GR-IHP06
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PART 4

Weak gravitational fields

(I) Schwarzschild solution & the "classical tests"

1. Geodesic equation

$$\begin{cases} g = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-2m/r} + r^2 d\Omega^2 & \text{Schw coord.} \\ \quad = -\left(\frac{1-m/2\bar{r}}{1+m/2\bar{r}}\right)dt^2 + \left(1+\frac{m}{2\bar{r}}\right)^4\left(d\bar{r}^2+\bar{r}^2 d\Omega^2\right) & \text{isotropic coord.} \\ \quad = -\left(\frac{\bar{r}-m}{\bar{r}+m}\right)dt^2 + \left[\left(1+\frac{m}{\bar{r}}\right)^2\delta_{\alpha\beta} + \left(\frac{\bar{r}+m}{\bar{r}-m}\right)\frac{m^2}{\bar{r}^4}x_\alpha x_\beta\right]dx^\alpha dx^\beta & \text{(harmonic coordinates)} \end{cases}$$

We shall start by working in Schwarzschild coordinates.

geodesic equation: $D_u u = 0$: $\frac{d^2x^i}{d\lambda^2} + \Gamma^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = 0$

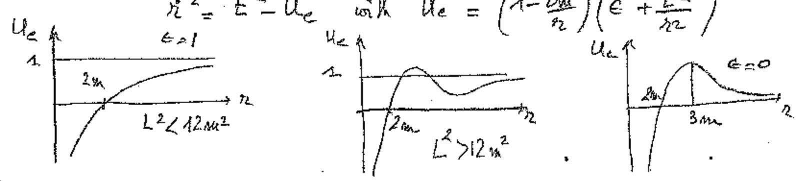
(eom of a "test particle" that is e.g. Mercury in the field of the Sun).

stationarity $\Rightarrow$ $u_0 = \text{cte}$ ie: $\left(1-\frac{2m}{r}\right)\dot{t} = E$

spherical symmetry $\Rightarrow$ $\theta = \pi/2$ ("plane" motion) and $u_\varphi = \text{cte}$, ie $r^2\dot{\varphi} = L$

$g(u,u) = -\epsilon$ ($\epsilon = +1$ for massive particles, $\epsilon = 0$ for photons) yields $\dot{r}$:

$$\dot{r}^2 = E^2 - U_e \quad \text{with} \quad U_e = \left(1-\frac{2m}{r}\right)\left(\epsilon + \frac{L^2}{r^2}\right)$$



(photons can orbit at $r = 3m$ (cf $R_{star} < 3m$!)).

2. Post-newtonian trajectory & perihelion shift

A standard way to obtain $r(\varphi)$ is to form the ratio $\dot{r}^2/\dot{\varphi}^2 \equiv (dr/d\varphi)$ to set $u = \frac{1}{r}$ and differentiate wrt φ ("Binet method").

Then obtaining:

$$\frac{d^2u}{d\varphi^2} + u = \epsilon\frac{m}{L^2} + 3mu^2$$ which can be integrated in terms of elliptic functions.

However if $m/r \ll 1$ then $3mu^2 \ll m/L^2$.

Hence the solution is found by iteration.

$\underline{\epsilon = 1}$ Newtonian order: $\frac{d^2u}{d\varphi^2} + u = \frac{m}{L^2}$ $\Rightarrow$ $u = \frac{1+e\cos\varphi}{p}$ $\quad p \equiv \frac{L^2}{m}$

imposing that this ellipse solves: $\dot{r}^2 = E^2 - 1 + \frac{2m}{r} - \frac{L^2}{r^2}$

yields $e^2 = 1 + \frac{L^2}{m^2}\underbrace{(E^2-1)}_{\simeq 2E_N}$ (Kepler's ellipse)

PN order $\frac{d^2u}{d\varphi^2} + u = \frac{m}{L^2} + \frac{3m^3}{L^4}(1+e\cos\varphi)^2$

whose solution can be written, at that order, under the form:

$$u \sim \frac{m}{L^2}\left\{1 + e\cos\left[\varphi\left(1-\frac{3m^2}{L^2}\right)\right]\right\} + \underbrace{\text{periodic terms in }\varphi}_{\frac{m^3}{L^4}\left(3+\frac{3e^2}{2}-\frac{1}{2}e^2\cos 2\varphi\right)}.$$

NB: the φ-dependence of the orbit depends of course on the radial coordinate. In harmonic coordinates for ex: $\frac{1}{\bar{r}} \sim \frac{1+e'\cos[(1-3m^2/L^2)\varphi]}{p'}$

The time dependence can be obtained along similar lines. However it proves fruitful to proceed as follows:

$$\left(\frac{dr}{dt}\right)^2 = \frac{\dot{r}^2}{\dot{t}^2} \simeq \underbrace{A + \frac{2B}{r} + \frac{C}{r^2}}_{\text{Newton-like term}} + \underbrace{\frac{D}{r^3}}_{\text{1st PN correction.}} + \ldots$$

(ABCD can be expressed in terms of E and L / see Damour-Deruelle 1985)

$$\Rightarrow \quad \frac{dt}{dr} \sim \frac{1}{(A+2B/r+C/r^2)^{1/2}}\left[1 - \frac{D}{r^3(A+2B/r+C/r^2)}\right]$$

which integrates as: $$\boxed{\begin{cases} n(t-T) = \eta - e_t\sin\eta \\ r = a_r(1-e_r\cos\eta) \end{cases}}$$ with is "quasi-Newtonian"

when $\begin{cases} n = (-A)^{3/2}/B & ; \quad e_t = \left[1 - \frac{A}{B^2}\left(C - \frac{BD}{C}\right)\right]^{1/2} \\ a_r = -\frac{B}{A} + \frac{D}{2C} & ; \quad e_r = \left(1 + \frac{AD}{2BC}\right)e_t \end{cases}$

Similarly $\frac{d\varphi}{dt} = \frac{\dot{\varphi}}{\dot{t}} \simeq \frac{H}{r^2} + \frac{I}{r^3}$ which integrates as:

$$\boxed{\tan\left[(1-3m^2/L^2)\varphi\right] = \sqrt{\frac{1+e_\varphi}{1-e_\varphi}}\tan\eta/2}$$ also "quasi-Newtonian".

with $e_\varphi = e_r(1 + AD/2BC - AI/BH)$.

See T. Damour for extensions and application to pulsar timing.

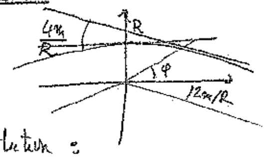
The dominant correction to Kepler's ellipse is the secular term, that is the $\varphi\left(1-\frac{3m^2}{L^2}\right)$ dependence in the orbit $r(\varphi)$, which shows that the ellipse precesses by the amount

$$\Delta\omega = \frac{2\pi}{1-3m^2/L^2} - 2\pi \sim \frac{6\pi m^2}{L^2} = \frac{6\pi m}{p} = \frac{6\pi GM}{c^2 a(1-e^2)}$$

For Mercury $\Delta\omega = 43''$ / century - (Found by Le Verrier & Newcomb 1845, Einstein 1915 ("The greatest scientific emotion of my life").
see C. Will for today's precision (3×10^{-3}).

3. Bending of light & Shapiro effect

consider photons $(\epsilon=0)$: $\frac{d^2u}{d\varphi^2}+u = 3mu^2$.



At newtonian order $u = \sin\varphi/R$: straight line

Next order : $d^2u/d\varphi^2+u \sim 3m\sin^2\varphi/R^2$ with solution :

$$u \sim \frac{\sin\varphi}{R} + \frac{3m}{2R^2}\left(1+\frac{1}{3}\cos 2\varphi\right)$$

asymptotes : $u\to 0$; φ small : $0 \sim \varphi/R + \frac{3m}{2R^2}\left(1+\frac{1}{3}\right) \Rightarrow \varphi = -2m/R$

Hence a deflection of : $\Delta\phi = \frac{4GM}{c^2R}$.

It is instructive to perform the same calculation in isotropic coordinates ; imposing zero length at 1st order in $m/\bar r$ yields : (photon in $\bar x\bar y$ "plane" along $\bar x$)

$$\begin{cases} 0 \simeq -\left(1-\frac{2m}{\bar r}\right)dt^2 + \left(1+\frac{2\gamma m}{\bar r}\right)(d\bar x^2+d\bar y^2) & (\gamma=1) \\ 0 \simeq \frac{d^2\bar y}{d\lambda^2} + \Gamma^{\bar y}_{00}\left(\frac{dt}{d\lambda}\right)^2 + \Gamma^{\bar y}_{\bar x\bar x}\left(\frac{d\bar x}{d\lambda}\right)^2 & (\bar y \text{ component of geodesic eqn}) \end{cases}$$

Hence $\frac{dt}{d\lambda} \sim \frac{d\bar x}{d\lambda}$; $\frac{d^2\bar y}{d\lambda^2} \sim \frac{d^2\bar y}{d\bar x^2}\left(\frac{d\bar x}{d\lambda}\right)^2$; hence $\frac{d^2\bar y}{d\bar x^2} + \Gamma^{\bar y}_{00} + \Gamma^{\bar y}_{\bar x\bar x} = 0$

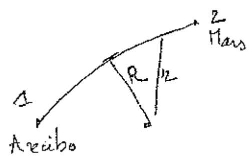
Now : $\Gamma^{\bar y}_{00} \sim m\bar y/\bar r^3$; $\Gamma^{\bar y}_{\bar x\bar x} \sim m\gamma\bar y/\bar r^3$. Hence :

$$\Delta\phi \doteq \int_{-\infty}^{+\infty} \frac{d^2\bar y}{d\bar x^2}\,d\bar x \sim 2m(1+\gamma)\int_0^\infty \frac{R\,d\bar x}{(\bar x^2+R^2)^{3/2}} \simeq \frac{2m(1+\gamma)}{R}$$

One thus sees that if the curvature of the spatial sections is ignored $(\gamma\to0)$ then one gets 1/2 the right value (that is, the "newtonian" value; Einstein 1907)

For a light ray grazing the Sun $\Delta\phi = 1.75''$
(Eddington 1919) see C. Will for today's precision (3×10^{-4})
(gravitational lensing & multiple images: see Y. Mellier)

Shapiro effect : from $\begin{cases} \dot t\left(1-\frac{2m}{r}\right) = E \\ \dot r^2 = E^2 - \left(1-\frac{2m}{r}\right)\frac{L^2}{r^2} \end{cases}$ $(\epsilon=0$: photons$)$



rewrite $(dr/dt)^2$ as :

$$\frac{dr}{dt} = \left(1-\frac{2m}{r}\right)\left[1-\frac{1-2m/r}{1-2m/R}\left(\frac{R}{r}\right)^2\right]^{1/2}$$

$$\Rightarrow\quad t(r,R) \sim \underbrace{\sqrt{r^2-R^2}}_{\text{"Newtonian" straight line}} + 2m\ln\frac{r+\sqrt{r^2-R^2}}{R} + m\sqrt{\frac{r-R}{r+R}}$$

NB: this formula makes sense when the explicit trajectory $r=r(t)$ is plugged in & for return trips (with t converted into "Arecibo" proper time).
Cassini probe : sent to Saturn in 1997 ; arrived there in January 2005
precision 2×10^{-5}. (see C. Will)

4. The β & γ post-newtonian parameters

To compute the perihelion shift, the light bending & Shapiro effects, there is no need to use the full exact Schwarzschild metric. In fact, using isotropic coordinates & writing the metric as :

$$ds^2 = -\left(1-\frac{2m}{\bar r}+\frac{2\beta m^2}{\bar r^2}+O\left(\frac{1}{\bar r^3}\right)\right)dt^2 + \left(1+\frac{2\gamma m}{\bar r}+O\left(\frac{1}{\bar r^2}\right)\right)dl^2$$

one obtains : $\begin{cases} \Delta\omega = \frac{2-\beta+2\gamma}{3}\,\frac{6\pi GM}{c^2p} \;;\; \Delta\phi = \frac{2GM(1+\gamma)}{c^2R} \\ t(r,R) \sim \sqrt{r^2-R^2} + \frac{2GM}{c^2}(1+\gamma)\ln\frac{r+\sqrt{r^2-R^2}}{R} + m\sqrt{\frac{r-R}{r-R}} \end{cases}$

in GR $\beta=\gamma=1$. The tests are then reinterpreted as constraints on β & γ (Thus light bending & Shapiro effect measure the same parameter γ).

(This is the PPN approach developped by Nordtvedt & Will ; see Will).

(II). The motion of spinning particles

1. Newtonian tops

In Newtonian mechanics there is a well defined notion of the accelerated motion of a rigid body: The kinematics of an accelerated body is given by the motion of the cartesian frame in which it is at rest; its dynamics follows from Newton's 2nd Law, $f = ma$.

If we choose for the origin of the accelerated frame the point O' such that

$$\Sigma m R' = 0 \qquad (\text{where } R = OO' + O'P \quad R' \equiv O'P) \quad (O' \text{ is centre of mass})$$

then the eom are $M \ddot{d} = F$ where $M = \Sigma m$; $d \equiv OO'$; $F \equiv \Sigma f$

which gives the motion of the centre of mass of the rigid body.

And: $\dot{J} = K$ where $J \equiv \Sigma m R' \wedge v$ is the angular momentum of the body & $K \equiv \Sigma R' \wedge f$ is the torque.

To complete the description one must relate J to Ω, the rotation vector of the body. Since $v = \dot{d} + \Omega \wedge R'$ one finds:

$$J = J'^{\alpha} e'_{\alpha}$$ where e'_α are the basis vector of the frame attached to the body

and $J'_\alpha = I_{\alpha\beta} \omega'^{\beta}$ with $I_{\alpha\beta} = \Sigma m (\delta_{\alpha\beta} X'^{\gamma} X'_{\gamma} - X'_\alpha X'_\beta)$ being the tensor of inertia of the body.

From these equations one can deduce precession, nutation etc of tops

(If $K = 0$, $J = \text{cte}$, $\forall$ the motion of the centre of mass

If $f = mg$ (constant gravity field) $K = \Sigma R' \wedge f = \Sigma m R' \wedge g = 0$ hence $J = \text{cte}$ in a constant gravity field (equivalence principle).

2. "Spin" in special Relativity

Consider an ensemble of particles with masses m at $X^i = X^i(\lambda)$ in an inertial frame S. Then

$$M^{ij} \equiv \Sigma m (X^i U^j - X^j U^i) \qquad \text{with } U^i \equiv dX^i/d\lambda$$

can be seen as some generalization of $\vec{J}$. However, here, X^i is the position vector, in S and not in some S' where all points would be "at rest".

As a result M^{ij} does not transform as a tensor in the Poincaré group. Indeed if $X^i = \Lambda^i{}_k X'^k - d^i$, then $U^j = \Lambda^j{}_\ell U'^\ell$ and hence:

$$M^{ij} = \Lambda^i{}_k \Lambda^j{}_\ell M'^{k\ell} - (d^i P^j - d^j P^i) \quad \text{where } P^i \equiv \Sigma m U^i \text{ (total momentum)}$$

There $\exists$ however a quantity which is a genuine vector under Poincaré transformation

$$S_i = -\frac{1}{2} \epsilon_{ijk\ell} M^{jk} U^\ell \qquad \text{"intrinsic angular momentum of the system"}$$

where $U^i = \frac{P^i}{M}$ is the 4-velocity of the system considered as a whole.

This "spin" vector has only 3 independent components since $S_i U^i = 0$.

In the tangent inertial frame where $U^i = (1,0,0,0)$ the components of S_i reduce to $S_i = (0, J)$ with $J_x = M^{23} = \Sigma m (YU^z - ZU^y)$ etc.

If the system is subject to no external forces, it is natural to impose the eom to be the same as in Newtonian physics, that is: $\frac{dJ}{dt} = 0$ where t is the time measured by clocks at rest in the inertial tangent frame.

A covariant form of this equation is: $U^j \partial_j S_i = f U_i$ which indeed reduces to $dJ/dt = 0$ in the tangent frame. The function f is determined by the condition $S_i U^i = 0$ which implies: $S_i U^j \partial_j U^i = -f + S_i U^j \partial_j$

$$0 = \frac{d}{d\tau}(S_i U^i) = U^j \partial_j (S_i U^i) = (U^j \partial_j S_i) U^i + S_i U^j \partial_j U^i = -f + S_i U^j \partial_j$$

Hence the covariant equations: $\frac{dS_i}{d\tau} = U_i \left[S_j \frac{dU^j}{d\tau}\right]$ with $S_i U^i = 0$.

In any accelerated frame, or in the presence of gravity this generalized to

$$\frac{DS_i}{d\tau} = u_i \left(S_j \frac{Du^j}{d\tau}\right) \qquad \text{with } S_i u^i = 0.$$

Having obtained these equations of motion one may forget about the "scaffolding" which yielded to it and postulate them to be the eom of a relativistic "spinning body". In classical (non quantum) relativity such a spinning body is composite but there is no clear way (as in Newtonian mechanics) to relate this spin vector to some "angular velocity" of the body.

3. Thomas precession

Consider a particle "with spin" which is forced on a circular orbit in an inertial frame (eg: e^- in atom, Thomas 1926).

We shall perform the calculation in the rotating frame where the particle is at rest. The Minkowski metric reads:

$$ds^2 = -[1-\Omega^2(x^2+y^2)]dt^2 + 2\Omega dt\,(x dy - y dx) + dx^2+dy^2+dz^2$$

the world line of the particle is $x^i = (t, R\cos\Psi, R\sin\Psi, 0)$ (R, Ψ const.)

The eq of motion of its spin is: $\frac{Ds_i}{d\tau} = u_i\, s_j \frac{Du^j}{d\tau}$ with $s_i u^i = 0$

with $u^i = \gamma(1,0,0,0)$ $\left(\gamma = \frac{1}{\sqrt{1-R^2\Omega^2}}\right)$; $u_i = \left(-\frac{1}{\gamma}, 0, 0, \gamma\Omega r\right)$

The Christoffel symbols are, on the trajectory: $\Gamma^x_{tt} = -R\Omega^2$; $\Gamma^y_{tx} = -\Gamma^x_{ty} = -\Omega$

The constraint $s_i u^i = 0 \Rightarrow s_0 = 0$.

Recalling that $\frac{Ds_i}{d\tau} = \frac{ds_i}{d\tau} - \Gamma^j_{ik} s_j u^k$ and $\frac{Du^i}{d\tau} = \frac{du^j}{d\tau} + \Gamma^j_{ik} u^i u^k$

The equations of motion become: $\frac{ds_z}{dt} = 0$ and $\frac{ds_x}{dt} = \Omega s_y$; $\frac{ds_y}{dt} = -\gamma^2\Omega s_x$

with solution: $s_0 = 0$; $s_x = S_1 \cos\gamma\Omega t$; $s_y = -\gamma S_1 \sin\gamma\Omega t$; $s_z = 0$.

One can then return to the inertial frame: $S_i = \frac{\partial x^j}{\partial X^i} s_j$ to obtain:

$$\begin{cases} S^X = S_1(\cos\Omega T \cos\Omega\gamma T + \gamma \sin\Omega T \sin\gamma\Omega T) \\ S^Y = S_1(\sin\Omega T \cos\Omega\gamma T - \gamma\cos\Omega T \sin\gamma\Omega T) \\ S^Z = 0 \;; \quad S^0 = -S_1 R\gamma\,\Omega \sin\gamma\Omega t \end{cases}$$

In the limit $R^2\Omega^2 \ll 1$ $\quad S^X + iS^Y \sim S_1 e^{-\frac{R^2\Omega^3}{2}T}$.

Hence this spin precesses with angular velocity $\omega_{Thomas} \approx \frac{R^2\Omega^2}{2c^2}$.

4. Geodetic precession & Lense-Thirring effect

Consider a spinning object (e.g the GPB spinning sphere) in orbit around a massive (non-spinning) object (e.g the Earth, neglecting its rotation for the time being).

The motion of the "centre of mass" of the "top" is a geodesic in Schwarzschild metric, see above. For a circular orbit ($\dot r = 0$, $\ddot r = 0$): $\frac{d\varphi}{dt} = \sqrt{\frac{m}{r^3}}$

and $\left(\frac{d\varphi}{d\tau}\right)^2 = \frac{m}{r^3}\frac{1}{1-3m/r}$. and $u^k = \left(\frac{1}{\sqrt{1-3m/r}}, 0, 0, \sqrt{\frac{m}{r^3}\frac{1}{1-3m/r}}\right)$

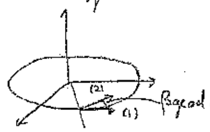
The eqn of the spin vector is: $\frac{Ds_i}{d\tau} = 0$ with $s_i u^i = 0$.

$s_i u^i = 0 \Leftrightarrow s^0 = \frac{\sqrt{mr}}{1-2m/r} s^\varphi$; $\frac{Ds^i}{d\tau} = 0 \Leftrightarrow \frac{ds^\varphi}{dt} + \sqrt{\frac{m}{r^5}}\, s^r = 0$; $\frac{ds^r}{dt} - \left(1-\frac{3m}{r}\right)\sqrt{\frac{m}{r}}\, s^\varphi = 0$

with solution: $s^\varphi = A\cos(\Omega t + \varphi)$; $s^r = r\sqrt{1-\frac{3m}{r}}\, A\sin(\Omega t+\varphi)$ with $\Omega = \sqrt{\frac{m}{r^3}}\sqrt{1-\frac{3m}{r}}$.

Since the orbital velocity is $\frac{d\varphi}{dt} = \sqrt{\frac{m}{r^3}}$, one thus sees that, after one orbit, the spin vector has precessed by the angle $\beta_{geod} = 2\pi\frac{\Omega}{\omega} - 2\pi$ that is:

$$\beta_{geod} = 2\pi\left(\sqrt{1-\frac{3m}{r}} - 1\right) \sim -\frac{3\pi m}{r}$$



equivalently: $\dot{\vec S} = \vec{\omega}_{geod} \wedge \vec S$ with $\omega_{geod} \sim \frac{3}{2}\sqrt{\frac{m^3}{r^5}}$

This is the "geodetic" precession effect.

For the Earth $\omega_{geod} \sim 8.4 \left(\frac{R_\oplus}{r}\right)^{5/2}$ arcs/y. See C. Will. & J. Hester.

NB: Had we written the Schwarzschild metric under the "PPN form":
$$ds^2 = -\left(1 - \frac{2m}{r} + 2\beta\frac{m^2}{r^2} + \cdots\right)dt^2 + \left(1 + 2\gamma\frac{m}{r} + \cdots\right)dl^2$$ we would have obtained: $\omega_{geod} \sim \frac{3}{2}(1+\gamma)\sqrt{\frac{m^3}{r^5}}$.

Hence measuring ω_{geod} (GPB) will constrain γ (already well-known, see Will)

BUT will also put to the test the eqns for spinning particles.

If now the central body is itself spinning then the metric is, at large distance
$$g \sim -\left(1-\frac{2m}{r}\right)dt^2 + \left(1+\frac{2m}{r}\right)dl^2 - \frac{4J}{r^3}(x dy - y dx)$$ where J is the angular momentum of the central body.

Solving $S_i u^i = 0$ & $\frac{DS^i}{d\tau} = 0$ in this metric yields (see eg Straumann, Will, Hester) a "Lense-Thirring" precession of S with pulsation: $\frac{1}{r^3}\left(-J + 3\frac{(J\cdot X)X}{r^2}\right)$

numerically: $\frac{\omega_{Lense-Thirring}}{\omega_{geodetic}} \simeq \frac{J_\oplus}{3M_\oplus^{3/2} R_\oplus^{1/2}} \sim 6.5\times 10^{-3}$ see Hester.

III. Gravitational waves

1. Electromagnetic waves

- The action describing an (isolated) ensemble of charges is (in an inertial frame)

$$S = S_{(p)} + S_{(i)} + S_{(f)} \quad \text{with} \quad S_{(p)} = -\sum mc^2 \int d\tau \,;$$

$$S_{(i)} = \sum \frac{e}{c}\int A_i dX^i = \frac{1}{c^2}\int A_i j^i d^4X \,; \quad S_{(f)} = -\frac{1}{4\mu_0 c}\int F_{ij}F^{ij} d^4X$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$; $j^i = \rho \frac{dX^i}{dt}$.

- The extremisation of S wrt path variation & potential configuration yields the Lorentz & Maxwell equations:

$$mc\frac{du^i}{d\tau} = e F^{ij} u_j \,; \quad \partial_j F^{ij} = \frac{\mu_0}{c} j^i . \qquad (\mu_0 \equiv 4\pi)$$

- $S_{(f)}$ is invariant in the "gauge transformation" $A_i \to A_i - \partial_i f$, $\forall f$. For S (that is $S_{(i)}$) to be invariant too in such gauge transformation we must have:

$$0 = S_{(i)}(A_i - \partial_i f) - S_{(i)}(A_i) = -\frac{1}{c^2}\int j^i \partial_i f \, d^4X = -\frac{1}{c^2}\int \partial_i (f j^i) d^4x + \frac{1}{c^2}\int f \partial_i j^i d^4X$$

the first term vanishes by Stokes theorem if $f = 0$ at the boundary. Hence $S_{(i)}$ is gauge invariant if $\partial_i j^i = 0$, that is if the current is conserved.

[we saw that, in SR, diffeomorphism-invariance, that is invariance of the matter action under changes of coordinates yields also a conservation law: $\partial_i T^{ij} = 0$

from $\partial_i j^i = 0$ one deduces $\int dV \partial_0 j^0 + \int dV \partial_\alpha j^\alpha = 0$. If there are no charges at infinity the 2nd term vanishes thanks to Stokes theorem. Hence

(Noether theorem) $\quad \frac{1}{c}\int dV j^0 = \int \rho \, dV = \sum e \quad$ is a constant.

[in SR $\partial_i T^{ij} = 0$ similarly yields the constancy of the 4-momentum P^i.]

- Electromagnetism being a linear theory, it can be formulated in an (explicitly) gauge-invariant way:

Decompose A_α as $\quad A_\alpha = \partial_\alpha A + \bar{A}_\alpha \quad$ with $\partial_\alpha \bar{A}^\alpha = 0$

(The decomposition is unique as $\Delta A = \partial_\alpha A^\alpha$ has a unique well-behaved solution).

Similarly write j^α as: $\quad j_\alpha = \partial_\alpha j + \bar{j}_\alpha \quad$ with $\partial_\alpha \bar{j}^\alpha = 0$

In Bardeen's terminology (1980) A_i & j_i are hence decomposed as "scalars" (A_0 & j_0) & "vectors" ($\bar{A}_\alpha$, $\bar{j}_\alpha$).

In a gauge transformation $A_i \to A_i - \partial_i f$: $\quad A_0 \to A_0 - \dot{f}$; $A \to A - f$; $\bar{A}_\alpha \to \bar{A}_\alpha$

and up to total divergencies the actions $S_{(i)}$ & $S_{(f)}$ can be rewritten as:

$$\begin{cases} S_{(f)} = -\dfrac{1}{2\mu_0 c}\displaystyle\int \left(-\partial_\alpha \Phi \partial^\alpha \Phi + \partial_i \bar{A}_\alpha \partial^i \bar{A}^\alpha\right) d^4X & \text{with } \Phi \equiv A_0 - \dot{A} \\ S_{(i)} = \dfrac{1}{c}\displaystyle\int \left(\Phi j^0 + \bar{A}_\alpha \bar{j}^\alpha\right) d^4X - \dfrac{1}{c}\displaystyle\int A \partial_i j^i d^4X & \end{cases}$$

where we recover that $S_{(i)}$ is gauge invariant if $\partial_i j^i = 0$.

- The extremisation of $S_{(i)} + S_{(f)}$ with respect to Φ and $\bar{A}_\alpha$ yields Maxwell's eqn rewritten in gauge invariant manner:

$$\Delta \Phi = \mu_0 j^0 \qquad \text{and} \qquad \Box \bar{A}_\alpha = -\mu_0 \bar{j}_\alpha .$$

The first eqn is an (elliptic) constraint equation; the other is the eom for the vector $\bar{A}_\alpha$, ie for the 2 dynamical degrees of freedom (indeed the 3-components of $\bar{A}_\alpha$ are restricted to be divergenceless, $\partial_\alpha \bar{A}^\alpha = 0$).

- In vacuo : $\Delta \Phi = 0 \iff \Phi = A_0 - \dot{A} = 0$

$\Box \bar{A}_\alpha = 0$: $\bar{A}_\alpha$ is a linear superposition of monochromatic plane waves

$$\bar{A}^\alpha_{k} = e^\alpha_{k} \cos(k_i X^i + \varphi_{k}) \quad \text{with } k_i k^i = 0 \text{ \& } k_\alpha e^\alpha_{k} = 1$$

In particular: $A_\alpha = (f_1(ct - z), f_2(ct - z), 0)$ describes a plane wave propagating along the z-axis with velocity c.

- The solution then obtained determines an equivalence class only. To "fix the gauge" consists in choosing an element of that class. For example the "Lorentz gauges" restrict A_i by means of the condition $\partial_i A^i = 0$. To fix the gauge completely another condition must be added (indeed $\partial_i A^i = 0 \iff \partial_0 A^0 = -\Delta A$ only).

The "Coulomb" (or "radiative") gauge consists in choosing $A^0 = 0 \Rightarrow A = 0$.

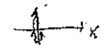
The 4-vector A^i_{k} describing a plane monochromatic wave propagating along the z-axis is then:

$$A^i_{k} = e^i_{k} \cos[k(z - ct) + \phi] \quad \text{with } e^i = (0, e^1, e^2, 0).$$

- To detect an electromagnetic waves one looks at the motion of charges in the electromagnetic field they create, which is determined by the Lorentz equation:

$$mc\frac{du^i}{d\tau} = e F^{ij} u_j \qquad \text{with } F_{ij} = \partial_i A_j - \partial_j A_i .$$

If $e_2 = 0$  the particle oscillates along the X-axis around the origin

If $e_1 = 0$  : idem along the Y-axis.

2. The linear approximation of GR

- A gravitational field is described, in vacuo, that is outside the masses, by the 10 components of the metric g_{ij}. The equivalence principle, that is the demand that the field equations be covariant when changing the 4 coordinates, imply that, a priori, only 6 components of g_{ij} are gauge invariant, that is remain the same in coordinate changes. Suppose we have obtained their explicit expressions. Then Einstein's equations for those 6 "gauge invariant quantities" will split into 4 constraint equations (one for each gauge invariance) and 2 evolution equations. The Cauchy problem being well-posed in GR (see Y. Choquet-Bruhat) the solutions of these 2 hyperbolic equations should represent the 2 dynamical d° of freedom of "gravitational waves". Unfortunately such a program is impaired by the non-linearity of Einstein's equations. It can however be fully achieved at the linear approximation.

- Hence consider an almost flat spacetime in quasi-minkowskian coordinates. The metric reads: $g_{ij} = \eta_{ij} + h_{ij}$ with $|h_{ij}| \ll 1$.

  (see JPU for perturbation of a cosmological background).

  $$ds^2 = -(1+2A)(dX^0)^2 + 2B_\alpha dX^\alpha dX^0 + [(1+2C)\delta_{\alpha\beta} + 2E_{\alpha\beta}]dX^\alpha dX^\beta$$

  with $E^\alpha{}_\alpha = 0$ (B_α & $E_{\alpha\beta}$ are 3-D euclidean vector & tensor)

  Hence $h_{00} = -2A$; $h_{0\alpha} = B_\alpha$; $h_{\alpha\beta} = 2(C\delta_{\alpha\beta} + E_{\alpha\beta})$.

  As in electromagnetism, decompose: $B_\alpha = \partial_\alpha B + \bar{B}_\alpha$ and write:

  $$E_{\alpha\beta} = \partial^2_{\alpha\beta} E - \frac{1}{3}\delta_{\alpha\beta}\Delta E + \partial_\alpha \bar{E}_\beta + \partial_\beta \bar{E}_\alpha + \bar{E}_{\alpha\beta}$$

  with $\partial_\alpha \bar{B}^\alpha = 0$; $\partial_\alpha \bar{E}^\alpha = 0$; $\partial_\alpha \bar{E}^{\alpha\beta} = 0$; $\bar{E}^\alpha{}_\alpha = 0$

  In Bardeen's terminology we replaced the 10 functions h_{ij} by 4 "scalars" (ABCE), 2 "vectors" ($\bar{B}_\alpha$ & $\bar{E}_\alpha$) (that is 4 functions) and a "tensor" $\bar{E}_{\alpha\beta}$ (2 functions).

- In a "gauge transformation", that is in an infinitesimal coordinate change $x^i \to x^i + \xi^i$ the metric components change by: $g_{ij}(x^k) \to g_{ij}(x^k) + \partial_i \xi_j + \partial_j \xi_i$

  so that, decomposing ξ^i as $\xi^i = (\xi^0; \partial^\alpha \xi + \bar{\xi}^\alpha)$ $(\partial_\alpha \bar{\xi}^\alpha = 0)$:

  $$\begin{cases} A \to A + \dot{\xi}^0; \quad B \to B + \dot{\xi} - \xi^0; \quad C \to C + \frac{1}{3}\Delta\xi; \quad E \to E + \xi \\ \bar{B}_\alpha \to \bar{B}_\alpha + \dot{\bar{\xi}}_\alpha; \quad \bar{E}_\alpha \to \bar{E}_\alpha + \bar{\xi}_\alpha; \quad \bar{E}_{\alpha\beta} \to \bar{E}_{\alpha\beta} \end{cases}$$

One thus sees that the following quantities:

$$\left.\begin{aligned} \Phi_A &\equiv A + \dot{B} - \ddot{E}, \quad \Phi_C \equiv C - \frac{1}{3}\Delta E \\ \Phi_{B\alpha} &\equiv \bar{B}_\alpha - \dot{\bar{E}}_\alpha; \quad \bar{E}_{\alpha\beta} \end{aligned}\right\} \text{ are gauge invariant.}$$

We have thus obtained the 6 "gauge invariant perturbations" (GIP): 2 scalars, 1 vector (2 independent components) and 1 tensor (2 independent components)

- Einstein's equations being invariant under coordinate changes, their linearization must be expressible in terms of the 6 GIP just introduced.

  One first notes that the 3 groups of GIP (scalar, vector, tensor) can be considered separately, that is: the Einstein equations for one group can be obtained by setting the 2 others to zero, indeed writing Einstein's equations consists in taking ∂_i derivatives, an operation which does not change the 3-D SVT character of a perturbation; for example: $\partial_i \bar{B}_\alpha = (\dot{\bar{B}}_\alpha, \partial_\beta \bar{B}_\alpha)$ whose (separate) components are still of the "vector" type $(\partial_\alpha(\partial_i \bar{B}^\alpha) = 0)$.

Consider first scalar perturbations. Note that if we start from the set (ABCE) we can perform the coordinate change $x^i \to x^i + \xi^i$ with $\xi^0 = (B - \dot{E})$, $\xi = -E$ so that (see above): $E \to 0$ and $B \to 0$ so that, in that "longitudinal gauge" $\Phi_A = A$; $\Phi_C = C$. Hence an efficient way to obtain the linearized Einstein equations for Φ_A & Φ_C is to work in longitudinal gauge in which the line element reduces to:

$$ds^2 = -(1+2A)(dX^0)^2 + (1+2C)\delta_{\alpha\beta}dX^\alpha dX^\beta$$

and to compute Einstein's tensor: $g_{00} = -(1+2A)$; $g_{\alpha\beta} = (1+2C)\delta_{\alpha\beta}$

$$\Rightarrow g^{00} = -(1-2A); \quad g^{\alpha\beta} = (1-2C)\delta^{\alpha\beta} \quad \text{(at linear order)}$$

$$\Rightarrow \Gamma^i_{jk} = \ldots; \quad R_{ij} = R^k{}_{ikj} = \partial_k \Gamma^k_{ij} - \ldots; \quad R = g^{ij}R_{ij} = \ldots$$

so that, eventually:

$$\begin{cases} G_{00} = -2\Delta C; \quad G_{0\alpha} = -2\partial_\alpha \dot{C} \\ G_{\alpha\beta} = [-2\ddot{C} + \frac{2}{3}\Delta(A + C)]\delta_{\alpha\beta} - \partial^2_{\alpha\beta}(A + C) \end{cases}$$

At this stage one replaces A & C by Φ_A & Φ_C (since they are equal in the chosen gauge). One then argues that Einstein's equations ARE gauge invariant so that:

$$\begin{cases} G_{00} = -2\Delta\Phi_C; \quad G_{0\alpha} = -2\partial_\alpha \dot{\Phi}_C \\ G_{\alpha\beta} = [-2\ddot{\Phi}_C + \frac{2}{3}\Delta(\Phi_A + \Phi_C)]\delta_{\alpha\beta} - \partial^2_{\alpha\beta}(\Phi_A + \Phi_C) \end{cases}$$

in ANY coordinate system.

In vacuo the Einstein equations for the scalar perturbations Φ_A & Φ_C split into two constraint equations (as expected):

$$\Delta\Phi_C = 0 \ , \ \partial_\alpha \dot{\Phi}_C = 0 \quad \text{whose regular solution is } \Phi_C = 0 \ ;$$

the last one $\Rightarrow$ $G^\alpha_\alpha = \Delta\Phi_A = 0 \iff \Phi_A = 0$ as well.

- Turning to vector perturbations one goes to the gauge where $E_\alpha = 0$ so that the line element reads: $ds^2 = -(dX^0)^2 + 2\bar{B}_\alpha dX^0 dX^\alpha + \delta_{\alpha\beta} dX^\alpha dX^\beta$.
  Computing Einstein tensor yields: $\begin{cases} G_{00} = 0 ; \ G_{0\alpha} = -\frac{1}{2}\Delta \bar{B}_\alpha ; \\ G_{\alpha\beta} = -\frac{1}{2}(\partial_\alpha \dot{\bar{B}}_\beta + \partial_\beta \dot{\bar{B}}_\alpha) \end{cases}$
  "Covariantization" yields: $G_{00} = 0$; $G_{0\alpha} = -\frac{1}{2}\Delta\Phi_{\bar{B}_\alpha}$; $G_{\alpha\beta} = -\frac{1}{2}(\partial_\alpha \dot{\Phi}_{\bar{B}_\beta} + \partial_\beta \dot{\Phi}_{\bar{B}_\alpha})$
  and, again, the vacuum Einstein equations impose $\Phi_{\bar{B}_A} = 0$.

- The line element for tensor perturbations is: $ds^2 = -(dX^0)^2 + (\delta_{\alpha\beta} + \bar{E}_{\alpha\beta})dX^\alpha dX^\beta$.
  and: $G_{00} = G_{0\alpha} = 0 \qquad G_{\alpha\beta} = \ddot{\bar{E}}_{\alpha\beta} - \Delta \bar{E}_{\alpha\beta}$. The Einstein equations for the tensor perturbations $\bar{E}_{\alpha\beta}$ are therefore evolution equations:

$$\Box \bar{E}_{\alpha\beta} = 0.$$

  This wave equation is solved as in electromagnetism.

$$\bar{E}^{\vec{k}}_{\alpha\beta} = e^{\vec{k}}_{\alpha\beta} \cos(k_i X^i + \varphi_{\vec{k}}) \quad \text{with } k_i k^i = 0 \ \& \ e^\alpha{}_\alpha = 0 ; \ k^\alpha e_{\alpha\beta} = 0.$$

  The equivalence class of metrics thus defined ($\Phi_A \equiv A + \dot{B} - \ddot{E} = 0$; $\Phi_C = C - \frac{1}{3}\Delta E = 0$; $\Phi_{\bar{B}_\alpha} \equiv \bar{B}_\alpha - \dot{\bar{E}}_\alpha = 0$ & $\bar{E}_{\alpha\beta}$) represents a (linear) gravitational wave propagating on a flat background.

- As in electromagnetism one can fix the gauge by choosing a particular coordinate system. The "transverse, traceless" gauge is such that $B = E = E_{\bar{\alpha}} = 0$
  In that gauge the metric perturbations read (for a plane wave along z axis):

$$h^{TT}_{ij} = e_{ij} \cos[k(Z - cT) + \varphi] \quad \text{with} \begin{cases} e_{XX} = -e_{YY} \equiv e_+ \\ e_{XY} = e_{YX} \equiv e_\times \end{cases}$$

  so that: $\begin{cases} ds^2 = -(dX^0)^2 + (1+f_+)dX^2 + (1-f_+)dY^2 + 2f_\times dX dY + dZ^2. \\ \quad \text{with } f_{(+\times)} = e_{(+\times)} \cos[k(Z-cT) + \varphi]. \end{cases}$

3. Detecting gravitational waves

- The motion of a test mass in the field of the plane, monochromatic, wave propagating along the Z axis is given by the geodesic equation, the metric being given, in the TT gauge, by the expression above.

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{jk} \frac{dx^j}{d\tau} \frac{dx^k}{d\tau} = 0.$$

  Since, at linear order: $2\Gamma^i_{00} \sim \eta^{ij}(2\partial_0 h_{0j} - \partial_j h_{00}) = 0$, one sees that a particle, initially at rest, will remain at rest. This, of course, does not mean that gravity waves are mere coordinate effects (in which case they would "travel at the speed of thought" as Eddington put it!) but only that the TT coordinate system is comoving.

- To grasp better the action of GW on free test particles, let's go to a locally inertial frame in free fall with a given test particle (Fermi coordinates).
  The transformation is the following: $x^i \to \tilde{x}^i = x^i + \frac{1}{2}h^i{}_j x^j + \mathcal{O}(x^2)$.
  Indeed $\tilde{x}^i = 0 \iff x^i = 0$ is a geodesic and $\tilde{g}_{ij} = \eta_{ij} + \mathcal{O}(x^2)$
  [Demo: $x^0 = \tilde{x}^0$; $x = \tilde{x} - \frac{1}{2}(f_+ x + f_\times y)$; $y = \tilde{y} - \frac{1}{2}(-f_+ x + f_\times x)$; $z = \tilde{z}$
  $\tilde{g}_{ij} = g_{k\ell} \frac{\partial x^k}{\partial \tilde{x}^i} \frac{\partial x^\ell}{\partial \tilde{x}^j}$; $\tilde{g}_{00} = g_{00}$; $\tilde{g}_{0\alpha} = 0$; $\tilde{g}_{xx} = g_{xx}\left(\frac{\partial x}{\partial \tilde{x}}\right)^2 = (1+f_+)(1-\frac{1}{2}f_+)^2 \simeq 1$ etc]
  In these Fermi coordinates the geodesic equation becomes:

$$\frac{d^2\tilde{x}^\alpha}{d\tau^2} \simeq \frac{d^2\tilde{x}^\alpha}{d\tilde{t}^2} = -\tilde{\Gamma}^\alpha_{00}(\tilde{x}) = -\underbrace{\tilde{\Gamma}^\alpha_{00}(0)}_{=0 \text{ in this locally inertial frame}} - (\partial_\beta \tilde{\Gamma}^\alpha_{00})\tilde{x}^\beta$$

  On the other hand $\tilde{R}^\alpha{}_{0\beta 0} \simeq \partial_\beta \tilde{\Gamma}^\alpha_{00} \neq 0$;
  and, at 1st order: $\tilde{R}^\alpha{}_{0\beta 0} = \frac{\partial \tilde{x}^\alpha}{\partial x^\ell} \frac{\partial x^i}{\partial \tilde{x}^0} \frac{\partial x^j}{\partial \tilde{x}^\beta} \frac{\partial x^k}{\partial \tilde{x}^0} R^\ell{}_{ijk} \simeq R^\alpha{}_{0\beta 0}$

$$= -\frac{1}{2}\ddot{h}^\alpha_\beta \quad \text{in the TT gauge.}$$

  Therefore: $\frac{d^2\tilde{x}^\alpha}{dt^2} \simeq -R^\alpha{}_{0\beta 0}\tilde{x}^\beta \simeq \frac{1}{2}\ddot{h}_\beta{}^\alpha x^\beta$.
  In this locally inertial frame attached to the geodesic at the origin a particle at distance x^β from the origin is accelerated. And:

$$\tilde{x}^\alpha \simeq \frac{1}{2} h^\alpha{}_\beta x^\beta.$$

  This equation allows to describe the evolution of a ring of test particles in the metric of a plane monochromatic wave propagating along the z axis.

NB: The result would also have been obtained by means of the geodesic deviation equation: $a^i \equiv \frac{D^2 n^i}{d\lambda} = R^i{}_{mjk} u^m n^j u^k$ $\left(n^i = \frac{\partial x^i}{\partial \rho};\ u^i = \frac{\partial x^i}{\partial \lambda}\right)$

This geodesic deviation caused by a GW is the guiding line to detection "gedanken experiments": (1) The distance between two "freely" falling mirrors varies when a GW goes by; hence the fringe pattern of light in a Michelson-Fabry-Perot type interferometer is modulated by the GW.
(2) the proper length of a bar is also thus modulated hence producing currents in piezoelectric devices... see Vinet, Schutz...

4. 3+1 decomposition of Einstein's equations.

We saw that the linearized Einstein equations split into 4 constraint equations and 2 evolution equations for the 2 dynamical degrees of freedom of gravitational radiation. This split is due to the "gauge invariance" of Einstein's equations and hence is not specific to their linear approximation.

Indeed, consider a "synchronous" coordinate system where the line element reads: $ds^2 = -dt^2 + \gamma_{\alpha\beta}(t, x^\gamma)\, dx^\alpha dx^\beta$.
Such a coordinate system always exists, at least locally.

It is an easy exercise to see that the $\binom{0}{0}$ & $\binom{0}{\alpha}$ components of the Einstein tensor G^i_j DO NOT depend on the second time-derivatives of $\gamma_{\alpha\beta}$. The 4 Einstein vacuum equations $G^0_i = 0$ are thus 4 constraint equations (elliptic). The remaining equations are evolution eqns (of hyperbolic type). Y. Choquet-Bruhat showed in 1952 (using harmonic coordinates such that $\Box x^j = 0 \Rightarrow \partial_i(\sqrt{g}\, g^{ij}) = 0$) that the Cauchy pb for such equations was "well posed" that is that, given initial data $g_{ij}(t=0)$, $\partial_0 g_{ij}(t=0)$ satisfying the constraints, a solution can be built iteratively. The extension of such results to a finite region was done only fairly recently: see Y. Choquet-Bruhat, Eric Gourgoulhon, the workshop "Geometry to numerics" (and Christodoulou - Klainerman).

IV Gravitational radiation from moving masses

1. Electromagnetic radiation & the dipole formulae

Consider a system of interacting charges whose motion are determined by the Lorentz & Maxwell equations which read in a Lorentz gauge and inertial frame:

$$\partial_i F^{ji} = \frac{4\pi}{c} j^i \iff \Box A^i = -\frac{4\pi}{c} j^i \quad ; \partial_i A^i = 0 \quad (c=1)$$

The current vector j^i is: $j^i = \sum e \int d\tau\, \delta_4 [X^i - z^i(\tau)]\, u^i$

(indeed: $j^0 = \sum e \int d\tau\, \delta(X^0 - z^0)\, \delta_3(X^\alpha - z^\alpha) \frac{dX^0}{d\tau} = \sum e\, \delta_3(X^\alpha - z^\alpha)$

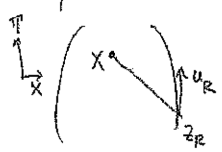
$j^\alpha = \sum e \int d\tau\, \delta(X^0 - z^0)\, \delta_3(X^\alpha - z^\alpha) \frac{dX^\alpha}{dX^0}\frac{dX^0}{d\tau} = \sum e\, \delta_3(X^\alpha - z^\alpha) V^\alpha$).

Hence the electromagnetic potential A^i created by the charges solves:

$$\Box A^i = -4\pi \sum e \int d\tau\, \delta_4 [X - z] u^i$$

whose solution can be obtained either by first writing $A^i = \sum A^i_{(e)}$ and going to the inertial frame where the charge e is temporarily at rest and where $\varphi' = e/r'$, $A' = 0$ and then returning to the original frame by means of a Lorentz transformation, or by using the propagator: $\Box D_R(X) = -4\pi \delta_4(X)$: $D_R(X) = \delta(X^0 - |\vec{X}|)/|\vec{X}|$; Both methods yield:

$$A^i = \sum e \left(\frac{u^i}{r}\right)_R \quad \text{"retarded potentials".}$$



where if z_R is such that $\eta_{ij}(X^i - z^i_R)(X^j - z^j_R) = 0$

then $r_R = -(X^i - z^i_R)\, u_{iR}$.

Since $\partial^i u^j_R = -\frac{du^j}{d\tau}(n^i_R + u^i_R)$ with $n^i_R = -u^i_R + (X^i - z^i_R)/r_R$

and $\partial^i r_R = n^i_R + \{r_R (n^i_R + u^i_R)\, n_{jR}\, du^j_R/d\tau\}$ one checks that $\partial_i A^i = 0$

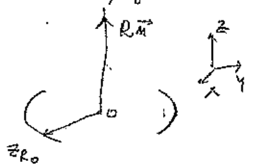
Introducing 3-D notation: $d\tau^2 = -dT^2(1 - V^2)$ where $V^\alpha = dX^\alpha/dT$ and $X^\alpha - z^\alpha_R \equiv X^\alpha_R$; $X^0 - z^0_R \equiv X^0_R$ $X_R \equiv \sqrt{\delta_{\alpha\beta} X^\alpha_R X^\beta_R}$ A^i reads:

$$A^0(T, \vec{X}) \equiv -\varphi = \sum \left[\frac{e}{X - (X \cdot V)}\right]_R \; ; \; \vec{A} = \sum \left[\frac{e \vec{V}}{X - (X \cdot V)}\right]_R$$

where $f_R \equiv f(t_R = T - X(t_R))$. (Lienard-Wiechert potentials)

Far away from ~~the~~ system: $\vec{X} = R\,\vec{n} - \vec{z}$ with $|z| \ll r_0$; if, moreover $v \ll 1$ (slow motion) then:

$$A^0 \simeq \frac{Q}{R_0} \quad ; \quad \vec{A} \simeq \left(\frac{\dot{\vec{d}}}{R}\right)_0$$



where $\begin{cases} Q = \Sigma e \text{ is the total charge} \\ \vec{d} = \Sigma e \vec{z} \text{ is the system dipole.} \end{cases}$

and where the index 0 means that the quantity is evaluated at the retarded time t_{R0} such that $t - t_{R0} = R_0$.

Similarly one obtains the electric & magnetic fields far away from the sources: $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla\varphi$; $\vec{H} = \nabla \wedge \vec{A}$

so that: $\vec{H} \sim \left.\frac{\ddot{\vec{d}} \wedge \vec{n}}{R}\right|_0$; $\vec{E} \sim \left.\frac{(\ddot{\vec{d}} \times \vec{n}) \times \vec{n}}{R}\right|_0$

The stress-energy tensor of this electromagnetic field is:

$$T_{ij} = \frac{1}{4\pi}\left(-F_{il}F^{l}{}_{j} - \frac{1}{4}\eta_{ij} F_{kl}F^{kl}\right)$$

and is such that $\partial_i T^{ij} = 0$ outside the charges. Hence the balance equation: $\frac{dP^i}{dt} = -\int T^{i\alpha} n_\alpha R_0^2 d\Omega$ where P^i is the 4-momentum of the field ($\int T^{0i} d^3X$) and the charges. Hence the energy radiated by the system is given by the $i=0$ component of that equation, that is:

$$\frac{dW}{dt} = -\frac{1}{4\pi}\int \epsilon_{\alpha\beta\gamma} E^\alpha H^\beta dS^\gamma = -\frac{2}{3c^3}(\ddot{d})_0^2$$

(dipole formula, see eg Landau-Lifchitz for more details). Hence a system of accelerated charges radiates electromagnetic waves and looses energy.

NB: $d = \Sigma e z$; com $d_{com} = \Sigma m z$; if m/e is the same for all charges $d_{com} \propto d$ hence $\ddot{d} = 0$ since the com has uniform motion. In that case the fields have to be Taylor expanded to next order and one finds (see e.g. Landau-Lifchitz):

$$\frac{dW}{dt} = \frac{1}{180c^5}\left(\dddot{D}_{\alpha\beta}\right)^2$$

where $D_{\alpha\beta} = \Sigma e\,(3 z_\alpha z_\beta - z^2 \delta_{\alpha\beta})$ is the quadrupole moment of the system.

The back reaction of this energy loss onto the motion of the charges is obtained by solving the Lorentz equations for each charge:

$$m\frac{du^i}{d\tau} = e\,F^i{}_j u^j \qquad F_{ij} = \partial_i A_j - \partial_j A_i$$

where $A^i = \Sigma e\left(\frac{u^i}{r}\right)_R$ has to be evaluated on the world line of the charge e.

now: $A^i = A^i_{self} + \sum_{e' \neq e} e'\left(\frac{u^i}{r}\right)_R$; the A^i_{self} part diverges on (L).

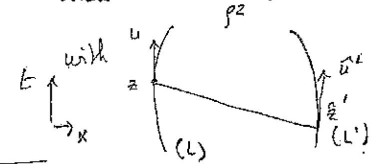
Its Taylor expansion near the line yields $F^{ij}_{self} = \frac{2e}{3\epsilon}[u^i \dot{u}^j - \dot{u}^j u^i] + \frac{2e}{3}[u^i \ddot{u}^j - \ddot{u}^i u^j]$ (after averaging over n^i: $x^\alpha - z^\alpha = \epsilon n^i$, $n.n = 1$, $n.u = 0$). Ignoring the divergent term (which can be interpreted as "renormalizing" the mass m) one obtains

$$(*) \quad m\frac{du^i}{d\tau} = ee'\,\hat{W}^i + ee'\,\hat{\dot{u}}'^j\,\hat{W}_j{}^i + e^2 W^i_A$$

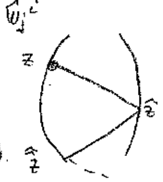
where $W^i_A = \frac{2}{3}\left(\ddot{u}^i - u^i \dot{u}^2\right)$ is the Lorentz-Abraham self-force.

where $\hat{W}^i = \frac{1}{\rho^2}(\varkappa u'^i - \omega v^i)$ and a similar expansion for $\hat{W}_j{}^i$

with



$$\begin{cases} \rho = -(z - \hat{z}')\,\hat{u}' \\ v^i = -\hat{u}'^i + \frac{z - \hat{z}'}{\rho} \\ \omega = (u\hat{u}') \qquad \varkappa = (v.u) \end{cases}$$



This equation of motion cannot be integrated straightaway as $\dot{u}^i$ at z depends on the position of the other charge at retarded time, which itself depends on the position of the particle we study at "retarded-retarded" time etc.

A technique developped by L. Bel and his school in the 70's consists in expanding (*) in the coupling constant ee'. ("Predictive mechanics")

A zeroth order: $du^i/d\tau = 0$: the 2 charges do not interact and their motion is uniform. At first order one solves (*) using the zeroth order trajectories in the right hand side: $m\frac{du^i}{d\tau} = ee'W^i$ where the retarded point $\hat{z}'$ can be easily calculated since the world line is taken to be a straight-line. Integration then yields the 1st order trajectory and the process can be iterated. The result is (see ND for details):

$$m\frac{du^i}{d\tau} = ee'\,W^i + (ee')^2\,\xi^i_{(2)a} + e^2 ee'\,\xi^i_{(2)b}$$

where W^i is $\hat{W}^i$ "without the hat" that is with $\hat{z}' \to z'$

and where $\xi^i_{(2)}$ also depend only on data at z' and NOT $\hat{z}'$.

The advantage of this transformation is that the eom remain manifestly Poincaré invariant.

Supposing then that the velocities remain small one performs a 3+1 split: $z^i - z'^i = (0, R\vec{N})$ and expands in powers of v. One obtains:

$$\begin{cases} m\dot{\vec{v}} = \vec{A}_0 + \frac{1}{c^2}\vec{A}_2 + \frac{1}{c^3}\vec{A}_3 + \dots \\ \vec{A}_0 = \frac{ee'}{R^2}\vec{N}; \quad \frac{1}{c^2}\vec{A}_2 = \frac{ee'}{c^2R^2}\left\{\vec{N}\left[\frac{1}{2}v'^2 - \frac{1}{2}v^2 (vv') - \frac{3}{2}(Nv')^2\right] - \vec{V}(Nv)\right\} + \frac{(ee')^2}{m'c^2}\frac{\vec{N}}{R^3} \quad (V = v' - v) \\ \frac{1}{c^3}\vec{A}_3 = \frac{e^2e'}{c^3R^3}\left(\frac{e'}{m'} - \frac{e}{m}\right)\left[2\vec{N}(NV) - \frac{2}{3}\vec{V}\right] \end{cases}$$

One may check that these equations are indeed Lorentz-invariant at $O(1/c^3)$).

One can also check that these eom can be obtained by Euler-variation of the Lagrangian of the charge e. Indeed: $S = \Sigma\int(-m_i\, d\tau_i + e A_i\, dX^i)$
$= \Sigma\int L\, dT$

with $L = [-m + A_0 u]\frac{d\tau}{dT}$ with $A^i = \Sigma\left(\frac{eu^i}{r}\right)_e$.

Regularising A^i (which diverges on (Z1)) and expanding in $1/c$ one obtains the "Darwin Lagrangian":

$$L = \Sigma m\left(\frac{1}{2}v^2 + \frac{1}{4}\frac{v^4}{c^2}\right) - \frac{ee'}{R} + \frac{ee'}{2c^2R}[(vv') + (Nv)(Nv')] + O(1/c^4)$$

whose Euler-variation yields $m\dot{v} = A_0 + \frac{1}{c^2}A_2$.

NB: To obtain the eom at order $O(1/c^4)$ included (apart from the $\frac{1}{c^3}$ radiative term) one must expand L to higher order. In that case L becomes a functional of z, v AND $a = \dot{v}$. One must NOT replace a by $a = \frac{ee'N}{mR^2}$ (that is by the Coulomb eom); one must vary L as:

$$0 = \frac{\partial L}{\partial z} - \frac{d}{dt}\frac{\partial L}{\partial v} + \frac{d^2}{dt^2}\frac{\partial L}{\partial a}.$$

- The existence of a Lagrangian yielding the $O(1/c^2)$ eom allows a definition of a conserved energy of the system at that order: $W = \Sigma pv - L$ with $p = \partial L/\partial v$.
- The explicit expression for W is:

$$W = \Sigma\left(\frac{1}{2}mv^2 + \frac{1}{2}\frac{ee'}{R}\right) + \frac{1}{c^2}\Sigma\left(\frac{3}{8}mv^4 + \frac{ee'}{R}\left[\frac{1}{4}(vv') + \frac{1}{4}(Nv)(Nv')\right]\right) \dots$$

One can also define the center of mass $\vec{K} = \vec{G} - t\vec{P}$ such that $\frac{d\vec{K}}{dt} = 0$ with $\vec{G} = \Sigma\vec{z}\left(m + \frac{1}{2c^2}mv^2 + \frac{ee'}{2c^2R}\right)$; $\vec{P} = \Sigma\frac{\partial L}{\partial v}$.

Now, the $\frac{1}{c^3}$ term in the eom cannot be deduced from a Lagrangian. It acts as an "external force" so that $\frac{d\vec{P}}{dt} = \Sigma\vec{F}$ and

$\frac{dW}{dt} = \Sigma(F \cdot v)$, with $F = \frac{1}{c^3}\vec{A}_3$

One can check therefore explicitly that this energy loss, obtained by solving the eom, is identical to the energy loss at infinity.

Furthermore one can compute the shrinking of the orbit due to this energy loss.

2. Outline of the pb in GR

(will be treated at length by T. Damour, L. Blanchet, B. Iyer, G. Schaefer ---).

A first difficulty, compared to electromagnetism, is that Einstein's equations are non linear. Indeed the Einstein tensor reads:

$$2|g|\left(R^{ij} - \frac{1}{2}g^{ij}R\right) = \Box h^{ij} - h^{kl}\partial^2_{kl}h^{ij} - \mathrm{II}^{ij} - \mathrm{III}^{ij} + O(h^4)$$

when $h^{ij} = \sqrt{-g}\,g^{ij} - g^{ij}$, $\mathrm{II}^{ij} = (\partial h)^2$; $\mathrm{III}^{ij} = (\partial h)^3$

in the harmonic gauges such that $\partial_i h^{ij} = 0$.

The stress energy-tensor is taken to be:

$$|g|T^{ij} = \Sigma m\int ds\, \delta_4(X - z)\frac{u^i u^j\sqrt{-g}}{(g_{ij}u^iu^j)^{1/2}} \qquad (\text{and } \eta_{ij}u^iu^j = -1)$$

Therefore it is only at linear order that we have:

$$\Box h^{ij} = \Sigma m\int ds\, \delta_4(X - z)u^iu^j \qquad (\text{similar to electromagnetism})$$

with solution $h^{ij} = \Sigma - 4m\left(\frac{u^iu^j}{r}\right)_R$.

When going to large distances & for small velocities one obtains (the computation is done as in electromagnetism):

$$h_{\alpha\beta} = \frac{2G}{c^4X_0}P_{\alpha\beta\gamma\delta}\ddot{Q}^{\gamma\delta}(t - X_0/c) \qquad (\text{1st quadrupole formula})$$

where $P_{\alpha\beta\gamma\delta}$ is some projection operator ($P = (\delta - nn)(\delta - nn)$) and where $Q_{\alpha\beta}$ is the quadrupole moment of the distribution.

A second difficulty is that there are no "obvious" conservation laws in GR. Indeed Bianchi's identity imply $D_i T^{ij} = 0$ and NOT $\partial_i T^{ij} = 0$. However (as discussed in Part 2) there $\exists$ conserved "pseudotensors", e.g. the Landau & Lifschitz one, such that:

$$\partial_i \left[(-g)(T^{ij} + t^{ij}) \right] = 0 \qquad \text{with} \quad t^{ij} = \Gamma\Gamma gg$$

which yields balance equation; eg $\dot{E} = -\int (-g)(T^{0\alpha} + t^{0\alpha}) n_\alpha X_0^2 \, dS$

$\left(E = \int (-g)(T^{00} + t^{00}) d^3 X \right)$.

Knowing the field at ∞, and using $\partial_i h^{ij} = 0$ one then obtains (see Landau. Lifschitz... Damour for details):

$$\frac{dE}{dt} = \frac{G}{5c^5} \left(\dddot{Q}_{\alpha\beta} \right)^2_{t - X_0/c} \qquad \text{2nd quadrupole formula.}$$

A third difficulty is to obtain the back reaction of this energy loss onto the motion of the masses. Indeed the linear approximation is not enough!

We have set $g_{..} = \eta_{..} + h_{..} \implies g^{..} = \eta^{..} - h + h^2 + h^3 + \dots$

$\implies \Gamma = g^{..} \partial g_{..} = (\eta + h + h^2 \dots) \partial h = \partial h + h \partial h + h^2 \partial h + \dots$

$\rightarrow$ Einstein's tensor: $G = \partial\Gamma + \Gamma\Gamma = \partial^2 h + (\partial h)^2 + h \partial^2 h + h^3 + \dots$

then energy tensor: $T = \Sigma m \int ds \, \delta_4 \frac{uu}{\sqrt{-g}} = \Sigma m \int ds \, \delta_4 \, uu \, (1 + h + h^2 + h^3 + \dots)$

Einstein's equation: $G = T$ (in harmonic gauge $\partial h = 0$)

1st order: $\partial^2 h = \Sigma m \int ds \, \delta_4 \, uu$ [precisely, see above: $\Box h^{ij} = 16\pi \Sigma m \int ds \, \delta_4(X - Z) u^i u^j$]

solution: $h = \Sigma \frac{m}{r} uu$ with $u^0 u^0 \sim 1$, $u^0 u^\alpha \sim v$, $u^\alpha u^\beta \sim v^2$

2nd order: $\partial^2 h = -(\partial h_1)^2 - h_1 \partial^2 h_1 + \Sigma m \int ds \, \delta_4 \, uu \, (1 + h_{(1)})$

$\hookrightarrow m \frac{uu}{R}$ regularized

sol (Bel et al 1982): $h = h_{(1)} + h_{(2)}$.

where there are terms of the type $\begin{cases} \frac{m}{r} v^2 & \text{coming from } h_{(1)} \\ \frac{m}{r} \frac{m (u^0)^2}{R} = \frac{m}{r} \frac{m}{R} & \leftarrow h_{(2)} \end{cases}$

Hence one can ignore the 2nd order if $m/R \ll v^2$: This is NOT the case for gravitationally bound objects (Eddington 1920).

To obtain the eom one must go to the 3rd iteration (at least).

Indeed the eom come from $\partial T = 0$.

Now $T = \Sigma m \int dz \, \delta_4 \, uu \, (1 + h + h^2 + h^3 + \dots)$

at 0th order $T = \Sigma m \int dz \, \delta_4 \, uu \implies \partial T = 0 \implies \dot{u} = 0$

1st order $T = \Sigma m \int dz \, \delta_4 \, uu \left(1 + \frac{m}{R}\right) \implies \partial T = 0 \implies \dot{u} = \frac{m}{R^2}$ (Newton).

etc hence $\dot{u} = \frac{m}{R^2} + \frac{m^2}{R^3} + \frac{m^3}{R^4}$

$h_{(1)}$ required, $h_{(2)}$ required, $h_{(3)}$ required.

Now, expand in $1/c$:

$$\dot{v} = \frac{m}{R^2} \left(1 + v^2 + v^4 + v^5 \dots \right) + \frac{m^2}{R^3} \left(1 + v^2 + v^3 \right) + \frac{m^3}{R^4} (1 + v)$$

The radiative force can be obtained only by iterating Einstein's equations 3 times.

When all this is done one can check explicitely that the loss of energy of the system ($\leftarrow$ shrinking of the orbit) is the same as the loss at ∞.

see Damour, Blanchet, Iyer, Schäfer... for explicit derivations.

The importance of this work: binary pulsar timings.