GR-IHP 06
N. Deruelle
- PART 1 -

Space, Time & Gravity from Newton to Einstein

(I) Newtonian Space & Time

1. Absolute space and Universal time

The intuitive notions of space & location are "skeletonized" by a mathematical ensemble of points : $\mathcal{E}_3$, the "absolute space". Each point is characterized by 3 numbers (its "coordinates").

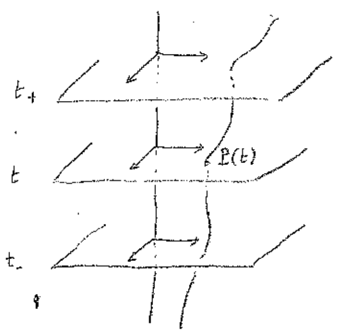
$\mathcal{E}_3$ is euclidean, that is ; among all possible labellings of points, $\exists$ some, X^α, such that the distance between 2 points is defined by means of Pythagoras theorem :

$$\left(\begin{matrix} & X^\alpha + dX^\alpha \\ dl \nearrow & \\ X^\alpha & \end{matrix} \right) \qquad dl^2 = dX^2 + dY^2 + dZ^2 = \sum_{\alpha,\beta} \delta_{\alpha\beta}\, dX^\alpha dX^\beta \equiv \delta_{\alpha\beta}\, dX^\alpha dX^\beta$$

The origin $(0,0,0)$ & the 3 axes (X, Y, Z) form a cartesian frame S.

The intuitive notion of time is represented by a real number : "universal time", t

Hence Newton's space-time is a "foliation", that is a succession of copies of $\mathcal{E}_3$ labelled by t. Each "fiber" represents a point in a state of "absolute rest".



$N_4 = \mathcal{E}_3 \times R$

The family of cartesian frames labelled by t forms an "absolute frame"

The motion of a point-like material object is represented by a "world-line" $\ell(t)$ in N_4

A cartesian frame is materialized by means of "rigid" objects (whose relative distances remain constant in time) using Pythagoras theorem (& consequences). Absolute time is materialized by repetitive phenomena : "clocks". A "good frame" and a "good clock" are ultimately defined by the fact that, at the precision available, euclidean properties of figures & laws of Newtonian dynamics are verified. If not, the frame & clock, a priori, are not good enough. But if mismatches persist, then Newtonian representations of space & time must be questioned (see letter of Einstein to Solovine).

An absolute reference frame must be at rest (so that fibers of N_4 can be identified with its elements). For Newton : solar system together with distant stars. Today : CMB ... Since the Universe appears mostly devoid of matter, most "fibers" are only "virtually materialized", hence the invention of "Ether".

Direct consequence of Newtonian representation of space & time : 2 travellers start somewhere & meet again after 2 $\neq$ journeys : the durations of their trips must be the same ...

2. Moving frames

Changing the labellings of points of $\mathcal{E}_3$ is postulated not to change the numerical value of their distances. Among relabellings some do not change either the form of the line element : rigid displacements, ie linear (time-dependent) 6 parameter group of rotations & translations :

$$X^\alpha \to X'^\alpha = R_\beta{}^\alpha(t)\,(X^\beta - d^\beta(t)) \quad \text{with } R_\mu{}^\alpha R_\nu{}^\beta \delta_{\alpha\beta} = \delta_{\mu\nu} \quad (\det R = +1)$$

$$\Rightarrow \; dl^2 = \delta_{\alpha\beta}\, dX^\alpha dX^\beta = \delta_{\alpha\beta}\, dX'^\alpha dX'^\beta$$

The $\exists$ of rigid displacements means that the choice of origin & orientation of axes is irrelevant, "Active version" : properties of figures do not depend on location ; the Universe is "neutral" that is "homogeneous & isotropic"

Kinematics: if $X^\alpha = X^\alpha(t)$ is the trajectory of P in (S); $V^\alpha = \frac{dX^\alpha}{dt}$ its velocity; then in (S'):

$$V'^\alpha \equiv \frac{dX'^\alpha}{dt} = R_\beta{}^\alpha V^\beta + \dot{R}_\beta{}^\alpha X^\beta - (R_\beta{}^\alpha d^\beta)^\cdot \qquad (\cdot \equiv \tfrac{d}{dt})$$

$$= R_\beta{}^\alpha (V^\beta - \dot{d}^\beta) - e^\alpha{}_{\beta\gamma}\, \omega'^\beta X'^\gamma$$

where $e_{\alpha\beta\gamma}$ is the Levi-Civita symbol and where the "instantaneous rotation" ω'^γ is defined by: $\dot{R}^\gamma{}_\alpha R_\gamma{}^\beta = e_\alpha{}^\beta{}_\gamma\, \omega'^\gamma \qquad (R^\gamma{}_\alpha R_\beta{}^\alpha = \delta^\gamma_\beta)$

Similarly: $$a'^\alpha \equiv \frac{d^2X'^\alpha}{dt^2} = R_\beta{}^\alpha (a^\beta - \ddot{d}^\beta) - e^\alpha{}_{\beta\gamma}\big(\underbrace{2\omega'^\beta V'^\gamma}_{\text{Coriolis}} + \dot{\omega}'^\beta X'^\gamma \underbrace{- e^\gamma{}_{\delta\epsilon}\, \omega'^\beta X'^\delta \omega'^\epsilon}_{\text{centrifugal}}\big)$$

(NB: $e_{ijk}\, e_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + mnk + nkm - mln - lnm - nml$)

3. Vector & tensor fields

When d^α & $R_\alpha{}^\beta$ do not depend on time a point in S' still is a fiber of N_4. It is hence natural to consider the ensemble of functions $V'^\alpha = R_\beta{}^\alpha V^\beta$ parametrized by $R_\beta{}^\alpha$ as an equivalence class: the velocity of P. Idem for $a'^\alpha = R_\beta{}^\alpha a^\beta$, the acceleration of P.

Objects which transform as $T'^\alpha(X'^\gamma) = R_\beta{}^\alpha T^\beta(X^\gamma)$ are vector fields

Vectors can also be defined in an intrinsic way: $T = T^\alpha e_\alpha$ where e_α is a basis of a vectorial space E.

A form λ acts on vectors to give numbers: $\lambda = \lambda_\alpha \epsilon^\alpha$ where ϵ^α is the conjugate basis of E of the dual space E^*: $\epsilon^\alpha(e_\beta) = \delta^\alpha_\beta$

A bi-linear form a acts on couples of vectors (V, W) to give a number if $V = V^\alpha e_\alpha$; $W = W^\beta e_\beta$; $a(V,W) = V^\alpha W^\beta \underbrace{a(e_\alpha, e_\beta)}_{\equiv a_{\alpha\beta}}$

hence the notation: $a \equiv a_{\alpha\beta}\, \epsilon^\alpha \otimes \epsilon^\beta$ where:

$$\epsilon^\alpha \otimes \epsilon^\beta (e_\gamma, e_\delta) \equiv \epsilon^\alpha(e_\gamma)\, \epsilon^\beta(e_\delta) = \delta^\alpha_\gamma \delta^\beta_\delta$$

This tensorial product $\otimes$ allows an "automatic" definition of multilinear applications. A type $\binom{p}{q}$ tensor is defined as:

$$T = \underbrace{T^{i_1 \dots i_p}{}_{j_1 \dots j_q}}_{\text{component}}\ \underbrace{e_{i_1} \otimes \dots \otimes e_{i_p} \otimes \epsilon^{j_1} \otimes \dots \otimes \epsilon^{j_q}}_{\text{basis of } E\otimes\dots\otimes E\otimes E^*\dots\otimes E^* \text{ which acts on a multiplet of p forms \& q vectors to give a number.}}$$

In this langage: { a vector is a "contravariant" tensor of degree 1
{ a form is a "covariant" tensor of degree 1
(see e.g. Lichnerowicz' book for developments)

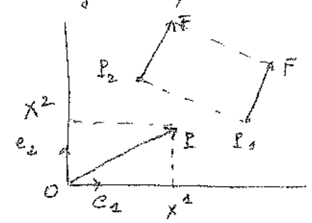
In a change of basis defined by $e_\alpha = R_\alpha{}^\beta e'_\beta$; $\epsilon'^\alpha = R_\beta{}^\alpha \epsilon^\beta$ the components of T transform as:

$$T'^{i_1 \dots i_p}{}_{j_1 \dots j_q} = T^{k_1 \dots k_p}{}_{\ell_1 \dots \ell_q} R_{k_1}{}^{i_1} \dots R_{k_p}{}^{i_p} R^{\ell_1}{}_{j_1} \dots R^{\ell_q}{}_{j_q}$$

(This law can be seen as a definition of a $\binom{p}{q}$ tensor).

The link between these vector spaces & Newton's absolute space is provided by "affine geometry"

An affine space is an ensemble of points $\mathcal{E}$ where couples (P, Q) are identified with vectors $PQ \in E$ such that $P_1P_2 = P_1Q + QP_2$ and such that if $P \in \mathcal{E}$, $F \in E$ then $\exists$ a unique point Q such that $PQ = F$.



$\{O, e_\alpha\}$: affine basis

$OP = X^\alpha e_\alpha$: position vector
↳ { components of OP
{ AND cartesian coordinates of P

$(P_1, F) \longrightarrow (P_2, F)$: parallel transport

Finally an euclidean space is an affine space endowed with the metric (bilinear form or covariant tensor of order 2):

$$e = \delta_{\alpha\beta}\, \epsilon^\alpha \otimes \epsilon^\beta$$

e acts on $dP = dX^\alpha e_\alpha$ to give:

$$e(dP, dP) = dX^\alpha dX^\beta e(e_\alpha, e_\beta) = \delta_{\alpha\beta}\, dX^\alpha dX^\beta = dl^2.$$

With these ingredients we can rewrite the transformation laws of velocities & accelerations as:

$$v' = v - \dot{d} - \Omega \wedge R' \;;\; a' = a - \ddot{d} - 2\Omega\wedge v' + \Omega \wedge (R' \wedge \Omega) - \dot{\Omega}\wedge R'$$

where $v' = \frac{dX'^\alpha}{dt} e'_\alpha$; $v = \frac{dX^\alpha}{dt} e_\alpha$; $\dot{d} = \dot{d}^\alpha e_\alpha$; $R' = X'^\alpha e'_\alpha = O'P$

$\dot{e}'_\alpha = \Omega \wedge e'_\alpha$ $\quad (\Omega = \omega'^\alpha e'_\alpha)$ and $(a\wedge b)^\alpha = \epsilon^\alpha{}_{\beta\gamma}\, a^\beta b^\gamma$

NB: $a' = a$ in "Galilean transformations" such that $\Omega = 0$, $\ddot{d} = 0$.

II Non-linear coordinates & covariant derivative

1. Metric & tensors in curvilinear coordinates

Consider $X^\alpha \to x^\alpha = x^\alpha(X^\beta)$ $\quad$ (and $X^\beta = X^\beta(x^\alpha)$)

then:

$$\underset{\text{invariant}}{dl^2} = \delta_{\alpha\beta}\, dX^\alpha dX^\beta = \delta_{\alpha\beta} \underbrace{\frac{\partial X^\alpha}{\partial x^\mu}\frac{\partial X^\beta}{\partial x^\nu}}_{\equiv\, e_{\mu\nu}} dx^\mu dx^\nu \qquad \left(dX^\alpha = \frac{\partial X^\alpha}{\partial x^\mu} dx^\mu\right)$$

example: $\begin{cases} X = r\cos\varphi,\quad Y = r\sin\varphi \quad (r = \sqrt{X^2+Y^2};\ \tan\varphi = \frac{Y}{X}) \\ e_{rr} = \left(\frac{\partial X}{\partial r}\right)^2 + \left(\frac{\partial Y}{\partial r}\right)^2 = 1 \;;\; e_{r\varphi} = 0 \;;\; e_{\varphi\varphi} = r^2 \\ \text{hence } \quad dl^2 = dr^2 + r^2 d\varphi^2 \end{cases}$

Pythagoras theorem as well as homogeneity & isotropy of E_3 are no longer manifest.

Exercise: show that the volume element $dV = dXdYdZ$ transforms as:

$$dV = dXdYdZ = \sqrt{\det e}\; dx\,dy\,dz.$$

In this enlarged framework it is natural to define a tensor field as an object whose components transform as:

$$T^{i_1\dots i_p}_{j_1\dots j_q} \longrightarrow t^{i_1\dots i_p}_{j_1\dots j_q} = \frac{\partial x^{i_1}}{\partial X^{k_1}} \cdots \frac{\partial x^{i_p}}{\partial X^{k_p}} \frac{\partial X^{l_1}}{\partial x^{j_1}} \cdots \frac{\partial X^{l_q}}{\partial x^{j_q}}\, T^{k_1\dots k_p}_{l_1\dots l_q}$$

(in a linear transformation: $\frac{\partial x^\alpha}{\partial X^\beta} = R_\beta{}^\alpha$, independent of X^γ)

2. Covariant derivative

In a non-linear change of coordinates, the derivative of a vector field transforms as:

$$\frac{\partial t^\alpha}{\partial x^\beta} = \frac{\partial}{\partial x^\beta}\left(\frac{\partial x^\alpha}{\partial X^\gamma} T^\gamma\right) = \underbrace{\frac{\partial x^\alpha}{\partial X^\gamma}\frac{\partial X^\mu}{\partial x^\beta}\frac{\partial T^\gamma}{\partial X^\mu}}_{\substack{\text{this is a } \binom{1}{1}\text{-tensor: } \tilde{D}_\beta t^\alpha \text{: the "covariant"} \\ \text{derivative of } t^\alpha \text{ wrt } x^\beta}} + \frac{\partial^2 x^\alpha}{\partial X^\mu \partial X^\gamma}\frac{\partial X^\mu}{\partial x^\beta} T^\gamma$$

Hence: $\tilde{D}_\beta t^\alpha = \frac{\partial t^\alpha}{\partial x^\beta} - \frac{\partial^2 x^\alpha}{\partial X^\mu \partial X^\gamma}\frac{\partial X^\mu}{\partial x^\beta} T^\gamma = \partial_\beta t^\alpha + \tilde{\Gamma}^\alpha_{\beta\gamma} t^\gamma$ $\quad \left(\partial_\beta \equiv \frac{\partial}{\partial x^\beta}\right)$

with $\tilde{\Gamma}^\alpha_{\beta\gamma} = \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial^2 X^\mu}{\partial x^\beta \partial x^\gamma}$ $\quad$ (obtained using $\frac{\partial X^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial X^\mu} = \delta_\alpha^\beta$)

The functions $\tilde{\Gamma}^\alpha_{\beta\gamma}$ are the "connexion coefficients".
symmetric in (β,γ) ; $\frac{n^2(n+1)}{2} = 18$ if $n = 3$

Example: $(X,Y) \to (r,\varphi)$: $\begin{cases} \tilde{\Gamma}^r_{\varphi\varphi} = \frac{\partial r}{\partial X}\frac{\partial^2 X}{\partial \varphi^2} + \frac{\partial r}{\partial Y}\frac{\partial^2 Y}{\partial\varphi^2} = -r \\ \tilde{\Gamma}^\varphi_{r\varphi} = 1/r. \end{cases}$

NB: the "tilde" is there to recall that $\exists$ special, cartesian, coordinates in which $\tilde{D} = d$ and all $\tilde{\Gamma} = 0$.

Definition of a constant field: $T^\alpha(X^\beta) = \text{Cst} \iff \frac{\partial T^\alpha}{\partial X^\beta} = 0 \iff \tilde{D}_\beta t^\alpha = 0$.

Example: $(X,Y) \to (r,\varphi)$: $\partial_\beta t^\alpha + \tilde{\Gamma}^\alpha_{\beta\gamma} t^\gamma = 0$ is integrable with solution: $t^r = a\cos(\varphi+\omega)$; $t^\varphi = -\frac{a}{r}\sin(\varphi+\omega)$ where a & ω are constants

(check: $T^\alpha = \frac{\partial X^\alpha}{\partial x^\beta} t^\beta$ yields $T^X = a\cos\omega$; $T^Y = -a\sin\omega$)

Parallel transport: $t^\alpha(x_1^\beta) \longrightarrow t_{/\!/}^\alpha(x_2^\beta)$ $\quad \tilde{D}_\beta t_{/\!/}^\alpha = 0$

Example:
$$\left.\begin{aligned} t^r(x_1,\varphi_1) &= a\cos(\varphi_1+\omega) \\ t^\varphi(x_1,\varphi_1) &= -\frac{a}{r_1}\sin(\varphi_1+\omega) \end{aligned}\right\} \to \left\{\begin{aligned} t_{/\!/}^r(x_2,\varphi_2) &= a\cos(\varphi_2+\omega) \\ t_{/\!/}^\varphi(x_2,\varphi_2) &= -\frac{a}{r_2}\sin(\varphi_2+\omega) \end{aligned}\right.$$

Parallel transport does not depend on path from P_1 to P_2 because $\tilde{D}_\beta t_{/\!/}^\alpha = 0$ is integrable, ie because $\exists$ cartesian coordinates where $\tilde{D}_\beta t_{/\!/}^\alpha = \frac{\partial T^\alpha}{\partial X^\beta} = 0$ ie because the "connexion is flat": $\tilde{\Gamma} \to 0$ when $x \to X$.

Covariant derivative of a form:

$\lambda_\alpha \omega^\alpha$ is a function of position. Hence if we set $\tilde{D}_\beta(\lambda_\alpha \omega^\alpha) = \partial_\beta(\lambda_\alpha\omega^\alpha)$ then we have, by Leibniz rule:
$$\tilde{D}_\beta(\lambda_\alpha\omega^\alpha) = (\tilde{D}_\beta\lambda_\alpha)\omega^\alpha + \lambda_\alpha(\cancel{\partial_\beta\omega^\alpha} + \tilde{\Gamma}^\alpha_{\beta\gamma}\omega^\gamma) = (\partial_\beta\lambda_\alpha)\omega^\alpha + \cancel{\lambda_\alpha\partial_\beta\omega^\alpha}$$

so that: $\tilde{D}_\beta\lambda_\alpha = \partial_\beta\lambda_\alpha - \tilde{\Gamma}^\gamma_{\alpha\beta}\lambda_\gamma$

Hence the covariant derivative of a $\binom{p}{q}$-tensor is:
$$\tilde{D}_\beta t^{i_1\ldots i_p}_{j_1\ldots j_q} = \partial_\beta t^{i_1\ldots i_p}_{j_1\ldots j_q} + \tilde{\Gamma}^{i_1}_{\beta k_1} t^{k_1 i_2\ldots i_p}_{j_1\ldots j_q} + \ldots - \tilde{\Gamma}^{\ell_1}_{\beta j_1} t^{i_1\ldots i_p}_{\ell_1\ldots \ell_q} - \ldots$$

Note that $\tilde{D}_\alpha\tilde{D}_\beta t^{i_1\ldots i_p}_{j_1\ldots j_q} = \tilde{D}_\beta\tilde{D}_\alpha t^{i_1\ldots i_p}_{j_1\ldots j_q}$

because this is true in cartesian coordinates where $\tilde{D}_\beta = \partial_\beta$:
"tilde" covariant derivatives commute

Exercise: in a change of curvilinear coordinates $x^\alpha \to x'^\alpha(x^\beta)$ the connexion coefficients transform as:
$$\tilde{\Gamma}^\alpha_{\beta\delta} \longrightarrow \tilde{\Gamma}'^\alpha_{\beta\gamma} = \frac{\partial x^\mu}{\partial x'^\beta}\frac{\partial x^\nu}{\partial x'^\gamma}\frac{\partial x'^\alpha}{\partial x^\lambda}\tilde{\Gamma}^\lambda_{\mu\nu} + \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial x'^\beta\partial x'^\delta}$$

The connexion coefficients $\tilde{\Gamma}^\alpha_{\beta\gamma}$ were introduced without reference to the coefficients $e_{\alpha\beta}$ of the metric in the coordinate system x^α. They are however related.

Indeed: $e_{\alpha\beta} = \frac{\partial X^\mu}{\partial x^\alpha}\frac{\partial X^\nu}{\partial x^\beta}\delta_{\mu\nu}$; $\tilde{\Gamma}^\alpha_{\beta\gamma} = \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial^2 X^\mu}{\partial x^\beta\partial x^\gamma}$

hence taking $\frac{\partial}{\partial x^\lambda}$ derivatives of $e_{\alpha\beta}$ & performing sum & difference:
$$\tilde{\Gamma}^\alpha_{\beta\gamma} = \frac{1}{2}e^{\alpha\mu}(\partial_\beta e_{\gamma\mu} + \partial_\gamma e_{\mu\beta} - \partial_\mu e_{\gamma\beta})$$
Under this form $\tilde{\Gamma}^\alpha_{\beta\gamma}$ are also called "Levi-Civita symbols"

Example: $(X,Y) \to (r,\varphi)$; $dl^2 = dX^2 + dY^2 = dr^2 + r^2 d\varphi^2$; $\tilde{\Gamma}^r_{\varphi\varphi} = \frac{1}{2}e^{rr}(-\partial_r e_{\varphi\varphi}) = -r$

$\underset{\sim}{\text{NB}}$: $\tilde{D}_\gamma e_{\alpha\beta} \equiv 0$ (because this is true in cartesian coordinates)

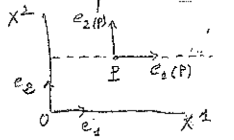
Kinematics in curvilinear coordinates
trajectory of P: $X^\alpha = X^\alpha(t)$ or $x^\alpha = x^\alpha(X^\beta(t))$.

Hence
$$\left\{\begin{aligned} v^\alpha &\equiv \frac{dx^\alpha}{dt} = \frac{\partial x^\alpha}{\partial X^\beta}V^\beta \\ \frac{dv^\alpha}{dt} &= \underbrace{\frac{\partial x^\alpha}{\partial X^\beta}a^\beta}_{\frac{\tilde{D}v^\alpha}{dt}} + \frac{\partial^2 x^\alpha}{\partial X^\gamma\partial X^\beta}V^\beta V^\gamma \end{aligned}\right.$$
so that: $\frac{\tilde{D}v^\alpha}{dt} = \frac{dv^\alpha}{dt} + \tilde{\Gamma}^\alpha_{\beta\gamma}v^\beta v^\gamma$

Example: $(X,Y) \to (r,\varphi)$: $\frac{\tilde{D}\dot{r}}{dt} = \ddot{r} - r\dot{\varphi}^2$; $\frac{\tilde{D}\dot{\varphi}}{dt} = \ddot{\varphi} + \frac{2\dot{r}}{r}\dot{\varphi}$

3. Tangent spaces, natural bases & triads

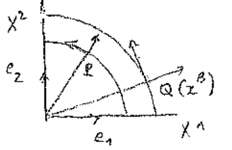
Consider the ~~affine~~ frame $\{O, e_\alpha\}$. The basis e_α can be seen as belonging to a vector space labelled by O : $e_\alpha \in E_{(O)}$



- // transport e_α to point P : $e_{\alpha(P)} \in E_{(P)}$ (isomorphic to $E_{(O)}$)
- $e_{1(P)}$ is the velocity of the trajectory $(X^1 = t, X^2)$ hence the tangent at P of the line coordinate X^1
- The operator which acts on the coordinate X^β of P to give the component $e^\beta_{1(P)} = \delta^\beta_1$ of e_1 can be written $\frac{\partial}{\partial X^1}$ since $\frac{\partial X^\beta}{\partial X^1} = \delta^\beta_1$

There is the same information in $e_{1(P)}$ and in $\frac{\partial}{\partial X^1}$.
Hence identify : $e_{1(P)} = \frac{\partial}{\partial X^1}$ and consider $\frac{\partial}{\partial X^\alpha}$ as a basis of $E_{(P)}$

In curvilinear coordinates the tangents to line coordinates vary from point to point and it makes (more) sense to distinguish $E_{(P)}$ from $E_{(Q)}$



$\boxed{}$ $\frac{\partial}{\partial x^\alpha}$ is the natural basis of the tangent space $E_{(P)}$ at P with coordinates x^α

A vector field t at P (or "tangent") $\in E_{(P)}$: $t = t^\alpha \frac{\partial}{\partial x^\alpha}$

A vector acts on a function f as: $t(f) = t^\alpha \frac{\partial f}{\partial x^\alpha}$

One can define an associate "form" df such that $df(t) = t(f)$
If $f = x^\beta$ then: $dx^\beta(t) = dx^\beta\left(t^\alpha \frac{\partial}{\partial x^\alpha}\right) = t^\alpha dx^\beta\left(\frac{\partial}{\partial x^\alpha}\right)$
$$= t(x^\beta) = t^\alpha \frac{\partial x^\beta}{\partial x^\alpha} = t^\beta$$

hence we must have: $dx^\beta\left(\frac{\partial}{\partial x^\alpha}\right) = \delta^\beta_\alpha$

Thus the forms dx^α are a basis of E^*_P the "cotangent" space at P.

More generally a tensor field of type $\binom{p}{q}$ decomposes as
$$T = t^{i_1 \dots i_p}_{j_1 \dots j_q} \, \partial_{i_1} \otimes \dots \partial_{i_p} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}$$

Euclidean metric in arbitrary coordinates:
$$e = \delta_{\alpha\beta}\, \epsilon^\alpha \otimes \epsilon^\beta = \delta_{\alpha\beta}\, \underbrace{dX^\alpha}_{\frac{\partial X^\alpha}{\partial x^\mu} dx^\mu} \otimes dX^\beta = \underbrace{\delta_{\alpha\beta} \frac{dX^\alpha}{\partial x^\mu} \frac{dX^\beta}{\partial x^\nu}}_{e_{\mu\nu}} dx^\mu \otimes dx^\nu$$

Example :
$$\begin{cases} (X,Y) \to (r,\varphi) : \begin{cases} \frac{\partial}{\partial r} = \cos\varphi \frac{\partial}{\partial X} + \sin\varphi \frac{\partial}{\partial Y} \\ \frac{\partial}{\partial \varphi} = r\left(-\sin\varphi \frac{\partial}{\partial X} + \cos\varphi \frac{\partial}{\partial Y}\right) \end{cases} \\ dr = \cos\varphi\, dX + \sin\varphi\, dY ; \quad d\varphi = \frac{1}{r}(-\sin\varphi\, dX + \cos\varphi\, dY) \\ e = dr^2 + r^2 d\varphi^2 \end{cases}$$

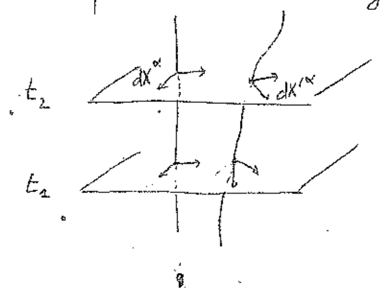
NB: dr as a differential 1-form must not be confused with dr the increment of the r coordinate.

"triads" : linear combinations of $\frac{\partial}{\partial x^\alpha}$: $h_\alpha = L^\beta{}_\alpha \frac{\partial}{\partial x^\beta}$
with associated form-basis : $\theta^\alpha = L_\beta{}^\alpha dx^\beta$

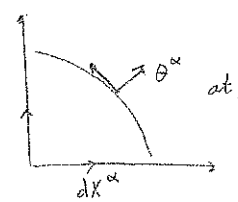
such that $e = \delta_{\alpha\beta}\, \theta^\alpha \otimes \theta^\beta$; Example : $\theta^r = dr$; $\theta^\varphi = r d\varphi$

NB : in general the triad θ^α is "non-holonomic", that is cannot be written as $\theta^\alpha = dx'^\alpha$; if yes the transformation $\frac{\partial}{\partial x^\alpha} \to h_\alpha$ reduces to the change of coordinates $x^\alpha \to x'^\alpha$.

Keep in mind the difference between going from a cartesian frame to a moving frame or to an orthonormal triad :



vs



at all t

4. Covariant derivative (II)

Consider a set of points P characterized by their coordinates x^α and the tangent spaces $E_{(P)}$ with natural bases $\frac{\partial}{\partial x^\alpha}$.

A "connexion" "connects" tangent spaces and hence defines "parallel transport" of a tensor from P to Q.

More precisely: an "affine connexion" D is: $v \xrightarrow{D} D_v$ such that

$$\forall (v,t) \in E_{(P)},\ \forall (a,b) \in R \quad \begin{cases} D_{av+bt} T = (a D_v + b D_t) T \\ D_v (T \otimes S) = D_v T \otimes S + T \otimes D_v S \end{cases}$$

$D_v T$ is the covariant derivative of T with respect to v;
T and $D_v T$ are tensorial fields of the same type.

In practice D is defined through its action on $\partial/\partial x^\alpha$:

$$D_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^\beta} \equiv D_\alpha \frac{\partial}{\partial x^\beta} \equiv D_\alpha \partial_\beta = \Gamma^\gamma_{\alpha\beta} \frac{\partial}{\partial x^\gamma}$$

where the $n^3 = 27$ functions $\Gamma^\gamma_{\alpha\beta}$ are the connexion coefficients which define D.

Thus: $D_v \omega = v^\alpha D_\alpha (\omega^\beta \partial_\beta) = v^\alpha (D_\alpha \omega^\beta) \partial_\beta$ with $D_\alpha \omega^\beta = \partial_\alpha \omega^\beta + \Gamma^\beta_{\alpha\gamma} \omega^\gamma$

Using $\partial_\alpha \otimes dx^\alpha = 1$ one obtains the covariant derivative of a 1-form:

$$D_v \lambda = v^\alpha D_\alpha (\lambda_\beta dx^\beta) = v^\alpha (D_\alpha \lambda_\beta) dx^\beta \quad \text{with } D_\alpha \lambda_\beta = \partial_\alpha \lambda_\beta - \Gamma^\gamma_{\alpha\beta} \lambda_\gamma$$

- The n^3 connexion coefficients $\Gamma^\gamma_{\alpha\beta}(x^\mu)$ are a priori arbitrary, each choice defining a connexion.
- If $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}$ the connexion is "torsion free": $\frac{n^2(n+1)}{2}$ component
- If we are given a torsion free connexion AND a metric g [that is $\frac{n(n+1)}{2}$ symmetric functions $g_{\alpha\beta}(x^\mu)$] which are "compatible", that is satisfy $\underset{g}{D}_v g = 0\ \forall v \Leftrightarrow D_\alpha g_{\beta\gamma} = 0$ then D is a "Levi-Civita" connexion
- If there $\exists$ coordinates X^α where all $\Gamma^\alpha_{\beta\gamma} \equiv \tilde{\Gamma}^\alpha_{\beta\gamma} = 0$, the connexion is "flat" and the X^α are cartesian coordinates.

III Newtonian inertial & gravitational forces

1. EOM, galilean relativity & inertial forces

Newton's "second law": in the absolute frame (S) the eom of a material point P are

$$m a = F \qquad \begin{cases} m: \text{"inertial mass"};\ a = \frac{d^2X^\alpha}{dt^2} e_\alpha \\ F = F^\alpha e_\alpha : \text{cartesian vector representing interactions of } P \text{ with other bodies} \end{cases}$$

Properties of force vectors F:

- if $f_{aa'}$ represents the interaction of P_a & $P_{a'}$
  then $F_a = \sum_{a'} f_{aa'}$ represents the interaction of P_a and $\{P_{a'}\}$
- $f_{12} = -f_{21}$ (Newton's "Third Law" of action & reaction)
- F is a priori a functional of the worldline $P = P(t)$; $ma = F$ remains a second order differential equation if $F = F(R, v, a)$ only
- "Copernican principle": F must not depend on the moving frame chosen to describe the interaction; since v & a are not represented by the same vectors in 2 ≠ moving frames, this translates in this Newtonian framework by $F = F(R)$

Galilean relativity.

A "free" particle is such that $F = 0$ (in all frames). Hence $a = 0$.

Hence the frame where free particles are in uniform motion should materialize Newton's absolute frame, and free motionless objects in that frame should materialize a "fiber" of New (that is a point in absolute state of rest (independently of the ∃ of distant stars).

But if $a = 0$ in (S) then $a' = a = 0$ in all Galilean frames; hence if a free body is at rest in a frame, there is no way to tell if it is in a state of absolute rest or moving together with the frame. (This would be possible if there was a law fixing the initial velocities of, say, distant stars, but Newton is silent on that point.

This "galilean invariance" of the law of motion for free particles extends to interacting particles if F satisfies Copernican principle

Therefore Newton's absolute space & universal time are, to some extent, "ghosts":

The geometrical properties of figures do not depend on whether the rest of the universe is empty or not

it is impossible to "anchor" absolute space: the only absolute entity is the equivalence class of all galilean frames.

On the other hand accelerated frames can be distinguished from the set of galilean frames since the eom read

$$m a' = F \underbrace{- m\ddot{d} + m\left(-2\Omega \wedge v' + \Omega \wedge (R' \wedge \Omega) - \dot{\Omega} \wedge R'\right)}_{\text{inertial forces}},$$

if $F=0$, a' does not depend on m: OK since the "inertial forces" are a pure "perspective effect" due to the motion of the frame only

if F is known (eg $F = eE$) the motion of P in S' allows a determination of $\ddot{d}$ & Ω.

Examples: $\begin{cases} \text{Foucault's pendulum at the Panthéon} \\ \text{Newton's bucket} \end{cases}$

2. Inertial & gravitational masses

The inertial mass appears in Newton's 2nd law: $F = m_I a$

its numerical value (compared to a reference body of mass $m_I = 1\text{kg}$) can in principle be obtained via elastic collisions, using momentum conservation laws.

"Gravity" being an experimental fact, the "gravitational mass" is another parameter associated with bodies submitted to, or creating "gravity"

the notion of gravitational mass is hence twofold:

The "Passive" gravitational mass enters in the "weight" of a body: $F = m_G\, g$ where F is the external gravity force & $g \equiv \left(\frac{F}{m_G}\right)_i$ is the same for all bodies (i). Hence, knowing F (via, say, a measurement of $a = F/m_I$, m_I being known), then g & m_G are also known.

The "Active" gravitational mass (or "charge") characterizes the object which creates gravity.

Hence the gravitational force F_{AB} between A & B is proportional to $m_A^{act}\, m_B^{pass} = m_A^{pass}\, m_B^{act}$ from Newton's 3rd law. They are therefore proportional and can be identified (the same is true for electric "active" & "passive" charges).

On the other hand, nothing in Newtonian physics imposes that "charges" should be equal to inertial masses. And, indeed, the ratio of the electric charge to m_I varies from body to body (and can be zero).

As known since Galileo: $m_I = m_G \equiv m$

For an account of all experiments (Galileo, Newton, Bessel, Eötvös, ..., Eöt-Wash, from a precision of 10^{-2} to 10^{-13}) cf C. Will.

3. Newton's law of gravity

The gravitational force on a material point P due to others P_a is represented by the vector:

$$F = -m \sum_a \frac{G\, m_a}{r_a^3}\, l_a \qquad P \xleftarrow{\;l_a\;} P_a \qquad r_a = \sqrt{l_a \cdot l_a}$$

G: Newton's constant

The distance r_a is numerically the same in all frames & all coordinate systems

If $l_a^\alpha = X^\alpha(t) - X_a^\alpha(t)$ are the components of F in a cartesian frame S

then its components in any moving frame are $l_a'^\alpha = R_\beta{}^\alpha(t)\, l_a^\beta$

and its components in any curvilinear coordinate system are

$$l_a'^\alpha = \frac{\partial x^\alpha}{\partial X^\beta}\, l_a^\beta \quad \text{where } \frac{\partial x^\alpha}{\partial X^\beta} \text{ is evaluated on the trajectory of } P$$

The measurement of G (Cavendish type experiments) is difficult: cf Will (and gr-qc/0602027)

"à la Faraday"
F can be seen as a vector field, that is a function of X^α rather than $X^\alpha(t)$. In which case it derives from the "gravitational potential" U:

$$F = -m\nabla U \qquad \text{with} \qquad U = -\sum \frac{G m_a}{r_a}$$

Hence the equations of motion of a material point in the gravitational field F are given by:

$$a = -\nabla U \quad \text{in any inertial frame}; \quad \tilde{D}_v v = -\nabla U \quad \text{in any coord. system}$$
$$a' = -\nabla U - 2\Omega \wedge v' + \Omega \wedge (R' \wedge \Omega) - \dot{\Omega} \wedge R' - \ddot{d} \quad \text{in any moving frame}$$

Neither m_I nor m_G appear since $m_I = m_G$

Hence:
In a sufficiently small region of space $\nabla U \equiv -g \sim$ constant. A particle in that field has a uniformly accelerated motion. Go to the uniformly accelerated (non rotating) frame where this particle is at rest. In that frame another particle, not too far so that the gravitational field it is submitted to is still $-g$, will have zero acceleration, $a' = 0$, and can be "mistaken" for a free particle. In S' the gravitational field is, locally, effaced.

Reciprocally the accelerated motion of a free particle in an accelerated frame ($a' = -\ddot{d}$) can be "mistakenly" interpreted as the motion in an inertial frame of a particle submitted to a gravitational field ($a = -g$).

This "accidental" (local) resemblance of inertial & gravitational forces will be the foundation of General Relativity under the name of (weak) equivalence principle.

An important quantity is the "potential energy" of a gravitational system of N point particles P_a, W, defined as

$$W = -\sum_{\substack{a, a' \\ a < a'}} \frac{G m_a m_{a'}}{r_{aa'}} = \frac{1}{2} \sum_a m_a U_a \quad \text{with} \quad U_a = -\sum_{a' \neq a} \frac{G m_{a'}}{r_{aa'}}$$

(The importance of W is that one deduces from the eom, $a_a = -\nabla_a U_a$, that the total energy of the system:

$$\mathcal{E} = \sum_a \frac{1}{2} m_a v_a^2 + W \qquad \text{is constant}).$$

Hence the Lagrangian of the system, from which we deduce Newton's law of gravity by Euler variation of the paths is:

$$L = \sum_a \frac{1}{2} m_a v_a^2 - W = \sum_a \frac{1}{2} m_a v_a^2 + \sum_{a < a'} \frac{G m_a m_{a'}}{r_{aa'}}$$

The gravitational potential created by a continuous distribution of matter with mass density ρ is

$$U(P) = -G \int_{V'} \frac{\rho(t, P')}{r_{PP'}} dV' \qquad (1)$$

Only by going to a continuum description can one prove "Gauss theorem", that is that the gravitational potential of a bounded spherically symmetric distribution of matter is the same as that of a point particle of the same mass, whatever its (radial) motion. (About the pb of "uniformly" distributing points on a 2-sphere, see e.g Saff et al.)

From (1) Laplace & Poisson deduced a differential relation between U & ρ:

$$\Delta U = 4\pi G\rho \qquad (2)$$

which is more general than (1) since (1) encodes an additional boundary condition.

Example: if $\rho = \rho(t)$ (1) is not defined, whereas a solution of (2) is $U = \frac{2}{3}\pi G \rho r^2$.

The gravitational potential energy of a continuous distribution is:

$$W = \frac{1}{2}\int \rho U \, dV = -\frac{1}{8\pi G}\int (\nabla U)^2 dV$$

where the second equality follows from $\Delta U = 4\pi G\rho$, assuming that $U \propto \frac{1}{r}$ at infinity.

The Lagrangian of the system can, at first sight, be written as $L = \frac{1}{2}\int \rho v^2 dV - W$ with $W = \int dV\left[\frac{1+c}{2}\rho U + \frac{c}{8\pi G}(\nabla U)^2\right]$ with c an arbitrary constant. However one must choose $c = 1$, that is:

$$L = \int dV \left(\frac{1}{2}\rho v^2 - \rho U - \frac{1}{8\pi G}(\nabla U)^2\right)$$

because its variation yields Euler's equation for a fluid in a gravitational field when one varies the paths of its elements:

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + (v.\nabla)v = -\frac{1}{\rho}\nabla p - \nabla U \quad ; \quad \frac{\partial \rho}{\partial t} + \nabla.(\rho v) = 0$$

AND because its variation yields Poisson's equation when one varies the configuration of the potential U. (with the condition $U \propto \frac{1}{r}$ at ∞)

4. Some unanswered questions

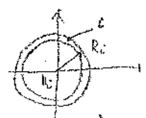
Newton's theory of gravity had 250 years of triumphs. However, as known to Newton himself, it is flawed:

- The interaction between distant objects is instantaneous ("hypotheses non fingo")
- The interaction is long-range and universal. Hence no material body can be considered as strictly free. But the definition (& construction) of inertial frames rely on the existence of free motion!
- The origin of the frame plays a special rôle in $F = ma$, and the (immaterial) centre of mass has an unknown uniform motion with respect to the inaccessible absolute frame. (Leibniz)

(IV) Newtonian "cosmology" & the pb of propagation of light

1. Newtonian "Hubble law" & "Friedmann equation"

Consider a distribution of matter with spherical symmetry wrt the origin of an inertial system. From Gauss theorem the eom of a shell (i) is



$$\ddot{r}_i = -\frac{GM_i(t)}{r_i^2} \qquad (1)$$

If there is no shell crossing M_i = constant, and if the motion is self-similar (that is, all shells reach maximum expansion and collapse at the origin simultaneously) then the solution is:

$$r_i = \frac{v_i}{c}\, a(t) \quad \text{with} \quad a(\eta) = a_0(1-\cos\eta)\ ;\ t - t_0 = \frac{a_0}{c}(\eta - \sin\eta)$$

(with v_i, c & a_0 constants) and the first integral of (1) reads

$$\frac{\dot{a}^2}{a^2} + \frac{c^2}{a^2} = \frac{8\pi G}{3}\rho \qquad \text{with} \quad \rho = \frac{3c^2 a_0}{4\pi G}\,\frac{1}{a^3}$$

As for M_i: $M_i = \frac{4}{3}\pi\rho(t) r_i^3$; self similar motion implies homogeneity.

The velocity distribution is isotropic with respect to all particles. Indeed $\vec{OP_i} = \frac{\vec{v_i}}{c}\, a(t) \Rightarrow \vec{P_iP_j} = \frac{\vec{v_j} - \vec{v_i}}{c}\, a(t)$

$$\Rightarrow \vec{v_{ij}} = H(t)\,\vec{P_iP_j} \quad \text{wit} \quad H(t) = \frac{1}{a}\frac{da}{dt} \ : \text{"Hubble parameter"}$$

However all particles are not equivalent: there is a centre (since the distribution is bounded) which is the only point in free motion.

One cannot let the external radius of the distribution go to ∞ because (1) relies on Gauss theorem which is not valid in that case.

Such models of an "Island Universe" are a disappointing aspect of Newton's theory of gravity (when based on Newton's law for gravity, that is $F \propto 1/r^2$).

The local Euler-Poisson version of Newton's gravity yields more satisfactory cosmological models. (Milne & McCrea 1934)

The equations are (in an inertial frame)

$$(2) \qquad \Delta U = 4\pi G\rho \; ; \quad \frac{\partial v}{\partial t} + (v.\nabla)v = -\frac{1}{\rho}\nabla p - \nabla U \; ; \quad \frac{\partial \rho}{\partial t} + \nabla.(\rho v) = 0$$

plus an equation of state for the fluid: $p = p(\rho)$.

Look for a solution of the form: $\rho = \rho(t)$, $p = p(t)$; $v = H(t)\,R$.

- Poisson equation $\rightarrow$ $U = \frac{2}{3}\pi G \rho(t) r^2$
- continuity equation $\rightarrow$ $\rho(t) = \frac{a_0}{a^3}$ with $H \equiv \frac{\dot a}{a}$
- Euler equation $\rightarrow$ $\frac{\dot a^2}{a^2} + \frac{Kc^2}{a^2} = \frac{8\pi G}{3}\rho$ (as before).

(3) for $K=0$ $\quad a \propto t^{2/3}$; $H = \frac{2}{3t}$; $\rho \propto \frac{1}{t^2}$ "Einstein-de Sitter" solution

In this infinite Universe there is no boundary to help determine a geometrical centre & all particles ("galaxies") are accelerated with respect to each other. There is therefore no way to determine the galilean reference frames in which the equations apply.

Note however that equations (2) are invariant under the larger group of "Milne transformations" defined by $X'^\alpha = X^\alpha - d^\alpha(t)$. Noting that:
$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t'} + \dot d^\alpha \frac{\partial}{\partial X'^\alpha}$; $\frac{\partial}{\partial X^\alpha} = \frac{\partial}{\partial X'^\alpha}$ one indeed sees that equations (2) are the same in (S) & (S') IF U transforms as $U \rightarrow U' = U + \ddot d_\alpha X'^\alpha$

Hence equations (2) apply in any (non rotating) "freely falling" frame attached to any given "galaxy".

(For further developments see e.g. B. Carter)

One can also perturb the background solution (3) and linearize (2) to find the equations of evolution for the perturbations and see under which conditions they can grow to form the observed large scale structure of the Universe: See J. Silk & J.P. Uzan.

2. Propagation of light in Newtonian physics

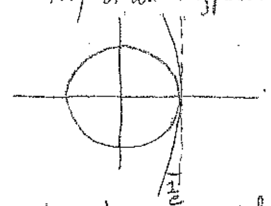
Astronomy is not a complete science without a theory of light...
A coherent theory of the propagation of "light particles" in gravitational fields was developped from Newton to the early 19th century (see J. Eisenstaedt).

The first integrals of the (planar) motion of a particle in the gravitational field of a spherical body are:

$$(1) \quad \begin{cases} r^2\dot\varphi = L & \text{(conservation of angular momentum)} \\ \frac{1}{2}\left(\dot r^2 + \frac{L^2}{r^2}\right) - \frac{GM}{r} = E & \text{(conservation of total energy)} \end{cases}$$

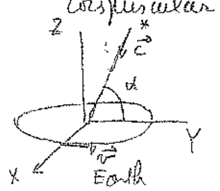
- A "light-particle" emitted with velocity c will escape and reach ∞ (with zero velocity) if the central body has a radius r_*
  $r_* \geqslant \frac{2GM}{c^2}$ (Michell, Laplace)
  If $r_* < \frac{2GM}{c^2}$ the body, as seen from infinity, is "black".

- Integrating (1) yields Kepler's conics: (2) $r = \frac{p}{1+e\cos\varphi}$; $p = \frac{L^2}{GM}$
  If the "light-particle" grazes the surface $r = r_*$ with velocity c then $L = r_* c$ (from (1)) & $e \sim \frac{r_* c^2}{GM} \gg 1$ from (2); the trajectory is an hyperbola with $1 + e\cos\varphi_a = 0 \Rightarrow \varphi_a \sim \pm\frac{\pi}{2} + \frac{1}{e}$



Hence a deflection $\Delta\varphi \simeq \frac{2}{e} = \frac{2GM}{r_* c^2}$
(Half the GR value) (Soldner 1801)

- The phenomenon of "light aberration" was also explained within a corpuscular theory of light by Bradley (1728)



in the Earth frame the velocity of light is $(0, c\cos\alpha - v, c\sin\ldots) \equiv (0, c'\cos\alpha', c'\sin\ldots$
so that, six months later $\Delta\alpha' \sim \frac{2v}{c}\sin\alpha'$
$\Delta\alpha'$ should vary (α' given) from star to star since c depends on the size of the star (see (1), negligible) AND its speed

In a wave-theory (Huygens, Fresnel...) the velocity of light does NOT depend on the velocity of the source ; this explains why $\Delta\alpha'$ is observed to be the same for all stars (α' given).

However v, in a wave theory, must be replaced by $|v - V_0|$ where V_0 is the speed of ether (in which light propagates) in the solar system frame S. Since aberration is observed $v - V_0 \neq 0$: the Earth does not "drag" ether ; there is an "ether wind" on Earth of about 30 kms/sec if ether is at rest wrt the Solar System.

Now, if light propagates in Ether as sound propagates in air one should be able to measure the velocity of Earth wrt Ether using the standard formulae:

$$\begin{cases} c_r = c_S + V_0 - v_r \ ; \ \tan\alpha_r = \dfrac{\tan\alpha}{1 - |v_r - V_0|/c_S \cos\alpha} \ ; \\[2ex] \nu_r = \nu_e \dfrac{1 + k\cdot(V_0 - v_r)/|k|c_S}{1 + k\cdot(V_0 - v_e)/|k|c_S} \quad \text{(Doppler formula)} \end{cases} \quad \begin{cases} r : \text{receptor} \\ e : \text{emitter} \\ c_S : \text{speed wrt Ether} \\ V_0 : \text{speed of Ether }/S \end{cases}$$

As is well-known... all experiments to measure V_0 or $V_0 - v_r$ failed. (Michelson-Morley 1881 & 1888) : experimentally $c_r = c_S$ $\forall V_0, \forall v_r$...

Ether had been introduced to "materialize" bodies in state of absolute rest and to be the medium in which light propagates. The velocity of Earth wrt Ether should have been measurable by kinematics experiments. But, as Poincaré put it : "It seems that this failure to exhibit the absolute motion of Earth is a general law of Nature". "An explanation was required ; it was found", [cf e.g. Lorentz' electron theory] ; "One always finds explanations : hypotheses are always in stock !".

There is no need to spend time on these brilliant explanations; They do not compare with Einstein's new vision of Space & Time.

(V.) Special Relativity in inertial frames

1. Minkowski Space-Time

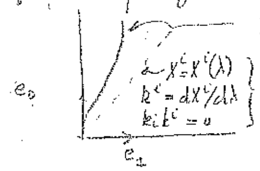
- In special relativity the intuitive notion of "to be somewhere at a certain time" is represented by a point, called "event". All events form a 4-dimensional space, Minkowski (absolute) space-time M_4 (which is postulated to be pseudo-euclidean that is : among all possible labellings of the events P, there $\exists$ some, X^i, called Minkowski coordinates such that the "infinitesimal "distance" between 2 events X^i & $X^i + dX^i$ is given by

$$ds^2 = -(dX^0)^2 + dX^2 + dY^2 + dZ^2 = \eta_{ij}\, dX^i dX^j$$

In the language of differential geometry : M_4 is equipped with a Lorentzian metric :

$$\ell = \eta_{ij}\, dX^i dX^j$$ where, here, dX^i are the differential forms conjugate to the natural basis $e_i = \dfrac{\partial}{\partial X^i}$ of the tangent space at P.

- The motion of a point like material object is represented by a world-line, that is a curve in M_4, which is postulated to lie entirely with its "light-cone" that is world lines of zero length which are postulated to represent the trajectories of light rays



$X^i = X^i(\lambda)$; $U^i = \dfrac{dX^i}{d\lambda}$

if the parameter λ is normalized by

$$U^i U_i = \eta_{ij} U^i U^j = -c^2 \quad (c = \text{speed of light}$$

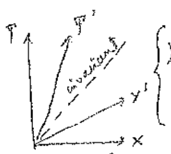
then λ is usually denoted by τ

- A Minkowskian frame is materialized by means of rigid objects whose relative spatial distances $(\Delta X^2 + \Delta Y^2 + \Delta Z^2)$ do not change with time $T = X^0/c$ as measured by clocks WHICH ARE AT REST with respect to the spatial frame (within, of course, the available precision).

  There is no need to introduce the notion of "absolute rest"

  A moving clock DOES NOT! "materialize" T.

Changing the labellings of the points of M_4 is postulated not to change the numerical value of their distances. The class of relabellings which does not change either the form of the line element is the Poincaré group:



$$\begin{cases} X^i \to X'^i = \Lambda_j{}^i (X^j - d^j) \quad \text{with } \Lambda_k{}^i \Lambda_\ell{}^j \eta_{ij} = \eta_{k\ell} \quad (\det \Lambda = +1;\ \Lambda_0{}^0 > 1) \\ \Longrightarrow \quad ds^2 = \eta_{ij}\, dX^i dX^j = \eta_{k\ell}\, dX'^k dX'^\ell . \end{cases}$$

$$\begin{cases} \text{Example:} \quad T' = \dfrac{T - V_0 X/c^2}{\sqrt{1 - V_0^2/c^2}} \ ; \quad X' = \dfrac{X - V_0 T}{\sqrt{1 - V_0^2/c^2}} \\ \quad \text{and } X = cT \iff X' = cT' \text{ : i.e. } \underline{\underline{c' = c}} \end{cases} \qquad \begin{cases} \Lambda_i{}^j = \begin{pmatrix} \cosh\psi & -\sinh\psi \\ -\sinh\psi & \cosh\psi \end{pmatrix} \\ V_0/c = \tanh\psi \end{cases}$$

A Lorentz-Poincaré transformation is materialized by a rotation of the rigid spatial frame together with uniform translation. T' represents time as measured by clocks at rest wrt (S').

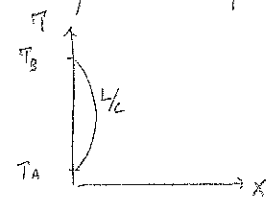
In this Einsteinian vision of space & time the equivalence of inertial frames is no longer a consequence of the laws of dynamics but merely geometrical: it is a rotation of the (T, X) axes, as "irrelevant" as a rotation of the (X, Y) axis. There is no more privileged time than there is a privileged X-axis: this may be psychologically difficult to accept but rewarding!

A consequence is that the transformation laws of velocities & accelerations become trivial:

$$U'^j = \frac{dX'^j}{d\tau} = \Lambda_i{}^j U^i \ ; \quad \gamma'^j = \frac{dU'^j}{d\tau} = \Lambda_i{}^j \gamma^i$$

The propagation of light becomes "crystal clear": the aberration of stars, Doppler frequency shifts etc are all obtained kinematically, without any need for any "Ether", and some experiments, like the Michelson-Morley ones, become ... pointless.

A last question of kinematics to be answered is: what is the mathematical representation of time measured by an accelerated clock?



If the clock is at rest in (S), the time elapsed between events A & B is $T_B - T_A$ which is also (up to c) the length of its world-line.

The length of a world-line is an invariant (that is: does not depend on the coordinate system chosen to label events).

Hence the (self-evident) postulate: "time measured by a moving clock is the length of its world line (up to c)":

$$L = \int_A^B \sqrt{-ds^2} = \int_{T_A}^{T_B} \sqrt{1 - V^2/c^2}\, c\, dT = \int_{\tau_1}^{\tau_2} \sqrt{-\ell(u,u)}\, d\tau = c\,\Delta\tau,$$

if $U^i U_i = -c^2$; Hence the name "proper time" for τ.

Two travellers start somewhere & meet again after 2 ≠ journeys: contrarily to the Newtonian prediction, the duration of their trips as measured by their clocks is NOT the same.

This consequence of Einstein's representation of space & time was popularized in Paris in 1922 by Langevin during Einstein's lectures at the Collège de France. It is now so commonly observed that it is taken for granted (cf. C. Will)

Exit Newton's absolute time.

2. Relativistic Dynamics & electromagnetism

The relativistic eom of a point particle in a Minkowskian (ie inertial) frame is taken to be the 4D analogue of Newton's second law:

$$m\,\gamma = F \qquad \begin{cases} m: \text{inertial mass} \\ \gamma = \dfrac{d^2X^i}{d\tau^2} e_i \ : \ 4\text{-acceleration; } \tau \text{ being proper time} \\ F \equiv F^i e_i : \ 4\text{-force vector} \quad (\text{s.t. } F.U = 0) \end{cases}$$

This law is invariant under Lorentz transformation.
Uniform translation of free bodies & equivalence of inertial frames are automatically guaranteed.

At the end of 19th century the eq of motion of a charge q in an electro-magnetic field (E, B) had been obtained under the form (Lorentz).

$$ma = q\sqrt{1-\frac{V^2}{c^2}}\left(E + \frac{V}{c}\wedge B - \frac{1}{c^2}V(V.E)\right) \qquad (1)$$

where $V^\alpha = \frac{dX^\alpha}{dt}$ are the cartesian components of its 3-velocity, t is Newton's universal time, and $a^\alpha = dV^\alpha/dt$. E & B are cartesian 3-vectors evaluated on the trajectory of the charge. Finally c is numerically equal to the speed of light.

If E & B were invariant when going to another inertial frame (just like gravity) then (1) would not be invariant. Hence the 19th century hope that this would allow to determine experimentally the "absolute" reference frame where (1) holds.

Experiments failing to pick up that frame, it was inferred (Lorentz Poincaré) that in order for (1) to remain invariant the electric & magnetic fields had to be represented by ≠ vectors in 2 ≠ inertial frames: "$E_x = E'_x$; $E_y = \frac{E'_y + V_0/c\, B'_z}{\sqrt{1-V_0^2/c^2}}$, $E_z = \frac{E'_z - V_0/c\, B'_y}{\sqrt{1-V_0^2/c^2}}$ (and similar expressions for B) AND THAT "auxiliary variables" had to be introduced:

$$X' = \frac{X - V_0 t}{\sqrt{1-V_0^2/c^2}}\ ; \quad t' = \frac{t - V_0 X/c^2}{\sqrt{1-V_0^2/c^2}}\ ;$$

Then, and only then did (1) become invariant:

$$ma' = q\sqrt{1-\frac{V'^2}{c^2}}\left(E' + \frac{V'}{c}\wedge B' - \frac{1}{c^2}V'(V'.E')\right)$$

with $V'^\alpha = dX'^\alpha/dt'$ & $a'^\alpha = dV'^\alpha/dt'$.

The physical relevance of the transformation laws for E & B had been confirmed experimentally (by measurements of E', B' in (S')) but only Einstein in 1905 gave t' its status as "time, pure and simple" as measured by a clock at rest in (S').

In Minkowski representation of space-time (1) reads

$$m\gamma^i = F^i \quad \text{with} \quad F^i = \frac{q}{c}F^i{}_j U^j \ ; \quad F_{ij} = \partial_i A_j - \partial_j A_i$$

with $A^i = (\Phi; A)$ identified with electro-static & magnetic potentials. Again, invariance under Lorentz transformations is obvious.

Similar comments can be made about Maxwell's equations, which imply,

$$-\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} + \Delta E = 0 \quad \text{outside the charges creating } E.$$

This equation is the same as the equation of propagation of sound waves with respect to the air in a frame where the air is at rest. Again the hope was to use its non invariance in the transformation: $X' = X - V_0 t$ to find the "absolute" frame where ether is at rest; this proved experimentally impossible and the Lorentz-Poincaré transformation was shown to leave it invariant.

Again this invariance is manifest in the relativistic framework, as the Maxwell equations read:

$$F_{[ij,k]} = 0 \quad ; \quad F^{ij}{}_{,j} = \frac{4\pi}{c} j^i \qquad (\text{NB: } {}_{,i} \equiv \partial_i \equiv \frac{\partial}{\partial x^i})$$

where $j^i = (c\rho, j)$ is identified with the density of charge & the electric current

(VI) General Covariance & accelerated frames

1. Accelerated frames & gaussian coordinates

In Newtonian mechanics we considered 4 ≠ frame/coordinate transformations

(1) $S \longrightarrow$ another cartesian system S'
(2) $S \rightarrow$ another inertial frame in uniform translation
(3) $S \rightarrow$ another, accelerated, rigid frame
(4) $X^\alpha \rightarrow x^\alpha$ is going from cartesian to general coord.

This complexity is due to the structure of Newton space-time $N_4 = \mathcal{E}_3 \times R$

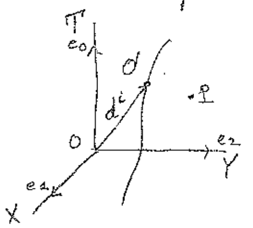
(1) & (4) are changes within $\mathcal{E}_3$ (& its tangent, isomorphic, spaces)
(2) & (3) define families of frames 1 for each sheet of N_4

In Minkowski space-time only coordinate transformations, $X^i = X^i(x^j)$, linear or not are at our disposal.

Thus the Lorentz-Poincaré transformations, which are linear, unify (1) & (2).

Non-linear transformations $X^i \to x^i = x^i(X^j)$ will hence define the passage from an inertial to an accelerated reference system.

Example Consider in an inertial frame a worldline representing the motion of a point-particle O'



its 4-position vector is $OO' = d^i(\tau)\, e_i$;
$u = U^i e_i$ with $U^i \equiv \dot{d}^i$ and $U^i U_i = -c^2$
is its 4-velocity and τ its proper time.
at O' introduce a "tangent inertial frame":
$$X'^j = \Lambda_i{}^j(\tau)\,(X^i - d^i(\tau)) \; ; \quad e_i = \Lambda_i{}^j e'_j$$
such that $e'_0 = u(\tau)$, which gives 3 components of the transformation ($\Lambda^i{}_0 = U^i(\tau)$) ; the 3 spatial vectors e'_α are determined by a choice of the 3-parameters defining a spatial rotation (e.g. ~~the 3~~ Euler angles), hence the 6-parameters of $\Lambda_i{}^j$ are known. (One may choose $e'_1 \,//\, \gamma$ since $\gamma \cdot u = 0$).

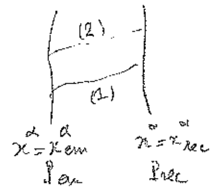
Consider now a point P : $OP = OO' + O'P$, that is : $X^i e_i = d^i e_i + X'^i e'_i$. If P is not too far from the worldline, there $\exists$! a unique τ such that $O'P$ is $\perp$ to u, that is such that :
$X^i e_i = d^i e_i + X'^\alpha e'_\alpha$, $\alpha = 1 2 3$

Hence the event P can be characterized either by its 4 Minkowskian coordinates X^i or by $x^0 \equiv \tau$ and $x^\alpha \equiv X'^\alpha$ which are related by :
$$X^i = d^i(x^0) + \Lambda^i{}_\alpha(x^0)\, x^\alpha \qquad (\text{since } e'_\alpha = \Lambda^i{}_\alpha e_i)$$
This relation is not linear ; hence in the x^i coordinate system the distances between two events is no longer manifestly given by Pythagoras theorem :
$$dl^2 = \ell_{ij}\, dx^i dx^j \qquad \left(\ell = \ell_{ij}\, dx^i dx^j \text{ with } dx^i \text{ conjugate to } \frac{\partial}{\partial x^i}\right)$$
with $\ell_{ij} = \dfrac{\partial X^k}{\partial x^i}\dfrac{\partial X^\ell}{\partial x^j}\eta_{k\ell} \neq \eta_{ij}$

How are (x^0, x^α) "materialized"?

- $t = x^0/c$ is a "time coordinate" ; the (proper) time measured by clocks at rest in (S') is $\tau = \int \sqrt{-\ell_{00}}\, dt$
- The notion of 3-D rigid reference frame looses meaning ; indeed, rigid with respect to which time ? and is abandonned.

Going to accelerated frames may simplify calculations (pirates in E_3 !). Ex :
Consider 2 bodies at rest in the reference system x^i, that is whose motion is represented by the 2 world lines $x^\alpha = x^\alpha_{rec}$, $x^\alpha = x^\alpha_{em}$.



Their proper times are related to $x^0/c \equiv t$ by :
$$d\tau_{\substack{em\\rec}} = \sqrt{-\ell_{00}(t, x^\alpha_{\substack{em\\rec}})}\, dt$$

Suppose P_{em} sends light signals to P_{rec} with period $\Delta\tau_{em}$

light travels along paths of zero length : $x^i = x^i(\lambda)$ with $\ell_{ij}(\dot{x}^i \dot{x}^j) = 0$. After integration of this equation with the appropriate initial conditions one can compute the P_{rec}-proper time interval between the 2 reception events.

If the metric is "stationary", that is does not depend on x^0, then (2) is just the time-translated of (1). $\Delta t_{rec} = \Delta t_{em}$ and :
$$\Delta\tau_{rec} = \frac{\sqrt{-\ell_{00}(rec)}}{\sqrt{-\ell_{00}(em)}}\,\Delta\tau_{em} \iff \nu_{rec} = \sqrt{\frac{\ell_{00}(em)}{\ell_{00}(rec)}}\,\nu_{em}$$
This is a Doppler effect, which of course can also be calculated in an inertial frame X^i where light follows light cones but where the world-lines of P_{em} & P_{rec} are not $X^\alpha = \text{const}$.

2. Covariant form of the eqn & geometrization of inertia

To express the laws of dynamics & electromagnetism in a form valid in any accelerated frame $x^i = x^i(X^j)$ is the 4D analogue of going from cartesian to curvilinear coordinates in E_3.

They read : $m\,\tilde{D}_u u = F \iff m\dfrac{\tilde{D}u^i}{d\tau} = \underbrace{\dfrac{\partial x^i}{\partial X^j} F^j}_{\text{components of } F \text{ in the } x^i \text{ system}}$

where $u^i = \dfrac{dx^i}{d\tau}$; $u^i u_i = \ell_{ij} u^i u^j = -c^2$

and $\dfrac{\tilde{D}u^i}{d\tau} = \dfrac{du^i}{d\tau} + \tilde{\Gamma}^i_{jk}\dfrac{du^j}{d\tau}\dfrac{du^k}{d\tau}$ $\begin{cases} \text{with } \tilde{\Gamma}^i_{jk} = \dfrac{\partial^2 X^m}{\partial x^j \partial x^k}\dfrac{\partial x^i}{\partial X^m} \\ \text{or } \tilde{\Gamma}^i_{jk} = \frac{1}{2}\ell^{im}(\partial_j \ell_{mk} + \partial_k \ell_{mj} - \partial_m \ell_{ij}) \end{cases}$

From a mathematical point of view going from d to $\tilde{D}$ is rather trivial. From a physical point of view this "general covariance" of the laws of mechanics, that is the fact that they have the same form in all reference frames encodes an aspect of general relativity: all coordinate systems are on the same footing.
Another important consequence of being able to write the eom as

$m\,\tilde{D}_u u' = F$ is that is replaces Newton's 2nd law written

in an accelerated frame: $m a' = F + m(-2\Omega \wedge v + \dots)$. Hence the covariant derivative "absorbs" the inertial accelerations which are "hidden" in the Christoffel symbols $\tilde{\Gamma}^i_{jk}$.

Minkowski spacetime which is an elegant framework to formulate electromagnetism yields as a "bonus" the geometrisation of inertial forces

3. The example of a rotating frame

Consider $(T, XYZ) \longrightarrow (t\,r\,\varphi\,z)$: $T = t$, $X = r\cos(\varphi + \Omega t)$; $Y = r\sin(\varphi + \Omega t)$, Z
Find that $ds^2 = \eta_{ij}\,dX^i dX^j = -dt^2(1 - r^2\Omega^2) + 2r^2\Omega\,dt\,d\varphi + dr^2 + r^2 d\varphi^2 + dz^2$.
Consider the world-line $x^i_a = (t, R, 0, 0)$ which represents an "observer" rotating with angular velocity Ω in the $(TXYZ)$ frame.

Consider 2 particles $x^i_\pm = (t, R, \omega_\pm t, 0)$ $\quad(\omega_+ > 0,\ \omega_- < 0)$
the coordinate time to go round the circle is $t_\pm = \pm 2\pi/\omega_\pm$
Since P_a-proper time is $d\tau_a = \sqrt{1 - R^2\Omega^2}\,dt$, the time interval between the arrivals of $P_\pm$ at P_a is $\Delta\tau_a = (t_+ - t_-)\sqrt{1 - R^2\Omega^2}$

$$= 2\pi\,\frac{\omega_+ + \omega_-}{\omega_+ \omega_-}\sqrt{1 - R^2\Omega^2}$$

Now the 4-velocities of $P_\pm$ are $u^i_\pm = \dfrac{dx^i_\pm}{d\tau_\pm}$ with $d\tau_\pm = dt\sqrt{1 - R^2(\Omega + \omega_\pm)^2}$

so that $u^i_\pm = \dfrac{(1, 0, \omega_\pm, 0)}{\sqrt{1 - R^2(\Omega + \omega_\pm)^2}}$.

Now consider the case when $\dfrac{\omega_+}{\sqrt{1 - R^2(\Omega + \omega_+)^2}} = -\dfrac{\omega_-}{\sqrt{1 - R^2(\Omega + \omega_-)^2}}$

which yields $\dfrac{\omega_+ + \omega_-}{\omega_+ \omega_-} = \dfrac{2R^2\Omega}{1 - \Omega^2 R^2}$

Hence find: $\Delta\tau_a = \dfrac{4\pi R^2 \Omega}{\sqrt{1 - R^2\Omega^2}}$ (true also for light-rays)

Let λ be the (de Broglie) wavelength of the particles:

$$\boxed{\ } \quad \frac{\Delta\lambda}{\lambda} = \frac{4S}{\lambda}\,\frac{\beta}{R\sqrt{1 - \beta^2}} \qquad \text{with } S = 4\pi R^2 \ \ \& \ \beta = \frac{\Omega R}{c}$$

This is the well-known Sagnac effect, the calculation having been performed in the rotating frame.

Comment: in the framework of a Newtonian corpuscular theory of light, $\Delta t = 0$ since the initial velocities are the same wrt the rotating frame.
in a Newtonian wave-theory of light, the speeds are the same with respect to Ether (not the rotating frame) and a fringe displacement should be observed. The experiment was suggested by O. Lodge (1897); Sagnac performed it in 1913 after detailed calculations made in 1905.
This experiment, just like Foucault's pendulum, measures the "absolute" rotation of a frame wrt all inertial frames.

Let us perform another coordinate transformation: $(t\,r\,\varphi\,z) \to (t\,x\,y\,z)$:

$$t = T,\quad x = r\cos\varphi,\quad y = r\sin\varphi;\quad z = z.$$

In this new system:

$$ds^2 = -[1 - \Omega^2(x^2 + y^2)]\,dt^2 + 2\Omega\,dt\,(x\,dy - y\,dx) + dx^2 + dy^2 + dz^2$$

The Christoffel symbols read:

$$\tilde{\Gamma}^x_{tt} = -\Omega^2 x - \frac{d\Omega}{dt}\,y;\quad \tilde{\Gamma}^y_{tx} = -\tilde{\Gamma}^x_{ty} = \Omega;\quad \tilde{\Gamma}^y_{tt} = -\Omega^2 y + \frac{d\Omega}{dt}\,x$$

so that the equation of motion of a free particle, $\tilde{D}_u u = 0$, reads

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= 2\Omega\frac{dy}{dt} + \Omega^2 x + \frac{d\Omega}{dt}\,y \\ \frac{d^2y}{dt^2} &= -2\Omega\frac{dx}{dt} + \Omega^2 y - \frac{d\Omega}{dt}\,x \end{aligned}\right\} \Longleftrightarrow \begin{aligned} &a' = -2\Omega \wedge v' + \Omega \wedge (R \wedge \Omega) - \frac{d\Omega}{dt} \wedge R, \\ &R = (x\,y\,0);\ v' = \frac{dR'}{dt};\ a' = \frac{dv'}{dt} \end{aligned}$$

Explicit example of "$\tilde{\Gamma}$ = inertia".

VII Inertial forces & gravity

1. Recapitulation

19th century physicists had tried to measure the speed of Earth wrt Ether and had failed: the speed of light is c wrt the solar system in which Ether was believed to be at rest (star aberration) AND wrt the Earth (Michelson Morley exp.)

Innumerable hypotheses had to be made on the "Electrodynamics of moving objects" to force Lorentz' & Maxwell's equations to abide by this impossibility to find the frame where Ether was at rest, within Newton's theory.

Einstein in 1905 showed how one could reconcile the observed constancy of the speed of light, the equivalence of all inertial frames & the law of composition of velocities by formulating the laws of Physics within a new representation of Space & Time: Special Relativity.

In Special Relativity the class of inertial systems was still privileged: only in an inertial system can a free particle be at rest. To maintain a free particle at rest in an accelerated frame one must subject it to inertial forces, which are a kind of reminder of the "absoluteness" of inertial frames

As soon as 1907 Einstein's ambition was to build a theory where no frames would be privileged, in which the laws of Physics would be the same in all frames, that is a theory of "general" relativity.

A first aspect of general relativity is "general covariance" which consist in writing the laws of dynamics in a form which holds in all frames. For example: $m\frac{du}{d\tau} = F \longrightarrow m\, D_u u = F$.

As already emphasized this covariant formulation of the laws of physics gives a geometrical origin to the inertial forces since the non-inertial character of the frame is encoded in the Christoffel symbols, that is in the coefficients of the metric.

A second aspect of Einstein's project, much richer, was to try and give a geometrical origin to ALL forces, inertial AND real; to, somehow, absorb the F vector into a D operator. Thus all forces would be a manifestation of geometry, all motions would be free, but would take place within spacetimes endowed with a more complicated structure than previously imagined.

We know that he succeeded to reduce gravity to geometry in 1916. Other forces still resist...

2. The equivalence principle

The "tool" that Einstein used to break into pieces Minkowski space-time was the curious fact that all bodies fall in the same way in a gravitational field, just like all free particles "fall" in the same way when observed in an accelerated frame.

This is due to the "accidental" equality of inertial & gravitational masses

Einstein decided to take this equality as the founding stone of his theory and to IDENTIFY gravity and inertia; this is the principle of equivalence.

Thus there must be ways to "ignore" gravity, at least approximately, by going to a "falling frame", and accelerated observers can legitimately attribute their observations (redshifts for example) to gravity rather than inertia. A consequence is that the notion of free particle looses its meaning: a free particle is submitted to an inertial force field in the frame of an accelerated observer; a particle falling in a gravity field appears to be free to a comoving observer. Hence the notion of inertial systems of reference must be abandoned too.

Let's be more accurate: The inertial forces acting on free particles can be cancelled by going to an inertial frame; this cancellation is "global" since in the inertial frame their motion is forever uniform. On the other hand the cancellation of gravity in a comoving frame can only be local: the distance between particles at rest slowly decreases due to the fact that their trajectories converge towards the attracting body.

We are thus led to a representation of the space time structure in which the laws of gravity will be formulated, consisting in a kind of Minkowski spacetime split into pieces, a "mosaic of Minkowski spacetime chips": in each "chip", that is locally, space-time is Minkowskian, gravity may be ignored & all laws of special relativity apply. (The melting polar cap at Spring turning in the icebergs in the sea!)

Hence in Einstein's geometrical vision of gravity, spacetime first reduces to a simple "continuum", an ensemble of points labelled in an arbitrary way, each neighbourhood of points being a small Minkowski "island".

3. A pseudo-Riemannian space-time

The continuum described above is a "manifold", that is an ensemble of points labelled with coordinates x^i; a "tangent" vector space being associated to each point. A each point one can therefore define vectors, forms & tensors.

In order to be able to compare quantities defined at ≠ points (i.e. in order to "connect" the M_4 "chips") the manifold must be endowed with an extra mathematical structure. This is the rôle of a connexion which defines parallel transport along a curve connecting two points. An "auto parallels" or "straight line" is thus defined as:

$$\begin{cases} P = P(\lambda) \quad [x^i = x^i(\lambda)] \quad ; u = \frac{dP}{d\lambda} \quad [u^i = \frac{dx^i}{d\lambda}] : \\ D_u u = 0 \quad \Longleftrightarrow \quad \frac{du^i}{d\lambda} + \Gamma^i_{jk} u^j u^k = 0 \end{cases}$$

Where the 64 functions $\Gamma^i_{jk}(x^m)$ define the connexion.
IF ~~there~~ ~~exist~~ other coordinates $X^i(x^m)$ such that they all vanish (which is very restrictive, because it implies that the 64 functions can all be expressed in terms of the 4 functions $X^i(x^m)$) then the manifold is said to be flat. Otherwise it is "curved".

In order now to measure distances between points, the manifold must be endowed with an additional structure, a metric:

$$g = g_{ij}\, dx^i dx^j$$

where the 10 functions g_{ij} define the distance between points. With a metric one can thus build "geodesics" that is curves of extremal length. In general g_{ij} cannot reduce to η_{ij} everywhere by $x^i \to x^i(X^j)$;

The notions of parallelism & distance are distinct but a metric g defines a unique (symmetric) connexion, called Levi-Civita connexion by the condition: $D_u g = 0 \quad \forall u$. The connexion coefficients Γ^k_{ij} are then called Christoffel symbols and are given in terms of g_{ij} by:

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} (\partial_j g_{km} + \partial_k g_{mj} - \partial_m g_{jk}) \quad (\partial_m = \frac{\partial}{\partial x^m})$$

Then autoparallels & geodesics coincide.

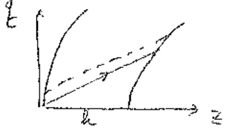
In General Relativity space & time are thus represented by a 4 Dimensional "pseudo-Riemannian" space, that is a manifold endowed with a metric & associated Levi-Civita connexion. This space time in general is curved since in general parallels intersect. In the exceptional case when it is flat then there ∃ special (Minkowski) coordinates in which the 10 functions g_{ij} can all be expressed in terms of 4 functions $X^i(x^j)$ and reduce to η_{ij} at all points. (I followed here Shroedinger's thread of argument, cf his book)

In newtonian physics & in special relativity the metric tensor is given a priori; in GR it will be determined, through Einstein's field equations, to the matter content of the Universe. In GR then space-time is no longer a neutral container of matter. In this way, GR embodies some ideas developped by Mach such as: the inertia of a particle should emerge from interactions with distant bodies — which translates in GR into the identification of inertial & gravitational forces. On the other hand GR fails to embody the Descartes-Mach idea that spacetime should not exist in the absence of matter since in GR empty space ∃ : M_4.

4. Gravitational redshift

Einstein's "gedanken experiment" 1907-1911 to illustrate the richness of the equivalence principle:

In an inertial frame consider a "tower" of lenght h accelerated along the z axis:



2 light signals are sent from the bottom to the top of the tower

$$\Delta\tau_{em} = \Delta t_{em}\sqrt{1-\frac{v_e^2}{c^2}} = \Delta t_{em} \quad (v_e = 0)$$

The signal arrives at $t \sim h/c$; the velocity of the tower is then $v \sim gh/c$ (g: acceleration of the tower)

$$\Delta t_{rec} \sim \Delta t_{em}\left(1+\frac{v}{c}\right) = \Delta t_{em}\left(1+\frac{gh}{c^2}\right) \sim \Delta\tau_{em} \text{ at first order}$$

Hence: $\Delta\tau_{rec} \sim \Delta\tau_{em}\left(1+\frac{gh}{c^2}\right)$ (Doppler effect).

The principle of equivalence states that acceleration = gravity

Hence the same redshift must be observed in a "tower" at rest in the Earth gravity field. Then $z=0$ is the bottom & $z=h$ the top of the tower and g is the Earth gravity field.

Introducing the Earth gravitational potential:

$$U_{bottom} = -\frac{GM_\oplus}{R_\oplus} \quad ; \quad U_{top} = -\frac{GM_\oplus}{R_\oplus + h} \sim -\frac{GM_\oplus}{R_\oplus}\left(1-\frac{h}{R_\oplus}\right)$$

$$\Rightarrow U_{bottom} - U_{top} = -\frac{GMh}{R_\oplus} \quad ; \quad \text{now}: \ g = \frac{GM}{R_\oplus^2}$$

hence $U_{bottom} - U_{top} = -gh$ so that:

$$\nu_{top} = \nu_{bottom}\left(1 \mp \frac{U_{bottom} - U_{top}}{c^2}\right). \quad (1)$$

- We saw how to compute redshifts in an accelerated frame (p28):

  $\eta_{ij} \longrightarrow g_{ij}$: $\nu_{top} = \nu_{bott}\sqrt{\frac{g_{00}(bottom)}{g_{00}(top)}}$

- Therefore $g_{00} \sim -\left(1+\frac{2U}{c^2}\right)$ where U is Newton's gravitational potential

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(1) was measured by Pound & Rebka, Vessot-Levine, Caroll-Alley.
GPS would not work if it was ignored (cf C. Will)