GR-IHP 06
N. Deruelle
PART 2

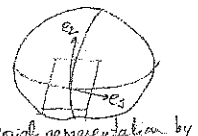
Einstein's gravitational field equations

I. Introduction to the Riemann tensor

1. Connected manifolds

Consider an ensemble of points characterized by their "coordinates" x^i (keeping in mind that one may require several "maps" to form a "chart" of the whole ensemble – cf the example of S_2).

To each point we associate a (4-D) vectorial space: the "tangent space" with "natural basis" $e_i \equiv \frac{\partial}{\partial x^i}$; The "cotangent" space of one-forms is spanned by the conjugate basis: dx^i such that $dx^i\left(\frac{\partial}{\partial x^k}\right) = \delta^i_k$



(pictorial representation by means of embedding)

Type $\binom{p}{q}$ tensors are then defined as:

$$T = t^{i_1 \dots i_p}_{\ \ j_1 \dots j_q}\ \partial_{i_1} \otimes \dots \otimes \partial_{i_p} \otimes dx^{j_1} \dots \otimes dx^{j_q}$$

where, e.g. $\partial_i \otimes dx^j$ acts on the pair of a 1-form $\lambda = \lambda_\ell dx^\ell$ & vector $u = u^m \partial_m$ to give the function $\lambda_i u^j$ evaluated at P.

In a change of coordinates, and hence of natural bases, the components of a tensor transform as:

$$t^{i_1 \dots i_p}_{\ \ j_1 \dots j_q} \longrightarrow t'^{i_1 \dots i_p}_{\ \ j_1 \dots j_q} = \frac{\partial x'^{i_1}}{\partial x^{k_1}} \dots \otimes \frac{\partial x^{l_1}}{\partial x'^{j_1}} \dots t^{k_1 \dots k_p}_{\ \ l_1 \dots l_q}$$

A "connected manifold" is endowed with a "covariant derivative", that is an operator which associates to a $\binom{p}{q}$ tensor T the $\binom{p}{q+1}$ tensor with components:

$$D_\ell t^{i_1 \dots i_p}_{\ \ j_1 \dots j_q} = \partial_\ell t^{i_1 \dots i_p}_{\ \ j_1 \dots j_q} + \Gamma^{i_1}_{\ell m} t^{m i_2 \dots i_p}_{\ \ j_1 \dots j_q} + \dots - \Gamma^m_{\ell j_1} t^{i_1 \dots i_p}_{\ \ m j_2 \dots j_q} + \dots$$

In a more intrinsic language: $D_{\partial_\ell} T \equiv D_\ell T = \left(D_\ell t^{i_1 \dots i_p}_{\ \ j_1 \dots j_q}\right) \partial_{i_1} \otimes \dots dx^{j_q}$

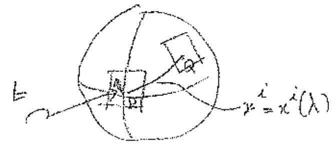
(then $D_\ell T$ is also a $\binom{p}{q}$ tensor).

Even more generally: $D_v T = v^i D_i T$.

The functions Γ^i_{jk} DEFINE D.

2. Connexion and parallel transport

A connexion associates to a tensor T, defined at P, another one, DT, also defined at P. But, just like any derivative operator [recall the familiar definition: $\frac{df}{dx} = \lim_{x \to 0} \frac{f(x+dx) - f(x)}{dx}$] it involves neighbouring points and can be used, via Taylor expansion, to "transport" a tensor from a point to another.



Consider a curve $x^i = x^i(\lambda)$ with tangent vector field $u^i = \frac{dx^i}{d\lambda}$. The equation of // transport of a vector field t along the curve is:

$$\frac{Dt^i}{d\lambda} = u^j D_j t^i = \frac{dt^i}{d\lambda} + \Gamma^i_{jk} u^j t^k = 0 \qquad (1)$$

or, equivalently, by:

$$D_u t = D_{u^j \partial_j}(t^i \partial_i) = u^j D_j (t^i \partial_i) = u^j [(\partial_j t^i)\partial_i + (D_j \partial_i) t^i]$$

$$D_j \partial_i \equiv \Gamma^k_{ji} \partial_k$$

$$= u^j (\partial_j t^i + \Gamma^i_{jk} t^k)\partial_i = 0.$$

Consider a vector field with components $t^i(x^k)$ at x^k, and components:

$$t^i(x^k + dx^k) = t^i(x^k) + \partial_j t^i dx^j + \frac{1}{2} \partial_{\ell j} t^i dx^\ell dx^j + \dots$$

at $(x^k + dx^k)$.

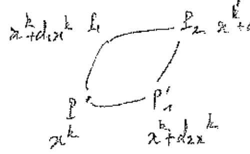
We want to compare $t^i(x^k + dx^k)$ with the vector $t^i_{/\!/}(x^k + dx^k)$ which is deduced from $t^i(x^k)$ by parallel transport from x^k to $x^k + dx^k$. We have $D_j t^i = \partial_j t^i + \Gamma^i_{jk} t^k$. In agreement with (1) we define $t_{/\!/}$ as the vector field such that, at x^k: $t_{/\!/}(x^k) = t(x^k)$ and $\partial_j t^i_{/\!/} = -\Gamma^i_{jk} t^k$, so that: $\partial_{\ell j} t^i_{/\!/} = -(\partial_\ell \Gamma^i_{jk}) t^k + \Gamma^i_{jk} \Gamma^k_{\ell m} t^m$. Hence we have, by Taylor expansion:

$$t^i_{/\!/}(x^k + dx^k) = t^i(x^k) - \Gamma^i_{jk} t^k dx^j + \frac{1}{2}\left(-\partial_\ell \Gamma^i_{jm} + \Gamma^i_{jk}\Gamma^k_{\ell m}\right) t^m dx^\ell dx^j + \dots$$

(In flat space & cartesian coordinates when $\Gamma^i_{jk} = 0$, we recover a familiar result).

3. Path dependence of // transport & Riemann tensor

Consider // transport along the two ≠ infinitesimal paths connecting p & P_2:



By iteration we obtain (easy exercise):

$$t_{//}^i(x^k+d_1x^k+d_2x^k)-t_{//}^i(x^k+d_2x^k+d_1x^k)=R^i{}_{mj\ell}\,t^m d_1x^\ell d_2x^k \quad (*)$$

(Riemann tensor) with: $\boxed{R^i{}_{mj\ell}=\partial_j\Gamma^i_{\ell m}-\partial_\ell\Gamma^i_{jm}+\Gamma^i_{jp}\Gamma^p_{\ell m}-\Gamma^i_{\ell p}\Gamma^p_{jm}.}$

$R^i{}_{mj\ell}$ is a type $\binom{1}{3}$ tensor. Indeed the left-hand side of $(*)$ is a vector field. so is the r.h.s which is the contraction of $R^i{}_{mj\ell}$ with 3 vectors (t^m, d_1x^ℓ, d_2x^k)

(Another, not very smart, way to prove that $R^i{}_{mj\ell}$ is a $\binom{1}{3}$-tensor is to see how it transforms in the coordinate change $x^i \to x'^i = x^i(x^j)$, knowing that $\Gamma \to \Gamma' = \frac{\partial}{\partial}\frac{\partial}{\partial}\frac{\partial}{\partial}\Gamma + \frac{\partial^2}{\partial\partial}\frac{\partial}{\partial}$ and seeing explicitely that all extra-terms cancel out so that: $R' = \frac{\partial}{\partial}\frac{\partial}{\partial}\frac{\partial}{\partial}\frac{\partial}{\partial}R$. This explicit calculation also shows that if there ∃ 4 functions $X^i(x^j)$ such that Γ reduces to the "extra term", that is $\Gamma^i_{jk}=\frac{\partial^2X^m}{\partial x^j x^k}\frac{\partial x^i}{\partial X^m}$ then $R^i_{j\ell m}\equiv 0$

In that case then the connexion is flat ($\Gamma^i_{jk}=0$ in the X^m system), the Riemann tensor is identically zero & parallel transport is path independent.)

4. Commutation of covariant derivatives & Riemann tensor

$D_if=\partial_if$ if f is a function can be seen as the i-component of the form $(D_if)dx^i$; Hence the components of its covariant derivative are:

$$D_iD_jf = \partial_{ij}f - \Gamma^k_{ij}\partial_kf \qquad \text{(components of a } \tbinom{0}{2}\text{-tensor)}.$$

Hence we have. $(D_iD_j-D_jD_i)f = -(\Gamma^k_{ij}-\Gamma^k_{ji})\partial_kf \equiv -T^k_{ij}\partial_kf$

or, equivalently: $D_j\partial_i=\Gamma^k_{ji}\partial_k \Rightarrow D_j\partial_i - D_i\partial_j = -T^k_{ij}\partial_k$

T^k_{ij} are the components of a $\binom{1}{2}$-tensor: the "torsion"

(In GR the connexion is torsion-free: $\Gamma^k_{ij}=\Gamma^k_{ji}$.)

let us apply now the operator D_iD_j on the basis vector ∂_k; the result is also a vector:

$$D_iD_j(\partial_k) = D_i(\Gamma^\ell_{jk}\partial_\ell) = (D_i\Gamma^\ell_{jk})\partial_\ell + \Gamma^\ell_{jk}\Gamma^m_{i\ell}\partial_m.$$

so that: $(D_iD_j-D_jD_i)\partial_k = R^m{}_{kij}\partial_m$ (these are vectors)

similarly: $(D_iD_j-D_jD_i)dx^k = -R^k{}_{mij}dx^m$ etc.

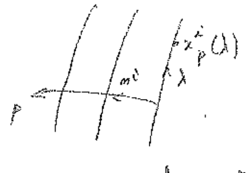
Remark: Let's compute in a pedestrian way $D_iD_jt^k$; t^k are the components of a vector; D_jt^k are the components of a $\binom{1}{1}$-tensor, hence $D_iD_jt^k$ are the components of a $\binom{1}{2}$-tensor given by:

$D_iD_jt^k = \partial_i(D_jt^k) - \Gamma^\ell_{ij}D_\ell t^k + \Gamma^k_{i\ell}D_jt^\ell$, which yields:

$\underline{(D_iD_j-D_jD_i)t^k} = R^k{}_{mij}t^m - T^\ell_{ij}D_\ell v^k$ where T^ℓ_{ij} is the torsion

components of $(D_iD_j-D_jD_i)t^k\,\partial_k\otimes dx^i\otimes dx^j$.

5. Geodesic deviation and Riemann tensor



Consider a family of curves parametrized by λ and labelled by p: $x^i=x^i_p(\lambda)$.

Introduce the tangent vector $u^i=\frac{dx^i}{d\lambda}$ and the separation vector $n^i=\frac{\partial x^i}{\partial p}$.

We have $\frac{Du^i}{dp}-\frac{Dn^i}{d\lambda} = n^jD_ju^i - u^jD_jn^i = n^j\partial_ju^i - u^j\partial_jn^i$

if $\Gamma^k_{ij}=\Gamma^k_{ji}$ (torsion free D)

In that case then: $\frac{Du^i}{dp}-\frac{Dn^i}{d\lambda} = \frac{\partial x^j}{\partial p}\frac{\partial}{\partial x^j}\frac{\partial x^i}{\partial\lambda} - \frac{\partial x^j}{\partial\lambda}\frac{\partial}{\partial x^j}\frac{\partial x^i}{\partial p} = 0$

Hence $a^i = \frac{D^2n^i}{d\lambda^2} = \frac{D}{D\lambda}\frac{Dn^i}{d\lambda} = \frac{D}{d\lambda}\frac{Du^i}{dp} = u^jD_j(n^kD_ku^i)$

$= (u^jD_jn^k)D_ku^i + u^jn^k\underbrace{D_{jk}u^i}_{=D_{kj}u^i+R^i{}_{mjk}u^m}$

Now, if the curves are autoparallels, we have $\frac{Du^i}{d\lambda}=0$ and hence:

$0=\frac{D}{dp}\frac{Du^i}{d\lambda} = n^jD_j(u^kD_ku^i) = n^ju^kD_{jk}u^i + \underbrace{(n^jD_ju^k)}_{=u^jD_jn^k}D_ku^i$

$= n^ku^jD_{kj}u^i + (u^jD_jn^k)D_ku^i$

So that: $a^i = R^i{}_{mjk}u^mu^jn^k$: Riemann tensor measures the tendency of // to intersect.

(II) Differential geometry & the curvature tensor

1. Lie brackets & Jacobi identity

Recall that a vector is a derivation operator : $v \equiv \partial_v = v^i \partial_i$.
The "commutator", or Lie bracket of 2 vectors v & w is defined as:

$$[v,w] \equiv v \circ w - w \circ v \equiv \partial_v \circ \partial_w - \partial_w \circ \partial_v$$
$$= v^i \partial_i \circ w^j \partial_j - w^i \partial_i \circ v^j \partial_j = (v^i \partial_i w^j - w^i \partial_i v^j)\, \partial_j \quad : \text{it is a vector.}$$

Note that $[\partial_i, \partial_j] = 0$; (being a vector the Lie bracket defines a derivation operation: the "Lie derivative", see below).
It is an easy exercise to show the Jacobi identity:

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0.$$

(2) Torsion of a covariant derivative

The torsion, T, of a covariant derivative D is a $\binom{1}{2}$ tensor which acts on pairs of vectors (v,w) to give another vector according to:

$$T(v,w) \equiv D_v w - D_w v - [v,w]$$

It is not a priori obvious that $T(v,w)$ is indeed a vector. To show it and make the link with the previous definition, decompose $v = v^i \partial_i$, $w = w^j \partial_j$, recall that $D_i \partial_j = \Gamma^k_{ij}$ and get:

$$T(v,w) = T^k_{ij}\, v^i w^j \partial_k \quad \text{with } T^k_{ij} \equiv \Gamma^k_{ij} - \Gamma^k_{ji}.$$

(3) Curvature of a covariant derivative

The curvature R (or Riemann-Christoffel tensor) of a covariant derivative D is a $\binom{1}{3}$ tensor which acts on triplets of vectors $(u\,v\,w)$ to give another vector according to:

$$R_{u,v}\, w \equiv [D_u, D_v]\, w - D_{[u,v]}\, w \qquad \text{where } [D_u, D_v]\, w = D_u(D_v w) - D_v(D_u w).$$

Again it is not a priori obvious that $R_{u,v}\, w$ is indeed a vector. This can be shown as before by decomposing u, v & w on a natural basis.

To make the link with the previous definition choose $u = \partial_i$; $v = \partial_j$; $w = \partial_k$ and get rightaway: (since $[\partial_i, \partial_j] = 0$):

$$R_{\partial_i, \partial_j}\, \partial_k = [D_i, D_j]\, \partial_k = D_i(D_j \partial_k) - D_j(D_i \partial_k)$$
$$= R^m{}_{kij}\, \partial_m$$

(4) Properties of the curvature tensor

$$\begin{cases} R^i{}_{jkl} = - R^i{}_{jlk} & \text{by definition} \\ R^i{}_{jkl} + R^i{}_{klj} + R^i{}_{ljk} = 0 & \text{if the torsion vanishes (this property follows from Jacobi identity on Lie brackets) (1}^{\text{st}}\text{ Jacobi identity)} \\ D_m R^i{}_{jkl} + D_k R^i{}_{jlm} + D_l R^i{}_{jmk} = 0 & \text{if the torsion vanishes (2nd Jacobi identity)} \end{cases}$$

The intrinsic versions of these properties read:

$$\begin{cases} R_{u,v}\, w = - R_{v,u}\, w \\ R_{u,v}\, w + R_{v,w}\, u + R_{w,u}\, v = 0 & (\text{if } T(u,v) = 0 \text{ i.e. if } D_u v = D_v u) \\ D_u R_{vw} + D_v R_{wu} + D_w R_{uv} = 0 & (\text{" " "}) \end{cases}$$

From the curvature tensor one can extract, by contraction a type $\binom{0}{2}$ tensor, the "Ricci tensor":

$$R_{ij} \equiv R^k{}_{ikj}$$

NB: at this stage one cannot say anything about the symmetry of R_{ij} & the tensor defined as $R^k{}_{kij}$ is a priori non zero.

III Cartan's equations of structure

1. Element of exterior calculus

A type $\binom{0}{p}$ completely antisymmetric tensor is a "p-form".

Example : Maxwell's tensor $F = F_{ij}\, dx^i \otimes dx^j$
with $F_{ij} = \partial_i A_j - \partial_j A_i$; dx^i being the natural basis of the cotangent space associated with the coordinates x^i.

The exterior product of 2 1-forms is given by:

$$\lambda \wedge \mu = \lambda \otimes \mu - \mu \otimes \lambda \qquad \text{it is a 2-form}$$

Hence 1-forms decompose as $\lambda = \lambda_i\, dx^i$ (etc)
2-forms " " $F = \frac{1}{2} F_{ij}\, dx^i \wedge dx^j$

The exterior derivative of a 1-form $A = A_i\, dx^i$ is defined as:

$$dA = \partial_j A_i\, dx^j \wedge dx^i = \frac{1}{2}(\partial_j A_i - \partial_i A_j)\, dx^j \wedge dx^i$$
$$= \frac{1}{2} F_{ji}\, dx^j \wedge dx^i = F.$$

An important property of the exterior derivative is : $d^2 = 0$

Example : $F = dA \Rightarrow dF = 0 \Leftrightarrow \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$

With this preliminaries one can show that: if ω is a 1-form, so that $d\omega$ is a 2-form which acts on pairs of vectors (X,Y) to give a function, then:

$$\boxed{\text{III}} \quad d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) \qquad (1)$$

demo: $\omega = \omega_k\, dx^k$; $d\omega = \partial_\ell \omega_k\, dx^\ell \wedge dx^k = \partial_\ell \omega_k (dx^\ell \otimes dx^k - dx^k \otimes dx^\ell)$

$X = X^i \partial_i$; $Y = Y^j \partial_j$: $d\omega(X,Y) = \partial_\ell \omega_k (X^\ell Y^k - X^k Y^\ell)$
$= (\partial_\ell \omega_k - \partial_k \omega_\ell) X^\ell Y^k$

now: $X\omega(Y) = X[\omega_k\, dx^k (Y^j \partial_j)] = X[\omega_k Y^k] = X^i \partial_i (\omega_k Y^k)$
$= X^i Y^k \partial_i \omega_k + \underline{X^i \omega_k \partial_i Y^k}$

$Y\omega(X) = Y^i X^k \partial_i \omega_k + Y^i \underline{\underline{\omega_k \partial_i X^k}}$

$\omega([X,Y]) = \omega[(X^j \partial_j Y^i - Y^j \partial_j X^i)\partial_i]$
$= \omega_k (\underline{X^j \partial_j Y^k} - \underline{\underline{Y^j \partial_j X^k}})$

Hence $X\omega(Y) - Y\omega(X) - \omega([X,Y]) = d\omega(X,Y)$ qed.

2. Torsion & Cartan's first structure equation

Let h_i be a basis of the tangent space at P. (h_i is not necessarily a coordinate basis, that is $h_i = L^j{}_i\, \partial_j$ with $L^j{}_i \neq \partial x^j / \partial x'^i$).
Let θ^i be the conjugate basis of the cotangent space: $\theta^i(h_j) = \delta^i_j$.

Keep in mind identity (1) in the particular case $\omega = \theta^k$; $X = h_i$; $Y = h_j$:

$$d\theta^k(h_i,h_j) = h_i\, \theta^k(h_j) - h_j\, \theta^k(h_i) - \theta^k([h_i,h_j])$$
$$= h_i \delta^k_j - h_j \delta^k_i - [h_i,h_j]^k \leftarrow \left(k^{th} \text{ component of } [h_i h_j]. \text{ ie } [h_i h_j] = [h_i h_j]^k h_k\right)$$

$$\boxed{\text{III}} \Rightarrow \underbrace{\{d\theta^k(h_i,h_j)\}}_{\text{a function}} \underbrace{h_k}_{\text{a vector}} = -[h_i h_j]$$

Introduce now a covariant derivative with its associated connexion:

$$D_{h_i} h_j = \underbrace{\gamma^k_{ij}}_{\text{a function}} h_k \qquad ; \ \gamma^k_{ij} \text{ are "Ricci's rotation coefficients"}$$
if $h_i = \partial_i$: $\gamma^k_{ij} \equiv \Gamma^k_{ij}$: connexion coefficients

Cartan's idea is to consider the function γ^k_{ij} as the result of the action of a 1-form on a vector, that is:

$$\gamma^k_{ij} = \omega^k{}_j(h_i) \Leftrightarrow \omega^k{}_j = \gamma^k_{ij}\, \theta^i \Rightarrow D_{h_i} h_j = \omega^k{}_j(h_i)\, h_k$$
these are called "connexion 1-forms"

One is now equipped to express the torsion of the covariant derivative in terms of $\omega^k{}_j$. Indeed, by definition:

$$T[X,Y] = D_X Y - D_Y X - [X,Y] \qquad ; \text{ Choose } X = h_i ; Y = h_j :$$

$$T[h_i,h_j] = D_{h_i} h_j - D_{h_j} h_i - [h_i,h_j] = \omega^k{}_j(h_i)\, h_k - \omega^k{}_i(h_j)\, h_k + \{d\theta^k(h_i,h_j)\}\, h_k$$

Now the action of the 1-form $\omega^k{}_j$ on the vector h_k can be seen as the action of the (contracted) 2-form $\omega^k{}_\ell \otimes \theta^\ell$ on the pair of vectors (h_i,h_j) : $\omega^k{}_j(h_i) = (\omega^k{}_\ell \otimes \theta^\ell)(h_i,h_j)$.

So that:

$$T[h_i,h_j] = \{(\underbrace{\omega^k{}_\ell \otimes \theta^\ell - \theta^\ell \otimes \omega^k{}_\ell + d\theta^k}_{})(h_i,h_j)\}\, h_k$$
$$= \boxed{d\theta^k + \omega^k{}_\ell \wedge \theta^\ell \equiv \Omega^k} \ : \text{connexion 2-form}$$

And : $T = \Omega^k \otimes h_k$ These are Cartan's first structure equations

3. Curvature & Cartan's 2nd structure equations

Let us now apply the identity (1) to the connexion 1-forms $\omega = \omega^i_j$ and $X = h_k$, $Y = h_\ell$. One gets (Exercise):

$$d\omega^i{}_j(h_k, h_\ell) = h_k\,\omega^i{}_j(h_\ell) - h_\ell\,\omega^i{}_j(h_k) - \omega^i{}_j([h_k, h_\ell]) \quad (*)$$

By definition the curvature tensor is $R_{XY}Z = D_X(D_Y Z) - D_Y(D_X Z) - D_{[X,Y]}Z$

choose $X = h_i$, $Y = h_j$, $Z = h_k$.

$$R_{h_i h_j} h_k = D_{h_i}(D_{h_j} h_k) - D_{h_j}(D_{h_i} h_k) - D_{[h_i,h_j]} h_k$$
$$= D_{h_i}(\omega^\ell_k(h_j) h_\ell) - D_{h_j}(\omega^\ell_k(h_i) h_\ell) - \omega^\ell_k([h_i,h_j]) h_\ell$$

Now: $\begin{cases} D_{h_i}(f) = h_i(f) & \text{(since } h_i \text{ is a derivative)} \\ \text{and } D_{h_i} h_\ell = \omega_\ell^m(h_i) h_m \end{cases}$

$$\Rightarrow R_{h_i h_j} h_k = \left(h_i(\omega^\ell_k(h_j)) - h_j\,\omega^\ell_k(h_i) - \omega^\ell_k([h_i,h_j])\right) h_\ell +$$
$$+ \omega^\ell_k(h_j)\,\omega^m{}_\ell(h_i) h_m - \omega^\ell_k(h_i)\,\omega^m{}_\ell(h_j) h_m$$
$$= [d\omega^\ell{}_k(h_i,h_j)] h_\ell + [\omega^\ell{}_m \wedge \omega^\ell{}_k (h_i,h_j)] h_m \quad \text{because of } (*)$$
$$= \{(\underbrace{d\omega^m{}_k + \omega^m{}_\ell \wedge \omega^\ell{}_k}_{\equiv\, \Omega^m{}_k \;:\; \text{curvature 2-forms}})(h_i,h_j)\} h_m$$

Therefore. $R = \Omega^m{}_k \otimes h_m \otimes \theta^k$ with $\Omega^m{}_k = d\omega^m{}_k + \omega^m{}_\ell \wedge \omega^\ell{}_k$ — Cartan's 2nd structure equations.

If $h_i = \partial_i$ & $\theta^i = dx^i$ are natural bases, then:

$$\omega^k{}_j = \Gamma^k_{ij}\, dx^i \quad ; \quad \Omega^k = \Gamma^k_{ij}\, dx^i \wedge dx^j = T^k_{ij}\, dx^i \otimes dx^j$$
with $T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$

and:
$$\Omega^m{}_k = d(\Gamma^m_{qk} dx^q) + \Gamma^m_{p\ell} dx^p \wedge \Gamma^\ell_{qk} dx^q$$
$$= (\partial_p \Gamma^m_{qk} + \Gamma^m_{p\ell}\Gamma^\ell_{qk})\, dx^p \wedge dx^q$$
$$= (\underbrace{\partial_p \Gamma^m_{qk} - \partial_q \Gamma^m_{pk} + \Gamma^m_{p\ell}\Gamma^\ell_{qk} - \Gamma^m_{q\ell}\Gamma^\ell_{pq}}_{R^m{}_{kpq}})\, dx^p \otimes dx^q$$

which makes the link with previous definitions.

IV. Metric manifolds

1. Metric tensor & geodesics

A metric tensor, g, is a type $\binom{0}{2}$ tensor, symmetric & non degenerate:

$$g = g_{ij}\, dx^i \otimes dx^j \qquad g_{ij} = g_{ji} \qquad g^{ik} g_{kj} = \delta^i_k$$

A metric associates forms & vectors in a bijective way:

if $v = v^i \partial_i$ is a vector, then $v_i dx^i$ where $v_i = g_{ij} v^j$ is a 1-form;
similarly: if $\lambda = \lambda_i dx^i$ is a 1-form, then $\lambda^i \partial_i$ with $\lambda^i = g^{ij}\lambda_j$ is a vector.

given g, one can always diagonalize it at a given point, that is find h_i such that
$g(h_i, h_j) = 0$ if $i \neq j$ & $g(h_i, h_i) = \pm 1$

In GR $g_{ij} \longrightarrow \eta_{ij} = (-1,+1,+1,+1)$. (the metric is "Lorentzian")

a metric defines a scalar product $\langle v, v' \rangle = v.v' = g(v, v') = g_{ij} v^i v'^j$
as well as the notion of lengths & distances:

$$L[p(\lambda)] = \int_{\lambda_1}^{\lambda_2} d\lambda \sqrt{-g(u,u)} = \int_{\lambda_1}^{\lambda_2} d\lambda \sqrt{-g_{ij}u^i u^j} = \int_{\lambda_1}^{\lambda_2} d\lambda \left(-g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\right)^{1/2}$$

Since L is invariant by reparametrisation, extremising it requires a little care:
Consider an ensemble of curves $p_s(\lambda)$ & compute δL when varying s:

$$\delta L = \int_{\lambda_1}^{\lambda_2} \frac{1}{2\sqrt{-g_{ij}u^i_s u^j_s}} \left(-2g_{ij}\frac{d\delta x^i_s}{d\lambda}\frac{dx^j_s}{d\lambda} - \partial_k g_{ij}\,\delta x^k_s \frac{dx^i_s}{d\lambda}\frac{dx^j_s}{d\lambda}\right)^{1/2} d\lambda$$

One can, now, choose the parametrisation, e.g. $g_{ij}u^i_s u^j_s = -1$, setting $\lambda = \tau$:

$$\delta L = \int_{\tau_1}^{\tau_2} \left(-g_{ij}\frac{d\delta x^i}{d\tau}u^j - \frac{1}{2}\partial_k g_{ij}\,\delta x^k u^i u^j\right) d\tau$$
$$= \int_{\tau_1}^{\tau_2} \left(-\frac{d}{d\tau}(g_{ij}u^j\delta x^i) + \frac{d}{d\tau}(g_{ij}u^j)\delta x^i - \frac{1}{2}\partial_k g_{ij}\,\delta x^k u^i u^j\right) d\tau$$
$$= \int_{\tau_1}^{\tau_2} \delta x^k d\tau \left(g_{kj}\frac{du^j}{d\tau} + \partial_i g_{kj} u^i u^j - \frac{1}{2}\partial_k g_{ij} u^i u^j\right) \quad \text{if } \delta x^i = 0 \text{ at } \tau_1 \text{ \& } \tau_2$$

Hence the curve which extremizes L, called a geodesic, satisfies the eq:

$$\boxed{\frac{du^i}{d\tau} + \left\{{}^{\,i}_{jk}\right\} u^j u^k = 0} \quad \text{with } \left\{{}^{\,i}_{jk}\right\} = \frac{1}{2} g^{i\ell}(\partial_j g_{k\ell} + \partial_k g_{\ell j} - \partial_\ell g_{jk})$$

where $\left\{{}^{\,i}_{jk}\right\}$ are called "Christoffel symbols".
they are symmetric in j and k.

2. Levi-Civita connexion

In order for an auto-parallel ($D_u u = 0 \iff \frac{du^i}{d\sigma} + \Gamma^i_{jk} u^j u^k = 0$) and a geodesic to be one & the same object one must identify the connexion coefficients Γ^i_{jk} & the Christoffel symbols $\{{}^{i}_{jk}\}$.

Given a metric g with coefficients g_{ij} in a coordinate system x^i, there $\exists$ one & only one torsion-free (ie symmetric) connexion "compatible" with g (that is such that $\Gamma = \{\,\}$).

Such a connexion is called a "Levi-Civita connexion". The intrinsic way to define it is to impose:

$$D_v g = 0 \iff \partial_v (g(u,w)) = g(D_v u, w) + g(v, D_w u)$$

(the equivalence can be shown by decomposing g, v, u, w in a basis; choosing $v = \partial_i$, $u = \partial_j$, $w = \partial_k$ the second equality reads:

$$\partial_i g_{jk} = \Gamma^m_{ij} g_{mk} + \Gamma^m_{ik} g_{mj} \quad).$$

3. Metric properties of the Riemann tensor

The Riemann tensor is type $\binom{1}{3}$. Using the metric we can associate to it the $\binom{0}{4}$ tensor defined by:

$$R_{ijkl} = g_{im} R^m{}_{jkl}.$$

If the connexion is Levi-Civita then we have (exercise): $\begin{cases} R_{ijkl} = R_{klij} \\ R_{ijkl} = -R_{jikl} \end{cases}$

This metric property of the Riemann tensor implies that the Ricci tensor is symmetric: $R_{ij} = R_{ji}$

The "scalar curvature" is defined as $s = g_{ij} R^{ij}$

Using the second Bianchi identities ($DR + DR + DR = 0$) one obtains another "Bianchi identity":

$$D_i G^{ij} = 0 \quad \text{where} \quad G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R$$

is "Einstein's tensor"

(NB: It seems that in 1916 Einstein (and Hilbert) were unaware of this Bianchi identity)

Finally one can decompose the Riemann tensor as:

$$R^{am}{}_{sq} = C^{am}{}_{sq} + \frac{1}{n-2}\left(g^a_s R^m_q + g^m_q R^a_s - g^m_s R^a_q - g^a_q R^m_s\right) - \frac{1}{(n-1)(n-2)}\left(g^a_s g^m_q - g^a_q g^m_s\right) R$$

where the "Weyl tensor" $C^{am}{}_{sq}$ possesses all the symmetries of $R^{am}{}_{sq}$ and is traceless: $C^{am}{}_{aq} = 0$.

Note also that 2 manifolds whose metrics are "conformally" related ie such that $\bar{g} = F(x^i) g$ have the same Weyl tensor.

④ Locally flat coordinate systems

In general relativity, space & time are represented by a 4-Dimensional manifold equipped with a metric g, which is imposed to reduce to η_{ij} at any point, and associated Levi-Civita connexion.

Locally, that is in the neighbourhood of any point, the special relativity representation of Space & Time is required to hold. Let us see to what extent that is true.

Let $g_{ij}(x^k)$ be the components of the metric in a coordinate system x^i. In another one $X^l = X^l(x^k)$ they are $f_{ij}(X^k) = \frac{\partial x^k}{\partial X^i}\frac{\partial x^l}{\partial X^j} g_{kl}$, which can be expanded in the vicinity of p_0 (X_0^L) as:

$$f_{ij}(X^k) = \frac{\partial x^k}{\partial X^i}\frac{\partial x^l}{\partial X^j} g_{kl}\Big|_0 + (X^m - X^m_0)\left(g_{kl}\frac{\partial^2 x^k}{\partial X^m \partial X^i \partial X^j}\frac{\partial x^l}{\partial X^j} + g_{kl}\frac{\partial^2 x^k}{\partial X^m \partial X^j \partial X^i}\frac{\partial x^l}{\partial X^i} + \frac{\partial x^k}{\partial X^i}\frac{\partial x^l}{\partial X^j}\frac{\partial g_{kl}}{\partial X^m}\right) \cdots$$

The question is: $\exists$? coordinate changes such that $f_{ij} = \eta_{ij} + O((X^m - X^m_0)^2)$?

The answer is yes: • at p_0 the data are the $\frac{n(n+1)}{2} = 10$ components of the metric $g_{kl}|_0$, the $n^2(n+1)/2 = 40$ components of $\partial_m g_{kl}|_0$, the $[n(n+1)/2]^2 = 100$ components of $\partial_{mn} g_{kl}|_0$ etc.

• the free parameters are the $n^2 = 16$ values of $\frac{\partial x^k}{\partial X^i}\big|_0$, the $\frac{n^2(n+1)}{2} = 40$ values of the 2nd derivatives, the $\frac{n^2(n+1)(n+2)}{3!} = 80$ values of the 3rd derivatives etc

. Hence the system $f_{ij}(X^0) = \frac{\partial x^k}{\partial X^i}\frac{\partial x^l}{\partial X^j} g_{kl}\Big|_0 = \eta_{ij}$ is a system of 10 equations for the 16 $\frac{\partial x^k}{\partial X^i}\Big|_0$; there is an infinite number of solutions parametrized by the 6 parameters of Lorentz transformations.

. Cancelling the coefficient of $(X^m - X^m_0)$ is a system of 40 equations for the 40 $\frac{\partial^2 x^k}{\partial X^i \partial X^j}\Big|_0$ with a unique solution.

. On the other hand one cannot in general cancel the second order term, since the system to solve is a system of 100 equations for 80 unknowns ; The difference (20) is the number of freely specifiable components of the Riemann tensor.

Expanding to 2nd order one finds (Riemann-Cartan theorem):

$$f_{ij}(X^k) = \eta_{ij} - \frac{1}{6}(X^m - X^m_0)(X^n - X^n_0)(R_{imjn} + R_{injm})\Big|_0 + \dots$$

Indeed, in this locally flat coordinate system (or "Minkowskian frame")

$$R_{ijkl} = \frac{1}{2}\left(\partial_{jk} g_{il} + \partial_{il} g_{jk} - \partial_{ik} g_{jl} - \partial_{jl} g_{ik}\right)$$

(in such a coordinate system the symmetry properties of R_{ijkl} are obvious).

so that $\partial_{mn} g_{ij} = \frac{1}{3}(R_{imjn} + R_{injm})$

In this "Riemann normal" coordinate system the geodesic equation $\frac{du^i}{d\tau} + \Gamma^i_{jk} u^j u^k = 0$ reads, since: $\Gamma^i_{jk} = -\frac{1}{3}(X^m - X^m_0)(R^i_{jkm} + R^i_{kjm})$:

$$\frac{du^i}{d\tau} = \frac{2}{3} R^i_{jkm} u^j u^k\Big|_0 (X^m - X^m_0)$$

(V) Motion of matter in a gravitational field

1. Test particles

The whole idea behind representing space & time by a curved manifold is that the gravitational force can be thus incorporated in a connexion, so that the motion of a test body (that is whose own gravity field can be ignored) is "free" motion represented by a "straight" line in curved space-time, that is : a geodesic:

$$(1)\quad \begin{cases} D_u u = 0 \iff \frac{du^i}{d\tau} + \Gamma^i_{jk} u^j u^k = 0\ ; & \Gamma^i_{jk} = \frac{1}{2} g^{il}(\partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk}) \\ u^i = \frac{dx^i}{d\tau}\ ; & u^i u_i = g_{ij} u^i u^j = -c^2 \end{cases}$$

where the components g_{ij} of the metric tensor describe at one and the same time the reference frame (that is the coordinate system used) AND the (exterior) gravitational field.

geodesics extremize lengths between 2 points. Hence the *action* of a particle in a gravitational field can be chosen to be

$$S_p = -mc^2 \int_A^B d\tau \qquad \text{with } c\,d\tau = \sqrt{-g_{ij}dx^i dx^j} \text{ being}$$

the proper time along the path $x^i = x^i(\tau)$;

The corresponding lagrangian is $L_p = -mc\sqrt{-g_{ij}u^i u^j}$;

When performing its Euler-Lagrange variation $\left(\frac{\partial L_p}{\partial x^i} = \frac{d}{d\tau}\frac{\partial L_p}{\partial u^i}\right)$ one must take into account the fact that there is a constraint : $L_p = -mc^2$ and hence set $L_p = -mc^2$ (and thus $\dot{L}_p = 0$) after variation.

This constraint (due to the reparametrization invariance of S_p) can be lifted by choosing the lagrangian of a particle to be:

$\mathcal{L} = g_{ij}u^i u^j$ whose Euler-Lagrange variation gives (1) straightaway not only for particles ($u^i u_i = -c^2$) but also for photons ($u^i u_i = 0$)

(NB: identify $\frac{\partial \mathcal{L}}{\partial x^i} = \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial u^i}$ with (1) is often an efficient way to obtain the Christoffel symbols Γ^i_{jk} of a given metric).

Suppose we are given a connexion ...., ie that we know the 40 functions $\Gamma^i_{jk}(x^\ell)$. In order to know if we are dealing with a genuine gravitational field or not, we must compute the Riemann tensor; if it is identically zero that space-time is flat and there exist Minkowskian coordinates X^i such that $g_{ij} = \eta_{ij}$ everywhere. In order to find these sets of four functions $X^i(x^j)$, recall the transformation law for connexion coefficients:

$$\frac{\partial^2 X^\ell}{\partial x^i \partial x^j} = \Gamma^m_{ij} \frac{\partial X^\ell}{\partial x^m}$$

These linear equations being solved (20 for $X^\ell(x^i)$), the initial conditions are constrained by imposing $g_{ij} = \frac{\partial X^k}{\partial x^i}\frac{\partial X^\ell}{\partial x^j}\eta_{k\ell}$ (10 constraints); so that 10 constants remain arbitrary, the 10 parameters of the Poincaré group.

In these new coordinate systems the eom thus reduce to $\frac{du^i}{d\tau} = 0$

If the Riemann tensor is not zero, a gravitational field is present.

There is no first integral of the geodesic equation, unless the gravitational field, that is the metric, exhibits some "symmetries". A way to express a symmetry is to say that there $\exists$ special coordinate systems such that the metric components do not depend on a given coordinate, say $x^{\underline{k}}$. To obtain a first integral, rewrite the geodesic equation in a covariant (rather than contravariant) form:

$$\frac{Du_j}{d\tau} = 0 \iff \frac{du_j}{d\tau} = \frac{1}{2} u^i u^k \partial_j g_{ik}$$

(Because $D_i g_{ij} = 0$ and by definition of Γ^i_{jk}).

Hence if g_{ij} does not depend on $x^{\underline{k}}$ ($\partial_{\underline{k}} g_{ij} = 0 \quad \forall i \,\&\, j$), then

$$u_{\underline{k}} = g_{\underline{k}i}\frac{dx^i}{d\tau} = \text{Const on the world line of the particle.}$$

- if $x^{\underline{k}} = t$ is a time coordinate and $\partial_t g_{ij} = 0$, the metric (and gravitational field) is said to be "stationary" and $u_0 = -E$ is interpreted as the energy (per unit mass) of the particle
- if $x^{\underline{k}} = \varphi$ is an angular variable $u_\varphi = L$ is the "angular momentum".

In newtonian mechanics the action of a particle in a gravitational field is:

$$S_{p(N)} = \int \left(-mc^2 + \frac{1}{2}mv^2 - mU\right) dt$$

where U is the gravitational potential created by the masses other than the particle under consideration (and is such that $S_{p(N)} \sim S_p$ in the absence of gravity and for $c \to \infty$). Here t is Newton's universal time.

Since $S_p = -mc^2\int d\tau$, there must be quasi-Minkowskian coordinate systems where, for weak gravity: $d\tau \sim dt\left(1 - \frac{v^2}{2c^2} + \frac{U}{c^2}\right)$.

Since $d\tau^2 = -ds^2 = -g_{ij}dx^i dx^j$, we thus have.

$$ds^2 \sim -\left(1 + \frac{2U}{c^2}\right)c^2 dt^2 + \delta_{\alpha\beta}\, dx^\alpha dx^\beta$$

and we recover the result obtained by Einstein in 1907 when predicting gravitational redshifts using the equivalence principle:

$$g_{00} \sim -\left(1 + \frac{2U}{c^2}\right)$$ where U is Newton's potential in a quasi inertial frame in the limit of weak field.

2. Charges & (non-gravitational) fields

eom of motion of a charge interacting with an em field and gravity:

$$m\frac{Du^i}{d\tau} = \frac{q}{c} F^i{}_j u^j \qquad u^i \equiv \frac{dx^i}{d\tau}; \qquad F_{ij} = D_i A_j - D_j A_i = \partial_i A_j - \partial_j A_i$$

which can also been obtained by extremising (with respect to path-variations) the action:

$$S_{pi} = -mc^2\int_a^b d\tau + \frac{q}{c}\int_a^b A_i dx^i$$

Maxwell's equation can be inferred by the "correspondance principle" $D \to \partial$ as:

$$\begin{cases} F_{ij;k} + F_{jk;i} + F_{ki;j} = 0 & \text{(1st group, which is a consequence of } F_{ij} = D_i A_j - D_j A_i) \\ D_j F^{ij} = \frac{4\pi}{c} j^i & \text{where } j^i \text{ is the 4-current.} \end{cases}$$

$$j^i = (c\rho, \vec{j})$$

NB: Maxwell's 2nd group of equations also read:

$$\frac{4\pi}{c} j^i = D_j (D^i A^j - D^j A^i) = -\underbrace{D_j D^j}_{\text{curved d'Alembertian}} A^i + D^i D_j A^j + R^i{}_j A^j$$

$$= \frac{1}{\sqrt{-g}} \partial_j \left(\sqrt{-g}\, F^{ij}\right)$$

($D_j A^j = 0$: generalized Lorentz gauge)

Maxwell's equation can also be derived from extremising (with respect to the field configurations A_i) the action:

$$S_{if} = \frac{q}{c} \int A_i dx^i - \frac{1}{16\pi} \int F_{ij} F^{ij} \, \underbrace{d\Omega}_{= \sqrt{-g}\, d^4 x}$$

where the interaction term $S_i = \frac{q}{c} \int A_i dx^i = \frac{1}{c} \int A_i j^i d\Omega$ with $j^i = \frac{\rho u^i}{\sqrt{-g}\, u^0}$.

(Indeed, setting $q = \int \rho\, d^3x$ we have:

$$\frac{q}{c} \int A_i dx^i = \frac{1}{c} \int \rho A_i u^i d\tau d^3x = \frac{1}{c} \int \rho u^i A_i \frac{dx^0}{u^0} d^3x = \frac{1}{c} \int \frac{\rho u^i}{u^0 \sqrt{-g}} A_i d\Omega)$$

— Maxwell's 2nd group implies:

$$\frac{4\pi}{c} D_i j^i = D_i D_j F^{ij}$$

Now: $D_i D_j F^{ij} = D_j D_i F^{ij} + R^i{}_{mij} F^{mj} + R^j{}_{mij} F^{im}$

$$= -D_j D_i F^{ji} + R_{mj} F^{mj} - R_{mi} F^{im}$$

$$= -D_i D_j F^{ij} + 2 R_{mj} F^{mj}$$

hence: $D_i D_j F^{ij} = 0$ since $R_{mj} F^{mj} \equiv 0$ ($R_{mj} = R_{jm}$; $F_{mj} = -F_{jm}$)

thus $D_i j^i = 0$: the 4-current is covariantly conserved.

— In flat spacetime & (3+1) notation, Maxwell's equations read:

$$\begin{cases} \nabla \cdot B = 0 \ ; & \nabla \wedge E = -\frac{1}{c} \frac{\partial B}{\partial t} \\ \nabla \cdot E = 4\pi\rho \ ; & \nabla \wedge B = \frac{1}{c} \frac{\partial E}{\partial t} + \frac{4\pi}{c} j \end{cases}$$

from which one deduces: $\frac{1}{c} \frac{\partial W}{\partial t} + c \nabla \cdot S = -j \cdot E$ with $W = \frac{E^2 + B^2}{8\pi}$; $S = \frac{E \wedge B}{4\pi c}$

One must not however be tempted to write the l.h.s as $\partial_i W^i$ and interpret $W^i = (W/c, S)$ as a 4-vector, as $j \cdot E$ is not a 4-scalar.

It is in fact impossible to build a 4-vector out of F_{ij} alone.

On the other hand the symmetric tensor:

$$T^{ij} = \frac{1}{4\pi} \left(F^i{}_k F^{jk} - \frac{1}{4} \eta^{ij} F_{kl} F^{kl} \right)$$

electromagnetic "stress-energy" tensor

is such that $T^{00} = W$ & $T^{0\alpha} = c S^\alpha$ and such that (using $F_{ij} = -F_{ji}$):

$$D_k T^{ik} = \frac{1}{4\pi} F^i{}_k D_j F^{jk} = -\frac{1}{c} F^i{}_k j^k$$

"on shell", that is if $D_i F^{ij} = \frac{4\pi}{c} j^i$

Reciprocally, the (covariant) conservation of the stress-energy tensor implies $D_j F^{jk} = 0$ that is is equivalent to Maxwell's equations outside the charges.

Note that these generalisations of the Lorentz & Maxwell equations to describe the interaction of charges & electromagnetic fields with gravity are in "harmony" with the conceptual framework & mathematical formalism of GR but their ability to describe actual phenomena must be assessed (cf JP Lasota).

A note on scalar fields. (cf JP Uzan, G. Esposito-Farese)

A standard action for the (still classic...) scalar fields is:

$$S = -\int \left[\frac{1}{2} g^{ij} \partial_i \phi \partial_j \phi + V(\phi) \right] \sqrt{-g}\, d^4x$$

whose extremisation with respect to variation of field configurations $\delta\phi$ yields the Klein-Gordon equation:

$$\Box \phi = D_i D^i \phi = \frac{1}{\sqrt{-g}} \partial_i \left(\sqrt{-g}\, g^{ij} \partial_j \phi \right) = 0$$

The associated stress-energy tensor is:

$$T_{ij} = \partial_i \phi \partial_j \phi - g_{ij} \left(\frac{1}{2} \partial_k \phi \partial^k \phi + V \right)$$

whose conservation: $D_i T^{ij} = 0$ is equivalent to the K.G equation (if $\partial_i \phi \neq$ ...

However one can also start from more complicated actions...

eg: $S = -\int \left[Z(\varphi) g^{ij} \partial_i \varphi \partial_j \varphi + 2V(\varphi) - F(\varphi) R \right] d^4x$

where R is the scalar curvature of space-time & $F(\varphi)$, $Z(\varphi)$ various functions of φ.

3. Perfect fluids

The Newtonian eqns of a perfect fluid in a gravitational field are the conservation & Euler equations:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho v) = 0 \quad ; \qquad \frac{dv}{dt} \equiv \frac{\partial v}{\partial t} + v.\nabla v = -\frac{1}{\rho}\nabla p - \nabla U .$$

(They follow from imposing the conservation of mass and from Newton's second law $F = ma$).

In the framework of Relativity one first looks for a symmetric tensor T_{ij} whose components T^{00} & $T^{0\alpha}/c$ may be interpreted as the energy density and linear momentum of the fluid.

The "flow" of a fluid is a vector field $u^i(x^k)$ normalized according to $g_{ij}u^iu^j = -c^2$ so that $u^i = \frac{dx^i}{d\tau}$ is the 4-velocity of the fluid element described by the world line $x^i = x^i(\tau)$.

In the local inertial frame where the fluid is (momentarily) at rest, the tensor is imposed to read:

$$(*) \qquad T^{00} = \epsilon \; ; \; T^{0\alpha} = 0 \; ; \; T^{\alpha\beta} = p\,\delta^{\alpha\beta} \qquad (u^i = (c,0,0,0))$$

which embodies the isotropic properties of perfect fluids, and where ϵ & p are ~~its~~ "~~proper~~ energy density" and pressure.

To infer the expression of T^{ij} in any frame, one builds by means of the scalars ϵ & p and the 4-velocity u^i, a tensor which reduces to (*) when $u^i = (c,0,0,0)$. The sought-for tensor is:

$$T^{ij} = (\epsilon + p)\frac{u^iu^j}{c^2} + p\,g^{ij}$$

Imposing the (covariant) conservation of T^{ij}: $D_iT^{ij} = 0$ then yields:

- the "conservation of baryonic nb": $D_i(nU^i) = 0$ with $\frac{dn}{n} = \frac{d\epsilon}{\epsilon + p}$ (n: "proper baryonic nb density")
- the relativistic Euler equation:

$$(\epsilon + p)\frac{DU^k}{d\tau} + u^k\frac{dp}{d\tau} + c^2\partial^k p = 0$$

(which reduces to Euler's equation when pressure can be neglected compared to $\epsilon = \rho c^2$ and at the Newton limit $g_{00} \sim -(1 + \frac{2U}{c^2})$).

(VI) Einstein's gravitational field equations

1. "Heuristic" approach

Special Relativity attributes a geometrical origin to the inertial accelerations and incorporates them in the connexion coefficients of the (flat) covariant derivative written in non-Minkowskian coordinates, that is in non-inertial frames.

The principle of equivalence postulates that $m_i = m_g$ and identifies inertia and gravitation.

Gravitation is thus incorporated in the covariant derivative of a curved space-time whose geometry is entirely determined by its metric g, with coefficients $g_{ij}(x^k)$ in a chosen coordinate system x^i.

If there is no gravity, the Universe is empty of matter and is represented by Minkowski space-time.

Hence: matter curves space-time;

The gravitational field equations must then relate geometry to matter, that is relate a tensor built out of Riemann's tensor to a tensor describing matter.

The tensorial object which completely describes the energy content of matter is its stress-energy tensor T_{ij}.

A tensor describing curvature which is of the same type is Ricci's tensor $R_{ij} \equiv R^a{}_{iaj}$.

Hence Einstein proposed in 1914 the following eqns: $R_{ij} \propto T_{ij}$.
These are indeed 10 differential equations for the 10 unknowns (g_{ij}).
However g_{ij} describes gravity AND the coordinate system (or frame).
Since one must be free to choose the coordinates at will, the equations should leave 4 functions arbitrary.
Moreover the stress energy tensor should be conserved ($D_iT^{ij} = 0$) which yields the matter equation of motion. However this implies that $D_iR^{ij} = 0$, that is extra conditions on g_{ij}.

In one word: $R_{ij} \propto T_{ij}$ are flawed equations.

Bianchi's identity : $D_i G^{ij} \equiv 0 \quad \forall g_{ij}(x^k)$ solves all problems.
Hence the equations proposed by Einstein on Nov 25^{th} 1915
(... and by Hilbert on Nov 20^{th} 1915...) :

$$\boxed{G_{ij} = \varkappa T_{ij}} \qquad \text{with} \quad G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R$$

to which he added in 1917 a "cosmological constant" term :

$$G_{ij} + \Lambda g_{ij} = \varkappa T_{ij} \qquad (\text{indeed } D_k g_{ij} \equiv 0)$$

which he considered later as the "biggest blunder" of his life but may be indeed required to explain recent cosmological observations (cf J. Silk & JP Uzan).

The coupling constant $\varkappa$ is determined by taking the small-velocity, weak-field limit of Einstein's equations.

At the newtonian limit & in quasi-Minkowskian coordinates :

$$ds^2 \sim -\left(1 + \frac{2U}{c^2}\right) c^2 dt^2 + \delta_{\alpha\beta}\, dx^\alpha dx^\beta$$

where U is Newton's gravitational potential

All Γ are of order $O\left(\frac{1}{c^2}\right)$ and their time derivatives of $O(1/c^3)$.

Hence at lowest order : $R_{00} \sim \partial_\alpha \Gamma^\alpha_{00} \sim -\frac{1}{2} \partial_\alpha (g^{\alpha\beta} \partial_\beta g_{00})$

$$\sim \frac{1}{c^2} \partial^\alpha (\partial_\alpha U) = \frac{1}{c^2} \Delta U$$

As for the stress-energy tensor of a fluid it reduces to $T^{00} \sim \rho c^2$, ρ being its mass-density.

If we now rewrite Einstein's equations under the equivalent form :

$$R_{ij} = \varkappa \left(T_{ij} - \frac{1}{2} g_{ij} T\right) \qquad (\text{with } \Lambda = 0)$$

we therefore obtain : $\Delta U \sim 4\pi G \rho$, that is Poisson's equation, if :

$$\boxed{\varkappa = \frac{8\pi G}{c^4}}$$

2. The Lagrangian for gravity

Maxwell's equation can be obtained by extremising the action $-\frac{1}{16} \int F_{ij} F^{ij} d\Omega$ when varying the configuration of the potential A_i.

In Einstein's theory the "gravitational potential" is the metric g, with component $g_{ij}(x^k)$. In order to describe a genuine gravitational field and not mere inertial forces, this metric must be non-flat, so that is it natural to suppose that the action for gravity will be a function of curvature. For the eom to be 2nd order differential equations S_g should, a priori, contain only the metric g_{ij} & its 1st derivatives $\partial_k g_{ij}$. But , one can always choose local cartesian coordinates where $g_{ij} = \eta_{ij}$ & $\partial_k g_{ij} = 0$ (principle of equivalence) and, as a consequence, the Riemann tensor contains 2nd order derivatives.

However there $\exists$ curvature scalars which - do - yield 2nd order differential equations. In $D=4$ dimensions the only curvature scalar having this property is :

$$S_g = \frac{c^4}{16\pi G} \int (R - 2\Lambda) \sqrt{-g}\, d^4x \qquad (\text{Hilbert } 1915)$$

where, recall $\begin{cases} R = g^{ij} R_{ij} ; \quad R_{ij} = R^k{}_{ikj} ; \quad R^\lambda{}_{jkl} = \partial_k \Gamma - \partial_l \Gamma + \Gamma\Gamma - \Gamma\Gamma \\ \Gamma = \frac{1}{2} g^{\cdot\cdot} (\partial_\cdot g + \dots) \end{cases}$

$$\delta S \equiv \delta \int R \sqrt{-g}\, d^4x = \int \left[(\delta R) \sqrt{-g} + R\, \delta(\sqrt{-g}) \right] d^4x$$

$$\begin{cases} \delta \sqrt{-g} = \frac{-\delta g}{2\sqrt{-g}} \quad ; \quad \delta g = -g\, g_{ij}\, \delta g^{ij} \\ \delta R = \delta(R_{ij} g^{ij}) = (\delta R_{ij}) g^{ij} + R_{ij}\, \delta g^{ij} \end{cases}$$

$$\Rightarrow \delta S = \int \left(R_{ij} - \frac{1}{2} g_{ij} R\right) \sqrt{-g}\, \delta g^{ij} d^4x + \int \sqrt{-g}\, \delta R_{ij} \cdot g^{ij} d^4x$$

We compute $\sqrt{-g}\, \delta R_{ij} \cdot g^{ij}$ in a locally minkowskian frame (ie st $\partial g = 0$)

$$\sqrt{-g}\, g^{ij} \delta R_{ij} = \sqrt{-g}\, g^{ij} \delta \left(\partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} \right) \qquad \text{change } j \leftrightarrow k \text{ in } 2^{nd} \text{ term :}$$

$$= \partial_k \left(\sqrt{-g}\, g^{ij} \delta \Gamma^k_{ij} - \sqrt{-g}\, g^{ik} \delta \Gamma^j_{ij} \right) \equiv \partial_k (\sqrt{-g}\, k^k)$$

Hence $\int_\Omega \sqrt{-g}\, \delta R_{ij}\, g^{ij} d^4x = \int_\Omega d^4x\, \partial_k (\sqrt{-g}\, k^k) = \int_{\partial\Omega} d^3x\, n_k k^k = 0$

IF not only the metric but also its 1st derivatives are held fixed at the boundary. Finally : $\delta S_g = \frac{c^4}{16\pi G} \int (G_{ij} + \Lambda g_{ij}) \sqrt{-g}\, \delta g^{ij} d^4x$

3. Matter stress-energy tensor revisited

- Noether's conservation laws in Minkowski space-time:

Consider a field action $S = \int \mathcal{L}\, d^4X$ where $\mathcal{L} = \mathcal{L}(\Phi, \partial_i \Phi)$

it does NOT depend explicitly on the minkowskian coordinates X^i

(ex : $\mathcal{L} = -\frac{1}{2}\eta^{ij}\partial_i\phi\,\partial_j\phi - V(\phi)$; $\mathcal{L} = -\frac{1}{16\pi}\eta^{ik}\eta^{jl}F_{ij}F_{kl}$)

The eom are $\dfrac{\partial\mathcal{L}}{\partial\phi} = \partial_i \dfrac{\partial\mathcal{L}}{\partial\partial_i\phi}$; Using them we have:

$$\frac{\partial\mathcal{L}}{\partial X^i} = \frac{\partial\mathcal{L}}{\partial\phi}\partial_i\phi + \frac{\partial\mathcal{L}}{\partial\partial_j\phi}\partial_{ij}\phi = \left(\partial_j\frac{\partial\mathcal{L}}{\partial\partial_j\phi}\right)\partial_i\phi + \frac{\partial\mathcal{L}}{\partial\partial_j\phi}\partial_{ij}\phi$$

$$= \partial_j\left(\frac{\partial\mathcal{L}}{\partial\partial_j\phi}\partial_i\phi\right).$$

Hence the Noether tensor $\Theta_i{}^j \equiv -\partial_i\phi\dfrac{\partial\mathcal{L}}{\partial\partial_j\phi} + \delta_i^j\mathcal{L}$ is conserved: $\partial_j\Theta_i{}^j = 0$

the tensor $\Theta_{ij} = \eta_{ik}\Theta_i{}^k$ is no, a priori, symmetric. It is if Φ is a scalar field : $\Theta_{ij} = \partial_i\phi\,\partial_j\phi - \eta_{ij}\left(\frac{1}{2}\eta^{kl}\partial_k\phi\,\partial_l\phi + V\right)$; it is not if $\phi = A_i$ is the em field: $\Theta_{ij} = \frac{1}{4\pi}\left(\partial_i A_k F_j{}^k - \frac{1}{4}\eta_{ik}F_{kl}F^{kl}\right)$.

- In the general case of arbitrary coordinates and/or non flat metric no conservation laws can be thus obtained because $\sqrt{-g}\,\mathcal{L}$ now depends explicitly on x^i.

Now the action being a scalar, it must be invariant in a coordinate change. In such a variation ϕ varies: $\delta\phi = \partial_i\phi\,\delta x^i$, but this variation can be ignored since the corresponding variation of the action vanishes by virtue of the eom. As for the contribution arising from δg_{ij}, it is:

$$\delta S = \frac{1}{2}\int T^{ij}\,\delta g_{ij}\sqrt{-g}\,d^4x \quad \text{with } \frac{1}{2}\sqrt{-g}\,T^{ij} \equiv \frac{\partial\sqrt{-g}\,\mathcal{L}}{\partial g_{ij}} - \partial_l\frac{\partial\sqrt{-g}\,\mathcal{L}}{\partial\partial_l g_{ij}}$$

NB: beware of signs! $-\frac{1}{2}\sqrt{-g}\,T_{ij} \equiv \dfrac{\partial\sqrt{-g}\,\mathcal{L}}{\partial g^{ij}} - \partial_l\dfrac{\partial\sqrt{-g}\,\mathcal{L}}{\partial\partial_l g^{ij}}$

Contrary to Noether's tensor these "stress energy tensors" are symmetric;

scalar field : $T_{ij} = \partial_i\phi\,\partial_j\phi - g_{ij}\left(\frac{1}{2}g^{kl}\partial_k\phi\,\partial_l\phi + V\right)$

electromagnetism : $T_{ij} = \frac{1}{4\pi}\left(F_{il}F_j{}^l - \frac{1}{4}g_{ij}F_{lm}F^{lm}\right)$

- Are such stress-energy tensors "conserved", and in what sense?

The invariance of S in a coordinate change does NOT imply that $T_{ij} = 0$ because all δg_{ij} are not independent. Since all quantities in the integrand of δS must be evaluated at x^k, the variation δg_{ij} induced by the coordinate change $x^i \to x'^i = x^i - \xi^i$ is $\delta g_{ij} = D_i\xi_j + D_j\xi_i$ (see demo below). Hence

$$\delta S = \frac{1}{2}\int T^{ij}\,\delta g_{ij}\sqrt{-g}\,d^4x = \int T^{ij}D_i\xi_j\sqrt{-g}\,d^4x$$

$$= \underbrace{\int D_i(T^{ij}\xi_j)\sqrt{-g}\,d^4x}_{= \int_\Omega \partial_i(\sqrt{-g}\,T^{ij}\xi_j)\,d^4x \;=\; -\int_{\partial\Omega}\sqrt{-g}\,T^{\alpha j}\xi_j n_\alpha\,d^3x \;=\; 0 \text{ since } \xi_j = 0 \text{ at } \partial\Omega} - \int (D_i T^{ij})\xi_j\sqrt{-g}\,d^4x$$

Hence $\delta S = 0 \implies D_i T^{ij} = 0$. Stress-energy tensors are "covariantly conserved".

(and one checks that this imply the eom for the matter field).

- The full Einstein field equation are thus derived from the Lagrangian:

$$S = \frac{c^4}{16\pi G}\int (R - 2\Lambda)\sqrt{-g}\,d^4x + \int \mathcal{L}_{matter}\sqrt{-g}\,d^4x$$

by extremising S when varying the gravitational field configuration, that is the metric components δg^{ij}

$$G_{ij} + \Lambda g_{ij} = \frac{8\pi G}{c^4}T_{ij}$$

4. Gravitational stress-energy "pseudo-tensors"

- The gravitational lagrangian density is $\sqrt{-g}\,R$ which can be rewritten as: $\sqrt{-g}\,R = \sqrt{-g}\,G + \partial_i(\sqrt{-g}\,w^i)$ with:

$$G = g^{ik}\left(\Gamma^m_{il}\Gamma^l_{km} - \Gamma^l_{ik}\Gamma^m_{lm}\right) \quad \text{and} \quad w^i = g^{jk}\Gamma^i_{jk} - g^{ij}\Gamma^l_{jl}$$

Hence $\sqrt{-g}\,R$ is quadratic in the first derivatives of the metric if one ignores the divergence term which integrates out if ∂g is zero at the boundary. (This is the reason why the Euler variation of $\sqrt{-g}\,R$ does not yield derivatives of the metric higher than the 2nd).

- G is not a scalar but $G = g^{ik}(\Delta^m_{il}\Delta^l_{km} - \Delta^l_{ik}\Delta^m_{lm})$ is, where $\Delta^i_{jk} \equiv \Gamma^i_{jk} - \bar{\Gamma}^i_{jk}$, $\bar{\Gamma}^i_{jk}$ being the Christoffel symbols of a "background spacetime" (that is: if $ds^2 = g_{ij}(x^k)dx^i x^j$; $d\bar{s}^2 = \bar{g}_{ij}(x^k)dx^i dx^j$)

Hence Einstein's (1918) idea to choose G as the Lagrangian for gravity. The metric coefficients can then be put on the same footing as, say, $\Phi(x)$ in M_4 and Noether theorem applies:

$$\Theta_i{}^j = -\partial_i g_{pq}\frac{\partial(\sqrt{-g}G)}{\partial\partial_j g_{pq}} + \delta_i^j\sqrt{-g}G \quad \text{is conserved: } \partial_j\Theta_i{}^j = 0$$

explicitely: $\sqrt{-g}\,\Theta_i{}^j = \delta_i^j G\sqrt{-g} + \Gamma^j_{lm}\partial_i(\sqrt{-g}g^{lm}) - \Gamma^l_{ml}\partial_i(\sqrt{-g}g^{mj})$

- A caveat: We defined (symmetric) matter stress-energy tensor in a gravitational field as the variational derivative of its lagrangian wrt g_{ij}, the eom for matter being satisfied. It would make no sense to proceed similarly to define a stress-energy tensor for the gravitational field itself as that "stress-energy" tensor would be the Einstein tensor! Indeed, up to divergence:

$$\frac{\partial G\sqrt{-g}}{\partial g^{ij}} - \partial_k\frac{\partial G\sqrt{-g}}{\partial\partial_k g^{ij}} = \sqrt{-g}(R_{ij} - \tfrac{1}{2}g_{ij}R) = 0 \text{ "on shell".}$$

- Remark; Freud (1939) showed that $\sqrt{-g}\,\Theta_i{}^j \doteq \partial_k(\sqrt{-g}F_i^{[jk]})$

with $\sqrt{-g}F_i^{jk} = \frac{1}{2}\frac{g_{il}}{\sqrt{-g}}\partial_m[-g(g^{jl}g^{km} - g^{kl}g^{jm})]$ ("superpotential")

Hence $\int_{V_4}\partial_j(\Theta_i{}^j\sqrt{-g})\,d^4x \to \int_{(\partial V)_3}\Theta_i{}^\alpha n_\alpha\sqrt{-g}\,d^3x \sim \int_{\Sigma_2}\sqrt{-g}F_i{}^{jk}n_j n_k d^2x$

That is the global conserved quantities (eg mass) are obtained as integrals over the 2-sphere at infinity (only the asymptotic values of the metric & its 1st derivatives are required).

The main drawback of the Einstein pseudo-tensor is that it is not symmetric and does not allow to define an angular momentum.

- in 1962 Landau & Lifchitz hence proposed another pseudo-tensor to represent the energy of a gravitational field:

In a locally inertial frame one finds easily that Einstein's tensor can be written as:

$$-g\,G^{ik} = \partial_l h^{ikl} \qquad \text{Note the factor } (-g) \text{ (NOT } \sqrt{-g})$$

with $h^{ijk} = \partial_l\lambda^{jkl}$ where $\lambda^{ijkl} = -g(g^{ij}g^{lk} - g^{ik}g^{jl})$

Hence Freud's superpotential $F_i^{[jk]}$ is related to LL's h^{ijk} by:

$$-g\,F_i{}^{[jk]}g^{il} = h^{ljk}$$

In a general coordinate system the equality $-g\,G^{ik} = \partial_l h^{ikl}$ does not hold any more and is replaced by:

$$-g(G^{ij} + t^{ij}) = \partial_l h^{ijl}$$

the explicit calculation of t^{ij} is fairly painstaking; the final result is $t^{ij} = \Gamma\Gamma gg$ (see Landau & Lifchitz).

This widely used gravitational stress-energy pseudo-tensor is symmetric, unlike Einstein's.

- The drawback of all such constructs is that they are NOT tensors and hence have no invariant meaning. The root of the problem is the equivalence principle which puts on the same footing gravity & inertia: Hence by going from an inertial to an accelerated frame in M_4 one "creates" fictitious gravitational energy.

- A way to go round this pb is (as alluded to above) to introduce a background ST (at least asymptotically since all conserved "charges", mass & angular momenta, are eventually computed as integrals over the 2-sphere at infinity). See J Katz et al and Deser's lectures

(VII) Symmetric spacetimes

1. Isometries and Killing vectors

A way to attack the problem of solving Einstein's equations is to restrict one-self (using proper physical arguments!) to subclasses of spacetimes, possessing "symmetries". We already mentioned that a way to express the fact that a ST had a symmetry was to say that there $\exists$ coordinate systems in which the components of the metric tensor do not depend on a given coord x^k. Let us be more systematic.

Let $g_{ij}(P)$ be the components of a metric g at point P in the coordinates x^k. At Q close to P with coordinates $x^k+\xi^k$, the components are $g_{ij}(P)+\xi^k\partial_k g_{ij}$ (at first order in ξ).

Let us now perform the coordinate change $x^k \to x'^k = x^k-\xi^k$ so that, in the new (primed) system the coordinates of Q are the same as that of P in the old (unprimed) one. (We "shift" the coordinate grid from P to Q).

In the new (primed) system the components of the metric at Q are:

$$g'_{kl}(Q) = \frac{\partial x^i}{\partial x'^k}\frac{\partial x^j}{\partial x'^l}\, g_{ij}(Q) = (\delta^i_k+\partial_k\xi^i)(\delta^j_l+\partial_l\xi^j)(g_{ij}(P)+\xi^m\partial_m g_{ij})$$

$$= g_{kl}(P) + g_{ml}\,\partial_k\xi^m + g_{km}\,\partial_l\xi^m + \xi^m\partial_m g_{ij} \quad \text{at 1st order}$$

The displacement from P to Q is an "isometry" if $g'_{kl}(Q)=g_{kl}(P)$ (which means that the variation $\xi^k\partial_k g_{ij}$ is "fallacious" since it can be compensated by a change of coordinates). The vector ξ^i is then called a "Killing vector" and must satisfy the Killing equation

$$g_{ml}\,\partial_k\xi^m + g_{mk}\,\partial_l\xi^m + \xi^m\partial_m g_{ij} = 0 \iff D_i\xi_j + D_j\xi_i = 0$$

(as can be shown as an exercise).

NB: We introduced the Lie bracket $[X,Y] = X\circ Y - Y\circ X$ of the 2 vectors and showed that it was also a vector. This bracket can be also seen as a directional derivative of Y along X and is then called "Lie derivative": $\mathcal{L}_X Y = [X,Y] = (X^i\partial_i Y^j - Y^i\partial_i X^j)\,\partial_j$

Similarly: $\mathcal{L}_X \lambda = (X^k\partial_k\lambda_i - \lambda_k\partial_i X^k)\,dx^i$

So that: $(\mathcal{L}_\xi g)_{ij} = D_i\xi_j + D_j\xi_i$.

Example: symmetries & Killing vectors of euclidean space $\mathcal{E}_3$.

The easiest way to solve $D_\alpha\xi_\beta + D_\beta\xi_\alpha = 0$ in $\mathcal{E}_3$ is of course to use cartesian coordinates X^γ so that the Killing equations reduce to $\partial_\alpha\xi_\beta + \partial_\beta\xi_\alpha = 0$, the general solution of which being

$$\xi^\alpha = d^\alpha + \omega_{\alpha\beta}X^\beta \quad \text{with } \omega_{\alpha\beta} = -\omega_{\beta\alpha} \quad ; d^\alpha, \omega_{\alpha\beta} \text{ cst.}$$

Hence $\mathcal{E}_3$ possesses 6 Killing vectors, 3 of "translations" proportional to $\xi^\alpha=(1,0,0)$, $\xi^\alpha=(0,1,0)$, $\xi^\alpha=(0,0,1)$, 3 of "rotation" proportional to $\xi^\alpha_{(1)} = (-Y,X,0)$, $\xi^\alpha_{(2)} = (0,Z,-Y)$, $\xi^\alpha_{(3)} = (Z,0,-X)$.

The existence of the 3 translation Killing vectors expresses the "homogeneity" of $\mathcal{E}_3$; the 3 rotation K.V its "isotropy".

Cartesian coordinates exhibit in an obvious manner the homogeneity of $\mathcal{E}_3$ since $dl^2 = dX^2+dY^2+dZ^2$ manifestly keeps the same form when $X\to X+\xi$ etc.;

To exhibit in a manifest way isotropy (around the Z-axis say) we go to spherical coordinates where

$$dl^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \quad \text{and} \quad \overset{\text{sph}}{\xi}{}^{\alpha}_{(1)} = (0,0,1)$$

which is invariant in the transformations $\varphi \to \varphi+\xi$.

We wrote the geodesic eqn $\frac{Du_i}{d\tau}=0$ as $\frac{du_j}{d\tau} = \frac{1}{2}u^i u^k\partial_j g_{ik}$ and deduced that if g_{ij} did not depend on a coordinate x^k then $u_k = C$.

A more intrinsic (that is coordinate independent) way to say the same thing is to say that if a spacetime admits a Killing vector ξ^i then $\xi^i u_i = $ cst along a geodesic.

Indeed
$$\frac{d}{d\tau}(\xi^i u_i) = \frac{D}{d\tau}(\xi^i u_i) = \underbrace{D_i\xi^i}_{=0 \text{ because } D_i\xi_j + D_j\xi_i = 0} + \xi^i\underbrace{\frac{Du_i}{d\tau}}_{=0 \text{ since it is a geodesic}} = 0.$$

(This is a first example of the relationship between Killing vectors and conservation laws in General Relativity.)

2. Maximally symmetric spacetimes

We saw that flat n-dimensional space possesses $\frac{n(n+1)}{2}$ Killing vectors. We now ask the question: what is the maximum nb of K.V a n-dimensional space may possess? (See Stephani's book for details)

A Killing vector obeys $D_i \xi_j + D_j \xi_i = 0$, and we have, for any vector field: $D_i D_j \xi_k - D_j D_i \xi_k = -R^m{}_{kij} \xi_m$, which yields:

$$D_i D_j \xi_k = R^m{}_{ijk} \xi_m .$$

Hence, once the n components of ξ_m and its $\frac{n(n-1)}{2}$ (antisymmetric) derivatives are known at a point $P(x^k)$ then its higher derivatives are also known & hence $\xi_\ell(x^k)$ is known everywhere (by Taylor expansion). Therefore the maximum number of independent Killing vectors is $\frac{n(n+1)}{2}$. A space possessing the max. nb of K.V is said to be "maximally symmetric", or equivalently "homogeneous & isotropic".

Not every space is maximally symmetric! Hence there are integrability conditions on the Killing equation $D_i \xi_j + D_j \xi_i = 0$.

Computing $D_i D_j D_k \xi_m - D_j D_i D_k \xi_m$ one indeed finds a relation of the type:

(*) $(DR - DR)\xi + (R - R + R - R) D\xi = 0$ (see Stephani).

which further restricts ξ & $D\xi$.

Now, for a space to be maximally symmetric (*) must not restrict ξ & $D\xi$. Hence the coefficients of ξ & $D\xi$ must be zero. This yields (Stephani) that the Riemann tensor of a maximally symmetric space must read:

$$R_{ijk\ell} = K\,(g_{ik} g_{j\ell} - g_{i\ell} g_{jk}) \quad \text{with } K \text{ a constant.}$$

(if $K=0$, this is flat space).

From that we get:
$$\begin{cases} R_{ij} = K(n-1) g_{ij} \,; \quad R = K n (n-1) \\ G_{ij} = \frac{K(n-1)(n-2)}{2} g_{ij} \end{cases}$$

so that maximally symmetric spacetimes are solution of the Einstein field equations if $T_{ij} = V g_{ij}$ where V is a constant and:

$$\frac{K(n-1)(n-2)}{2} + \Lambda = V$$

if $V=0$ (no matter) & $\Lambda = 0$, Minkowski spacetime is the only max. sym. soluti

It can now be shown (Eisenhart 1949 & S. Weinberg's book 1972) that, for a given dimension n, a given metric signature [$(+++\dots)$ for $\mathcal{E}_n$, $(-+++)$ for V_4] and a given value of K (which can be normalized by means of coordinate rescaling to ± 1 or 0), there is a unique maximally symmetric space [that is: the metric components of such spaces can all be deduced from one another by coordinate changes]

Hence n-dimensional maximally symmetric Riemannian $(+++\dots)$ and Lorentzian $(-+++\dots)$ are, as expected, generalizations of the sphere S_2 $(K=+1)$, $\mathcal{E}_n$ $(K=0)$ & the hyperboloid H_2 $(-++)$ whose Riemann tensors are of the form: $R = K(gg - gg)$.

Example: $n = 3$, signature $(+++)$:

$$d\sigma_3^2 = a^2 \left[d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2\theta\, d\varphi^2) \right]$$

where a is a constant; $\varphi \in [0, 2\pi]$, $\theta \in [0, \pi]$ and

$$\begin{cases} f(\chi) = \chi \text{ if } K=0 & (\mathcal{E}_3 \text{ in spherical coordinates}) \\ f(\chi) = \sin\chi \text{ if } K=1, \ \chi \in [0,\pi] & (S_3) \\ f(\chi) = \sinh\chi \text{ if } K=-1 & (H_3) \end{cases}$$

The volume of S_3 is $V = \int dV = \int \sqrt{g}\, d\chi d\theta d\varphi = 2\pi^2 a^3$ and is finite. The volume of $\mathcal{E}_3$ & H_3 is infinite UNLESS on proceeds to identification (example: the 2-dimensional flat torus).

3. Cosmological spaces

Consider a 4-dimensional pseudo-riemannian manifold V_4 and suppose there $\exists$ in V_4 a family of time-like curves defining nonintersecting spacelike 3-surfaces $\perp$ to them, which are maximally symmetric. Such spaces are Robertson-Walker spacetimes and their metric can be written as:

$$ds^2 = -dt^2 + a^2(t) \left[d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2\theta\, d\varphi^2) \right]$$

(see JP Uzan for a thorough analysis of such "cosmological" spacetimes).

Here we shall only consider the case (as illustrative simple examples of curved geometries) when V_4 not only is sliced by maximally symmetric 3-spaces, but is itself maximally symmetric.

Knowing that $R_{\alpha\beta\gamma\delta} = \frac{k}{a^2}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})$ (with $k = 0, \pm 1$) (since the t = constant surfaces are maximally symmetric) one obtains easily (using the definition of the Riemann tensor in terms of the Christoffel symbols and knowing those in terms of the metric):

$$\begin{cases} R_{\alpha\beta\gamma\delta} = \left[\frac{k}{a^2} + \left(\frac{\dot a}{a}\right)^2\right](g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \\ R_{0\beta0\delta} = -\frac{\ddot a}{a} g_{\beta\delta} \end{cases}$$

Therefore V_4 is maximally symmetric if $R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk})$ with K a number. Hence $a(t)$ must obey:

$$\frac{\ddot a}{a} = \left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = K.$$

if $K = 0$: $V_4 = M_4$; if $k = 0$ $a = 1 \Rightarrow ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

if $k = -1$: $a = t \Rightarrow ds^2 = -dt^2 + t^2[d\chi^2 + \sinh^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)]$

this is "Milne universe" a fallaciously expanding looking universe which is nothing but M_4 in disguise. [The coordinate change: $T = t\cosh\chi$; $r = t\sinh\chi$ brings the metric back to a familiar for

if $K = H^2$ (constant): V_4 is "de Sitter universe"

if we choose

$$\begin{cases} k = 0: & ds^2 = -dt^2 + e^{2Ht}(dx^2 + dy^2 + dz^2) \\ k = 1: & ds^2 = -dt^2 + \frac{\cosh^2 Ht}{H^2}[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)] \\ k = -1: & ds^2 = -d\tilde t^2 + \frac{\sinh^2 H\tilde t}{H^2}[d\tilde\chi^2 + \sinh^2\tilde\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)] \end{cases}$$

see Hawking & Ellis' book the relationships between the 3 coordinate systems.

De Sitter space is a 4-hyperboloid. It can be "visualized" as a 4-D hyperboloid embedded in a 5-D, pseudo-euclidean flat space. Indeed, if we consider in M_5 with metric: $dS_5^2 = dX^2 + dY^2 + dZ^2 + dW^2 - dV^2$ the 4D hyperboloid with equation: $X^2 + Y^2 + Z^2 + W^2 - V^2 = K^2$ one finds that the induced metric is indeed a de-Sitter metric (see Weinberg's book for details)

The various coordinate systems introduced above correspond to different slicings of this hyperboloid

== In contrast with Newton's theory where representations of the universe as a whole are deceptive, all solutions of Einstein's equations, each representing a different spacetime are "cosmological".