x}}_A^2}{c^2}}$$

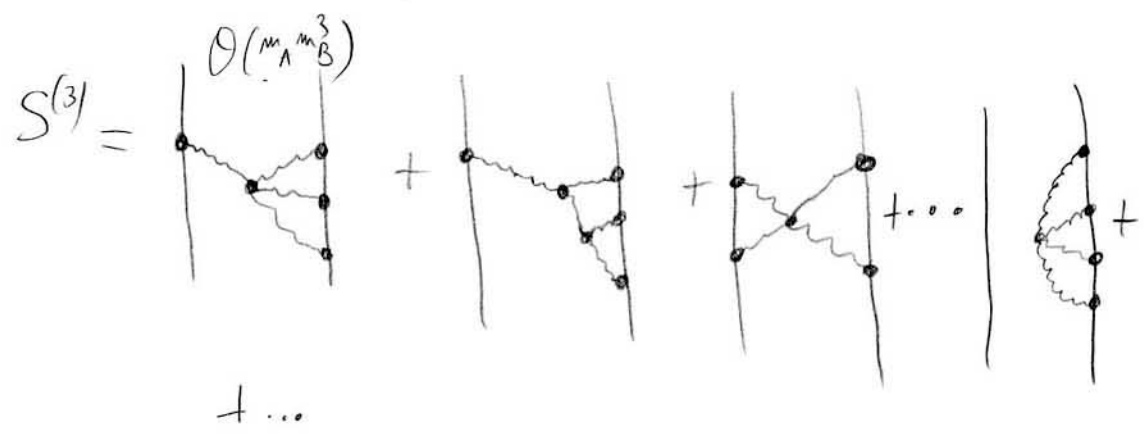
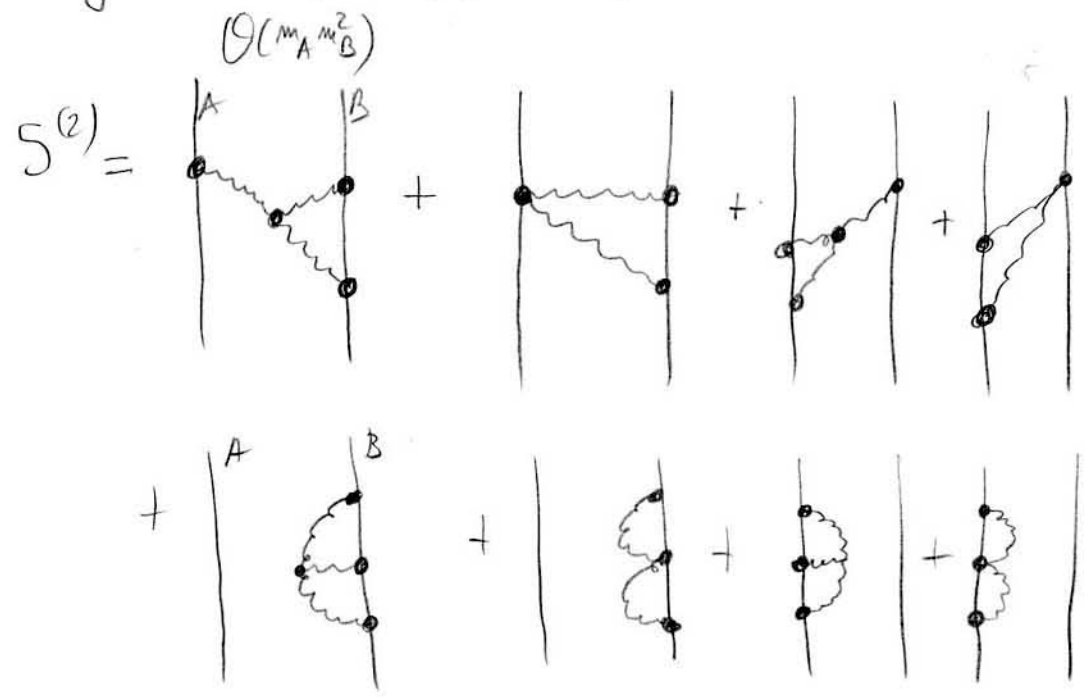
$$S^{(1)} = \frac{1}{4} \int \frac{d^{d+1}x}{c} h_{\mu\nu}^{(1)}(z) T^{\mu\nu}(z) = -4\pi G \int \frac{d^d x}{c} \frac{d^d y}{c} \Gamma^{\mu\nu}(z) \left(\frac{\eta_{\mu\alpha} \eta_{\nu\beta} - \frac{1}{d-1} \eta_{\mu\nu} \eta_{\alpha\beta}}{\square} \right) T^{\alpha\beta}(y)$$

↑
↑
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 FIRST-ORDER INTERACTION SOURCE PROPAGATOR SOURCE

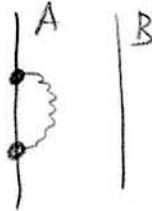
AGR3.17



higher-order interactions



3.13 Self-gravity effects

Meaning of  ?

Consider lowest (Newtonian) approximation

$$T^{00}(x) = \rho c^2 \text{ with } \rho = \sum_A m_A \delta^{(d)}(\vec{x} - \vec{z}_A)$$

$$\text{with } \square = \Delta - \frac{1}{c^2} \partial_t^2 \approx \Delta ; \square^{-1} \approx \Delta^{-1}$$

Define (Newtonian potential) $\Delta U = -4\pi G \frac{2(d-2)}{d-1} \rho$

$$S_{\text{Newton}}^{(1)} = \frac{1}{2} \rho \Delta^{-1} \rho = \int dt \left[\frac{1}{2} \int d^d x \rho(x) U(x) \right]$$

$$U(x) = \sum_A \tilde{k} \frac{G m_A}{|\vec{x} - \vec{z}_A|^{d-2}} \quad \tilde{k} = \frac{2(d-2)}{d-1} \tilde{k} \rightarrow \frac{1}{d=3}$$

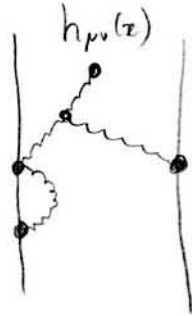
$$S_{\text{Newton}}^{(1)} = \int dt \frac{1}{2} \tilde{k}^2 \sum_A \sum_B \frac{G m_A m_B}{|\vec{z}_A - \vec{z}_B|^{d-2}}$$

SELF-GRAVITY TERM

$$G m_A^2 |\vec{z}_A - \vec{z}_A|^{2-d}$$

in $d=3$ infamous $\left[\frac{G m_A^2}{r} \right]_{r=0} = \infty$
 Analytic continuation in d $\left[\text{AC} \left[\frac{r^{2-d}}{d} \right]_{r=0} = 0 \right]$ because $\frac{1}{d} = 0$ for $d < 2$

More generally one checks that other self-gravity terms such as



are well-defined when $d \xrightarrow{AC} 3$
 of ^m spite
 of dangerous

'loop diagram' :

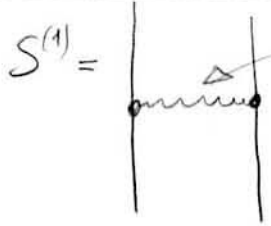


- At the 2PN approximation, it has been shown (Damour 80,83) that the analytic continuation $d \xrightarrow{AC} 3$ gives a finite result
- At the 3PN approximation, if one uses ADM coordinates, it has been shown that the Hamiltonian is finite as $d \xrightarrow{AC} 3$ (Damour Jaramowski Schäfer 2001)
- At the 3PN approximation, and in harmonic coordinates, it has been shown that the AC $d \xrightarrow{AC} 3$ generates some poles $\propto \frac{1}{d-3}$, which can, however, be absorbed in a shift $\delta z_A^i \propto \frac{G^2 m_A^2}{c^6 (d-3)} \ddot{z}_A^i$ of the 'bare worldlines' (Blanchet Damour Esposito-Farèse 2004)

Crucial point: the AC $d \xrightarrow{AC} 3$ defines, in $d=3$, a finite solution of $R_{\mu\nu}(g)=0$ in $d=3$, which satisfies the 'Dominant Schwarzschild boundary conditions' deduced from MATCHING. The uniqueness of this solution ensures that the formal AC generates the physically relevant metric outside N strongly self-gravitating bodies

3.14 Explicit results available for the dynamics AGR 3.20
of 2 condensed bodies

"one-graviton exchange"

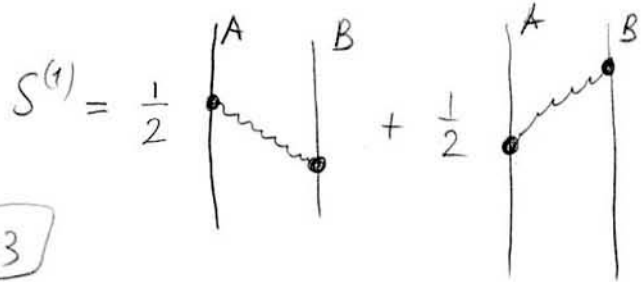


NB: physically relevant retarded propagator, but must use

$$\text{time-symmetric} = \frac{1}{2} (\text{adv} + \text{ret})$$

for deriving conservative dynamics

⇒ NON-LOCAL (in TIME) ACTION



symmetric scalar propagator

$w|d=3$

$$S^{(1)} \propto \iint ds_A ds_B \frac{dz_A^\mu}{ds_A} \frac{dz_A^\nu}{ds_A} \left(\eta_{\mu\alpha} \eta_{\nu\beta} - \frac{1}{2} \eta_{\mu\nu} \eta_{\alpha\beta} \right) \frac{dz_B^\alpha}{ds_B} \frac{dz_B^\beta}{ds_B} \delta \left(\left(z_A^\mu(s_A) - z_B^\mu(s_B) \right)^2 \right)$$

When using coordinate times specific (spin 2) velocity dependence

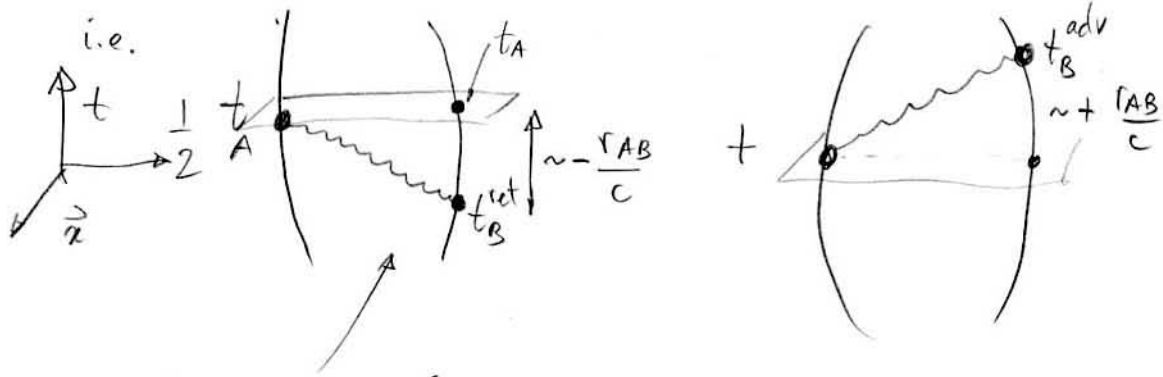
$$\iint dt_A dt_B \dots \delta \left((t_A - t_B)^2 - \frac{1}{c^2} \left(\vec{z}_A(t_A) - \vec{z}_B(t_B) \right)^2 \right)$$

$$\downarrow$$

$$\frac{1}{2} \frac{\delta \left((t_A - t_B) - \frac{1}{c} \left| \vec{z}_A(t_A) - \vec{z}_B(t_B) \right| \right)}{\left(1 - \vec{m}_{AB} \cdot \frac{\vec{v}_{AB}}{c} \right) r_{AB}} + \frac{1}{2} \frac{\delta \left((t_A - t_B) + \frac{r_{AB}}{c} \right)}{\left(1 - \vec{m} \cdot \frac{\vec{v}}{c} \right) r_{AB}}$$

AGR 3.21

To have an ordinary (time-local) action
 one needs to expand in retardation $\sim \frac{r_{AB}}{c}$



essentially a Taylor (actually Lagrange) expansion

→ introduces higher time derivatives of $\vec{z}_B(t)$: $\ddot{\vec{z}}_B, \dddot{\vec{z}}_B, \dots$

At 1PN, one can integrate by parts to eliminate $\ddot{\vec{z}}_B$

$$\Rightarrow \left(\text{wavy line} \right) = \int dt \frac{1}{2} \sum_{A+B} \frac{G m_A m_B}{r_{AB}} \left[\begin{array}{l} \text{specific of spin 2 propagator} \\ 1 + \frac{3}{2c^2} (\vec{v}_A^2 + \vec{v}_B^2) - \frac{7}{2c^2} (\vec{v}_A \cdot \vec{v}_B) \\ - \frac{1}{2c^2} (\vec{m}_{AB} \cdot \vec{v}_A) (\vec{m}_{AB} \cdot \vec{v}_B) + O(\frac{1}{c^4}) \end{array} \right]$$

↑ come from retardation

same as $\int_{2\text{-body}}^{LD} [\ddot{\vec{z}}_A(t) - \ddot{\vec{z}}_B(t), \dot{\vec{z}}_A(t), \dot{\vec{z}}_B(t)]$

by effacement:
 same as weakly
 self-gravitating case

At 2PN, i.e. $\mathcal{O}(\frac{v^4}{c^4})$ beyond Newton

one cannot get rid of higher-derivatives \ddot{z}

$$\Rightarrow L_{2\text{body}} [\vec{z}_A(t) - \vec{z}_B(t), \dot{\vec{z}}_A(t), \dot{\vec{z}}_B(t), \ddot{\vec{z}}_A(t), \ddot{\vec{z}}_B(t)]$$

In principle, even higher derivatives at 3PN, $\frac{v^6}{c^6}$, etc...

However, by suitable use of some tricks ('double-zero', ⁺integ. by parts) one can arrange to have

$$L [z_A - z_B, \dot{z}_A, \dot{z}_B; \ddot{z}_A, \ddot{z}_B]$$

LINEAR IN ACCELERATIONS
(see Damour Schäfer 1985 + ...)

n -graviton exchange with $n \geq 2$

e.g. $\mathcal{O}(m^3)$

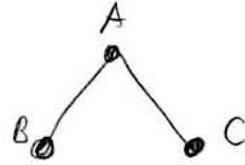
must expand $\frac{1}{\square_{\text{sym}}} = \frac{1}{\Delta - \frac{1}{c^2} \partial_t^2} = \frac{1}{\Delta (1 - \frac{1}{c^2} \partial_t^2 \Delta^{-1})} = \Delta^{-1} + \frac{1}{c^2} \partial_t^2 \Delta^{-2} + \dots$

1PN 2PN 3PN

eg. 1PN

$$L_{3\text{-body}} = -\frac{1}{2} \sum_{B \neq A \neq C} \frac{G^2 m_A m_B m_C}{c^2 r_{AB} r_{AC}}$$

ie. 3 masses

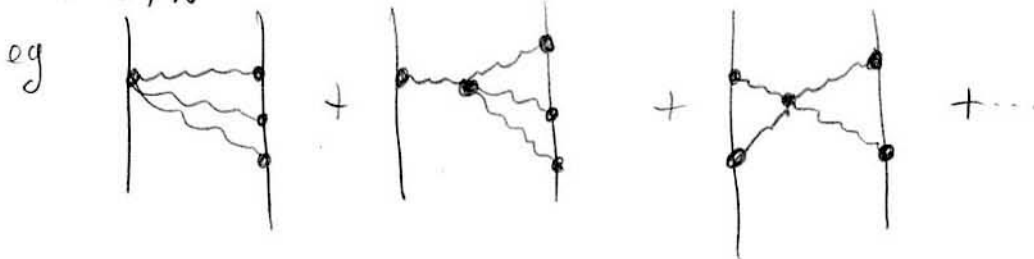


$$\sim \frac{m_A m_B^2}{r_{AB}^2} + \frac{m_B m_A^2}{r_{BA}^2}$$



when $B=C$
i.e.
2 bodies
only

For computing the integrals that appear at 2PN and 3PN



one needs to use:

- ① eliminate '1PN' cubic vertex $\sim h_{00} (\vec{\nabla} h_{00})^2$ by $(h_{00} - g_{00})$
- ② Riesz formula for

$$I(\alpha, \beta) = \int \frac{d^3x}{4\pi} |\vec{x} - \vec{z}_A|^{-\alpha} |\vec{x} - \vec{z}_B|^{-\beta} = (\text{product } \Gamma \text{ functions}) \times |\vec{z}_A - \vec{z}_B|^{-\alpha - \beta - 3}$$

Dynamics of two point masses in harmonic coordinates

$\left\{ \begin{array}{l} 2 \text{ PN level} : (v/c)^4 \\ 2.5 \text{ PN level} : (v/c)^5 \end{array} \right.$
Damour, Deruelle 81
Damour 82, 83
 actually obtained from EQS OF MOTION

$3 \text{ PN level} : (v/c)^6$
Blanchet-Faye '00
Blanchet Damour Esposito-Farese '04
Itoh, Futamase '03 '04

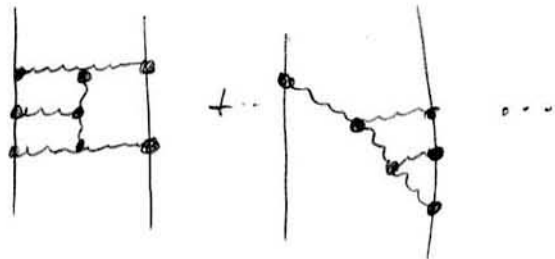
$3.5 \text{ PN} \quad (v/c)^7$
Königsdörffer, Faye, Schäfer '03
Pati Will '02 Nisanke Blanchet '05

Dynamics of two point masses in Arnowitt-Deser-Misner coords

$2 \text{ PN} \quad (v/c)^4$
Schäfer 195 '86
 $2.5 \text{ PN} \quad (v/c)^5$
[and Ohta, Okamura, ... 70's]

$3 \text{ PN} \quad (v/c)^6$
Jaramowski Schäfer 198 -
Damour Jaramowski Schäfer 101

up to '3 loop diagrams'

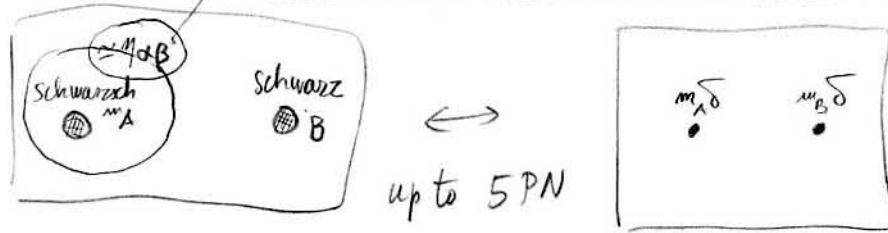


NB: $AC d \rightarrow 3 \Rightarrow$ finite results

Possible violation of 'effacement' outside GR

in GR

ONLY $\approx \eta_{\alpha\beta}$ in INTERMEDIATE DOMAIN



[to which one must add SPIN EFFECTS for spinning bodies]

3.15 Tensor-scalar gravity

$$S = \frac{c^4}{16\pi G_*} \int \frac{d^4x}{c} \tilde{g}^{1/2} [F(\Phi) \tilde{R} - Z(\Phi) \tilde{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - U(\Phi)]$$

$$+ S_m[\psi_m; \tilde{g}_{\mu\nu}]$$

\nearrow \uparrow
 'MATTER' 'PHYSICAL METRIC'

but redefinitions: $\Phi \rightarrow \varphi = f(\Phi)$

$$\tilde{g}_{\mu\nu} \rightarrow g_{\mu\nu}^* = F(\Phi) \tilde{g}_{\mu\nu}$$

$$S = \frac{c^4}{16\pi G_*} \int \frac{d^4x}{c} g_*^{1/2} [R_* - 2g_*^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi)] + S_m[\psi_m; e^{2a(\varphi)} g_{\mu\nu}^*]$$

('Einstein frame') POTENTIAL COUPLING FUNCTION

$$R^*_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi + 8\pi G_* (T^*_{\mu\nu} - \frac{1}{2}T^*g^*_{\mu\nu})$$

$$\square_{g_*}\phi = -4\pi G_* \alpha(\phi) T^*_{\mu\nu} - \frac{1}{4}\frac{\partial V(\phi)}{\partial\phi}$$

ASSUMED HERE TO BE SMALL AND NEGLIGIBLE
say $m_\phi^2 \sim H_0^2$

crucial

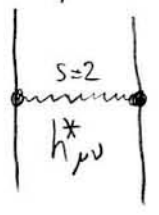
$$\alpha(\phi) \equiv \frac{\partial a(\phi)}{\partial\phi}$$

and

$$\beta(\phi) \equiv \frac{\partial \alpha(\phi)}{\partial\phi} \equiv \frac{\partial^2 a(\phi)}{\partial\phi^2}$$

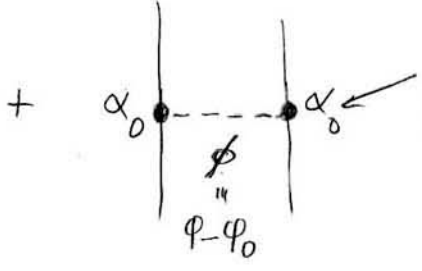
DIAGRAMS

$$g^*_{\mu\nu} = \eta_{\mu\nu} + h^*_{\mu\nu}$$



BACKGROUND VALUE OF ϕ

$$\phi = \phi_0 + \phi$$



$$\alpha_0 \equiv \alpha(\phi_0)$$

Newtonian level

$$\frac{G_* m_A m_B}{r_{AB}} + \frac{G_* (\alpha_0 m_A) (\alpha_0 m_B)}{r_{AB}}$$

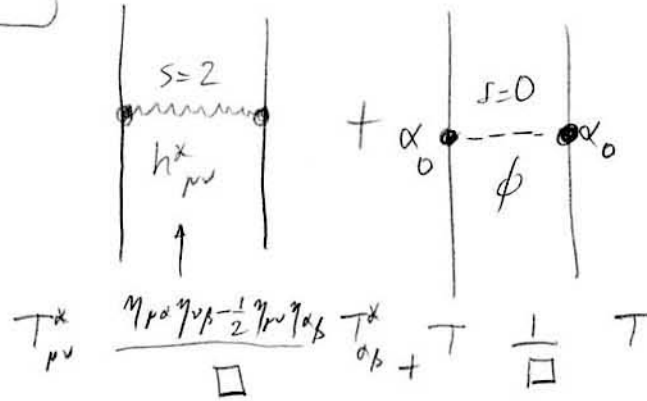
$$\Rightarrow G = G_* (1 + \alpha_0^2)$$

Cavendish
in Einstein units

$$\tilde{G} = G_* e^{2a(\phi_0)} (1 + \alpha_0^2)$$

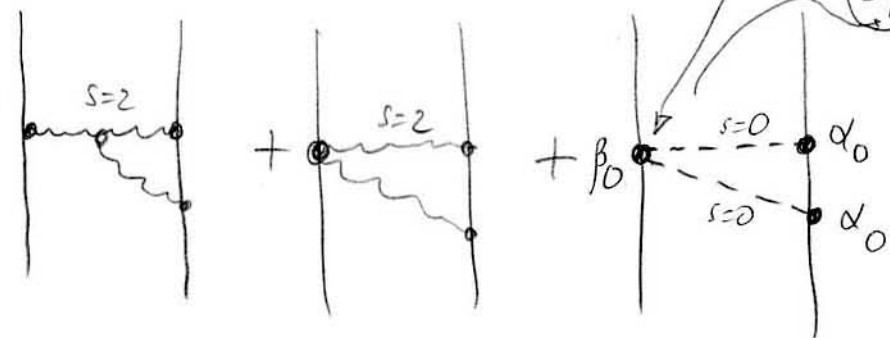
in physical units

1PN level



$$\bar{\gamma} \equiv \gamma^{PPN} - 1 = -2 \frac{\alpha_0^2}{1 + \alpha_0^2}$$

$$a(\phi_0 + \phi) = a_0 + \alpha_0 \phi + \frac{1}{2} \beta_0 \phi^2$$

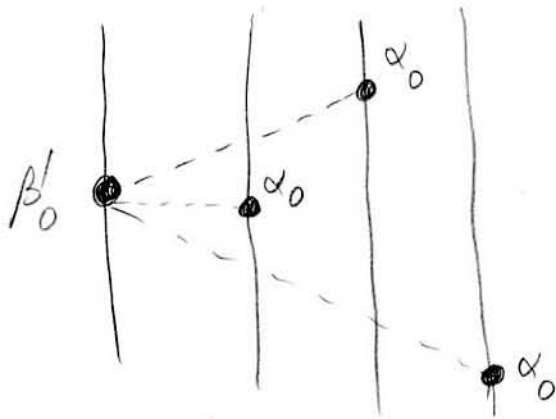


$$\bar{\beta} \equiv \beta^{PPN} - 1 = \frac{1}{2} \frac{\alpha_0 \beta_0 \alpha_0}{(1 + \alpha_0^2)^2}$$

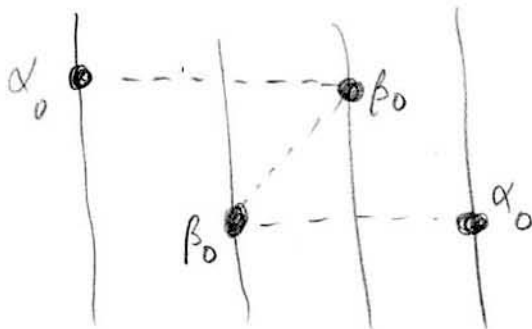
1PN 'POST-EINSTEIN' parameters

3.16 PARAMETRIZED 2PN POST-EINSTEIN

At 2PN need: $a(\phi) = \alpha_0 \phi + \frac{1}{2} \beta_0 \phi^2 + \frac{1}{6} \beta'_0 \phi^3$



$$\epsilon \equiv \frac{\beta'_0 \alpha_0^3}{(1 + \alpha_0^2)^3}$$



$$\zeta \equiv \frac{\alpha_0 \beta_0 \beta'_0 \alpha_0}{(1 + \alpha_0^2)^3}$$

DIAGRAMMATIC PROOF THAT

→ ONLY TWO 2PN POST-EINSTEIN PARAMETERS ϵ, ζ in tensor-scalar gravity

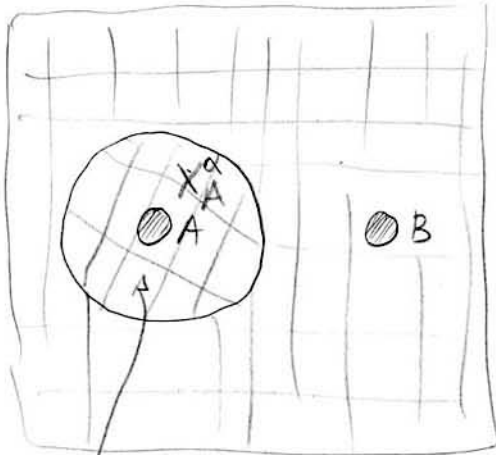
(Giancarlo Esposito-Farisei 196)

- Unmeasurable in solar system, even with parsec light deflection
- Accessible through self-gravity effects in motion and radiation of binary pulsars

(3.17) Violation of 'effacement' in tensor-scalar gravity

(Eardley '75, Will Eardley '77)

MODIFICATION OF (INTERNAL PROBLEM)



NEED BOUNDARY CONDITIONS
 ('FAR AWAY') $R_A \ll |\vec{X}_A| \ll d$
 OF A BOTH FOR

$$G_{\alpha\beta}^A(X) \approx \eta_{\alpha\beta}$$

AND

$$\Phi(X) \approx \Phi_{\text{asymptot } A}$$

$$\begin{cases} R_{\alpha\beta}^* = 2\partial_\alpha\partial_\beta\Phi + 8\pi G_* (T_{\alpha\beta}^* - \frac{1}{2}T^* G_{\alpha\beta}^*) \\ \square_{g_*} \Phi = -4\pi G_* \alpha(\Phi) T^* \end{cases}$$

$\Phi_{\text{as } A}$ changes the solution, within body A

obtained by matching
 $\Phi_{\text{as } A} = \Phi_0 + \text{contribution due to B at distance } r_{AB}$

Leading internal solution when $R_A \ll |\vec{X}_A| \ll d$

$$G_{\alpha\beta}^*(X) \approx \eta_{\alpha\beta} + \frac{2G_* m_A^*(\Phi_{\text{as } A})}{|\vec{X}|} \delta_{\alpha\beta}$$

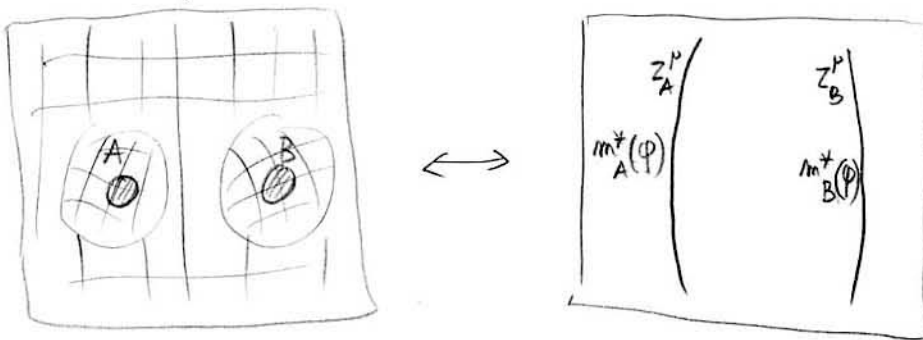
$$\Phi \approx \Phi_{\text{as } A} - G_* \alpha_A(\Phi_{\text{as } A}) \frac{m_A(\Phi_{\text{as } A})}{|\vec{X}|}$$

where (Damon, Esposito, Faraoni '92)

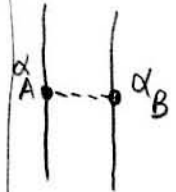
$$\alpha_A(\varphi_{as}) = \frac{\partial \ln m_A^*(\varphi_{as})}{\partial \varphi_{as}}$$

crucial 'master function', $m_A^* = m_A^*(\varphi_{as})$

MODIFIED SKELETONIZATION OF COMPACT BODIES



$$S^{skeleton} = \frac{c^4}{16\pi G_*} \int \frac{c^{d+1} x}{c} g_*^{1/2} [R_* - 2 g_*^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi] - \sum_A c \int m_A^*(\varphi(z_A)) \sqrt{-g_{\mu\nu}^*(z_A) dz_A^\mu dz_A^\nu}$$



• → strong self-gravity of neutron stars can introduce significant violation of 'effacement' → significant deviations for GR predictions
 $\propto \alpha_A \alpha_B$ etc...

• On the other hand, for Black Holes: 'no scalar hair' → $m_A^* = \text{const}$
 $\alpha_A = 0$
 e.g. 2 BHs in tensor-scalars \equiv 2 BHs in GR!