Equation of state for compact stars

Lecture 1

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Anticipation and prediction of neutron stars

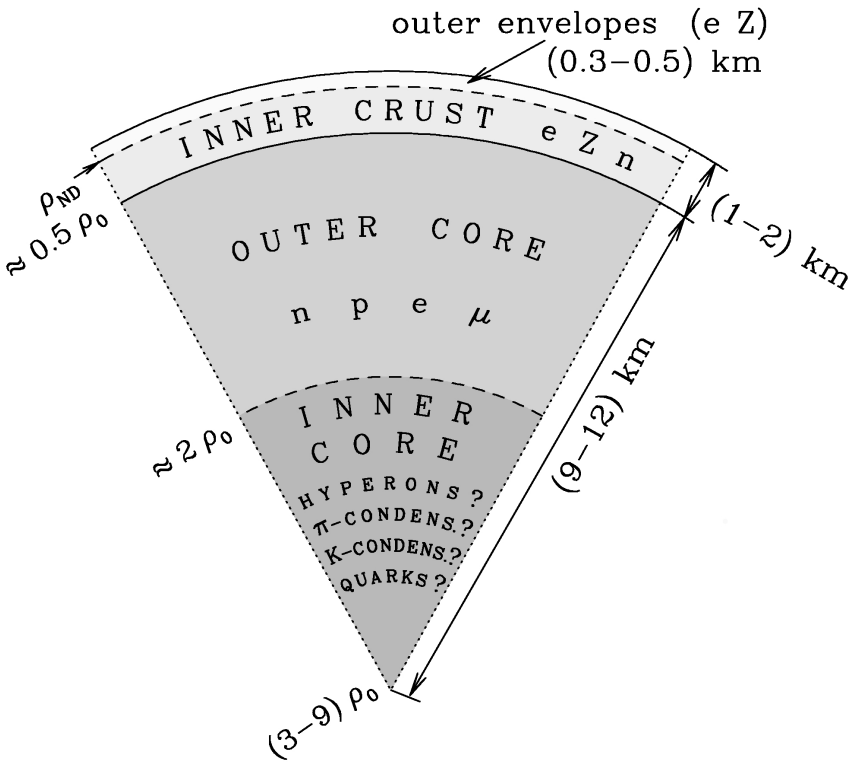
Anticipation - Landau February 1931
Widely known facts: Neutron discovered by J. Chadwick - Jan.1932, published in Nature on February 27, 1932 (Chadwick 1932). **A year before Chadwick paper - Lev Landau had written a paper devoted to dense stars!** Landau (23 at that time, graduate student of the Leningrad Physico-Technical Institute). Paper completed in February 1931 in Zürich, where Landau visited Pauli (Landau 1932). First part: brilliant (independent of Chandrasekhar) derivation of the mass limit for white dwarfs. Last part: for stars heavier than $1.5 M_\odot$ "*the density of matter becomes so great that atomic nuclei come in close contact, forming one gigantic nucleus.*" $\Longrightarrow$ correct description of dense matter in neutron star interiors, given before the discovery of the neutron.

Prediction - Baade and Zwicky December 1933

W. Baade (Mt. Wilson Observatory) and F. Zwicky (Caltech) analyzed observations of supernova explosions and proposed an explanation of an enormous energy release in these explosions. Results presented at the meeting of the American Physical Society at Stanford (December 15–16, 1933) and published in the 15 January issue of the *Physical Review* (Baade & Zwicky 1933) "*With all reserve we advance the view that supernovae represent the transitions from ordinary stars to* **neutron stars**, *which in their final stages consist of extremely closely packed neutrons*."

A brief summary of the neutron star prediction was presented to non-expert readers in the form of a cartoon in the *Los Angeles Times* on January 19th, 1934: *"Cosmic rays are caused by exploding stars which burn with a fire equal to 100 million suns and then shrivel from 1/2 million mile diameters to little spheres 14 miles thick, says Prof. Fritz Zwicky, Swiss Physicist."*

Neutron star interior: known and unknown



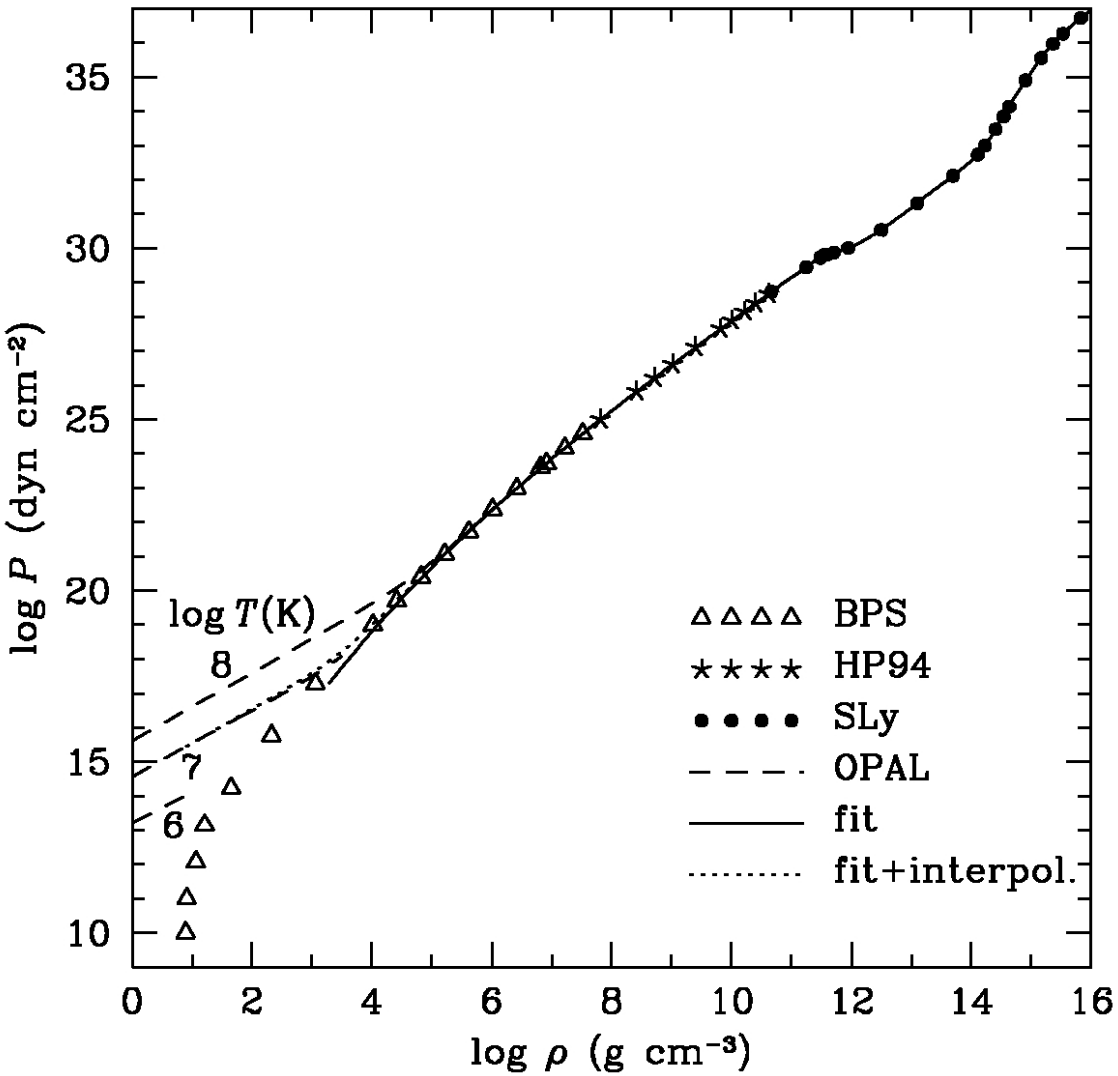
Plan

- White dwarfs and neutron star envelopes
- Neutron star crusts
- Neutron star cores: nucleons and hyperons
- Briefly - exotic neutron star cores, quark stars, etc.
- Briefly - confronting EOS models with mesured neutron star masses

Standard reference: S.L. Shapiro, S.A. Teukolsky *Black Holes, White Dwarfs, and Neutron Stars: The Physics of Compact Objects* (Wiley, 1983). Alas, 23-years old.

New advanced reference available soon: P. Haensel, A.Y. Potekhin, and D.G. Yakovlev *Neutron Stars 1. Equation of state and structure* (Springer, January 2007)

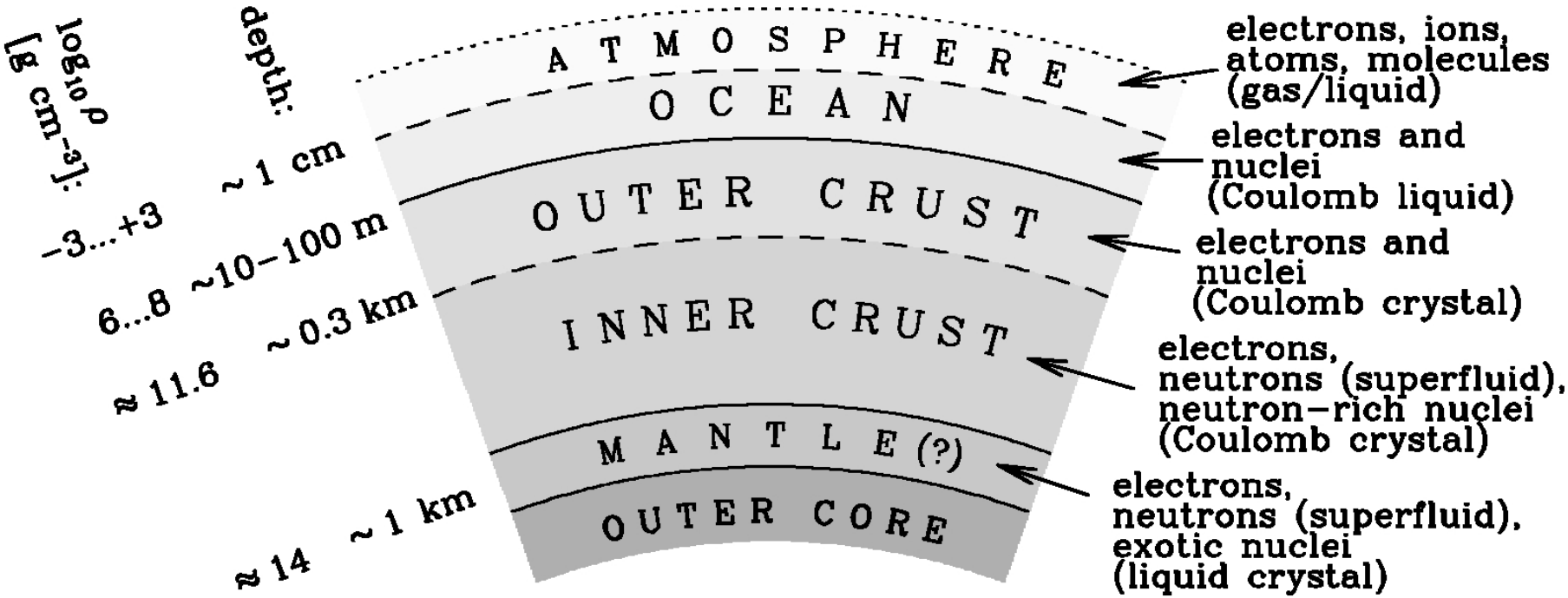
EOS - an overview



EOS of neutron star matter at $T = 0$: Baym et al. (1971) (triangles), Haensel & Pichon (1994)(HP94, stars), SLy - Douchin & Haensel (2001) (dots).

Finite T EOSs: OPAL (Rogers et al. 1996) at $T = 10^6$, 10^7, and 10^8 K (dashed lines), the fit (Haensel & Potekhin 2004)(the solid line) and the fit modified at low ρ (the dotted line). From Haensel & Potekhin (2004). For $\rho > 10^6$ g cm^{-3} T-effects invisible - matter fully degenerate, $T = 0$ approximation for the EOS is valid.

Structure of envelope and crust



Schematic structure of an envelope of a neutron star with the internal temperature $\sim 10^8$ K.

Plasma parameters

Outer envelope: one kind of nuclei (fully ionized atoms) (A, Z), $A = N + Z$. This is an approximation - one component plasma (OCP). Mean number density of nuclei (=ions) $n_{\mathcal{N}}$. (If several j-species of nuclei are present then mean number density of nuclei $n_{\mathcal{N}} = \sum_j n_j$.) Electron mean number density $n_e = Z n_{\mathcal{N}}$.

Outer envelope, outer crust: all nucleons bound in nuclei $\Longrightarrow$ mean nucleon (=baryon) number density $n_{\rm b} = A\, n_{\mathcal{N}}$

Inner envelope, inner crust $n_{\rm b} = A'\, n_{\mathcal{N}}$ where $A' \equiv A + A''$. Here A'' is the number of free (unbound) nucleons per one atomic nucleus. We have $A'' n_{\mathcal{N}} = n_n\,(1 - w)$, where w is the fraction of volume occupied by atomic nuclei.

Outer envelope $A' = A$, inner envelope $A' > A$.

Mass density $\rho = \mathcal{E}/c^2$, but in the envelope and crust $\rho \approx m_{\rm u} n_{\rm b}$, where $m_{\rm u} = 1.6605 \times 10^{-24}$ g is the atomic mass unit.

Electrons

Relativity parameter (often used instead of n_e)

$$x_{\rm r} \equiv \frac{p_{\rm F}}{m_e c} \approx 1.00884 \left(\frac{\rho_6 \langle Z \rangle}{A'} \right)^{1/3}, \tag{1}$$

where

$$p_{\rm F} = \hbar k_{\rm F} = \hbar \left(3\pi^2 n_e\right)^{1/3}, \tag{2}$$

is the electron Fermi momentum and $\rho_6 \equiv \rho/10^6$ g cm^{-3}. The Fermi energy

$$\epsilon_{\rm F} = c^2 \sqrt{(m_e c)^2 + p_{\rm F}^2} \tag{3}$$

Notice: electron rest energy $m_e c^2$ is included. The electron Fermi temperature is

$$T_{\rm F} = T_{\rm r} \left(\gamma_{\rm r} - 1\right), \tag{4}$$

where

$$T_{\rm r} = m_e c^2 / k_{\rm B} \approx 5.930 \times 10^9 \ {\rm K} \tag{5}$$

is the relativistic temperature unit, $\gamma_{\rm r} = \sqrt{1 + {x_{\rm r}}^2}$, and $k_{\rm B}$ is the Boltzmann constant. The electron gas is non-relativistic at $T \ll T_{\rm r}$ and $x_{\rm r} \ll 1$, and it is ultrarelativistic at $x_{\rm r} \gg 1$ or $T \gg T_{\rm r}$. It is nondegenerate at $T \gg T_{\rm F}$ and strongly degenerate at $T \ll T_{\rm F}$.

Electrons - continued

The density or temperature are called *relativistic* if the respective parameter $x_{\rm r}$ or

$$t_{\rm r} = T/T_{\rm r} \tag{6}$$

is large (> 1). The electron Fermi velocity $v_{\rm F} = \partial\epsilon_{\rm F}/\partial p_{\rm F}$.

One introduces **electron-sphere radius** = radius of a sphere of volume equal to a volume per one electron $= V/N_e = 1/n_e$,

$$a_e = [3/(4\pi n_e)]^{1/3} \ . \tag{7}$$

Electrons - continued

The strength of the electron Coulomb interaction in a plasma of nondegenerate electrons can be characterized by the electron Coulomb coupling parameter

$$\Gamma_e = \frac{e^2}{a_e k_{\rm B} T} \approx \frac{22.75}{T_6} \left(\rho_6 \frac{Z}{A'}\right)^{1/3} , \tag{8}$$

where $T_6 \equiv T/10^6$ K. We shall use this parameter Γ_e for any plasma conditions though Γ_e has no transparent physical meaning for degenerate electrons.
For further use, it is convenient to introduce the electron plasma temperature

$$T_{\rm pe} = \hbar\omega_{\rm pe}/k_{\rm B} \approx 3.300 \times 10^8 \, x_{\rm r} \sqrt{\beta_{\rm r}} \ {\rm K} \ . \tag{9}$$

where $\beta_{\rm r} = v_{\rm F}/c$ and **electron plasma frequency** is given by

$$\omega_{\rm pe} = \left(4\pi e^2 n_e/m_e^*\right)^{1/2} \tag{10}$$

Here, $m_e^* \equiv \epsilon_{\rm F}/c^2 = m_e \gamma_{\rm r}$ is the **effective dynamical mass** of an electron at the Fermi surface. Perturbations of electron density oscillate with a frequency $\omega_{\rm pe}$ - see Landau & Lifshitz *Stat. Phys. 1, Physical Kinetics*

Ions

$n_{\rm i} = n_{\mathcal{N}}$

The strength of the Coulomb interaction of ions is characterized by the Coulomb coupling parameter,

$$\Gamma = \Gamma_e Z^{5/3} = (Ze)^2/(a_{\rm i} k_{\rm B} T), \tag{11}$$

where

$$a_{\rm i} = a_e Z^{1/3} = (3n_i/4\pi)^{1/3} \tag{12}$$

is the **ion-sphere radius**.

At sufficiently high temperatures, the ions form a classical Boltzmann gas. With decreasing T, the gas gradually, without a phase transition, becomes a strongly coupled Coulomb liquid, and then (with a phase transition) a Coulomb crystal. The gas and liquid constitute the neutron star ocean, while the crystal is formed under the ocean.

Ions - continued

The gaseous regime of an OCP occurs for $\Gamma \ll 1$, or $T \gg T_{\rm l}$, where

$$T_{\rm l} = \frac{Z^2 e^2}{a_{\rm i} k_{\rm B}} \approx 2.275 \times 10^7 \, Z^2 \left(\frac{\rho_6}{A'}\right)^{1/3} \ {\rm K}. \tag{13}$$

Important feature: For a plasma of light elements (H, He,C) one has $T_{\rm l} < T_{\rm F}$, whereas for heavy elements (Fe, Ni, etc.) $T_{\rm l} \gtrsim T_{\rm F}$.
The ion-sphere radius $a_{\rm i}$, defined by Eq. (12), is the main ingredient of the **ion-sphere model** for strongly coupled Coulomb systems ($T \lesssim T_{\rm l}$) with nearly uniform electron background. In this model the matter is considered as an ensemble of ion spheres filled by the uniform electron background. It is supposed that any ion sphere contains an ion in its center. The radius $a_{\rm i}$ is chosen in such a way that electron charge within the sphere compensates the ion charge. Then the spheres are electrically neutral and can be treated as non-interacting between themselves.

Ions - continued

The scale-length that characterizes quantum-mechanical effects on thermodynamics of liquid and gaseous phases is the **ion thermal (de Broglie) wavelength**

$$\lambda_{\rm i} = \left(\frac{2\pi\hbar^2}{m_{\rm i} k_{\rm B} T} \right)^{1/2}, \tag{14}$$

where $m_{\rm i}$ is the ion (nucleus) mass. In the following, we shall also use also **electron thermal wavelength** λ_e, given by Eq. (14) with $m_{\rm i}$ replaced by m_e.

Ions - continued

The crystallization of a Coulomb plasma occurs at a well defined temperature

$$T_{\rm m} = \frac{Z^2 e^2}{a_{\rm i} k_{\rm B} \Gamma_{\rm m}} \approx 1.3 \times 10^5 \, Z^2 \left(\frac{\rho_6}{A'}\right)^{1/3} \frac{175}{\Gamma_{\rm m}} \ {\rm K}, \tag{15}$$

where $\Gamma_{\rm m}$ is the "melting value" of Γ. For a classical OCP, $\Gamma_{\rm m} \approx 175$.
At low T the quantum effects on ion motion become important. They are especially pronounced at $T \ll T_{\rm pi}$, where

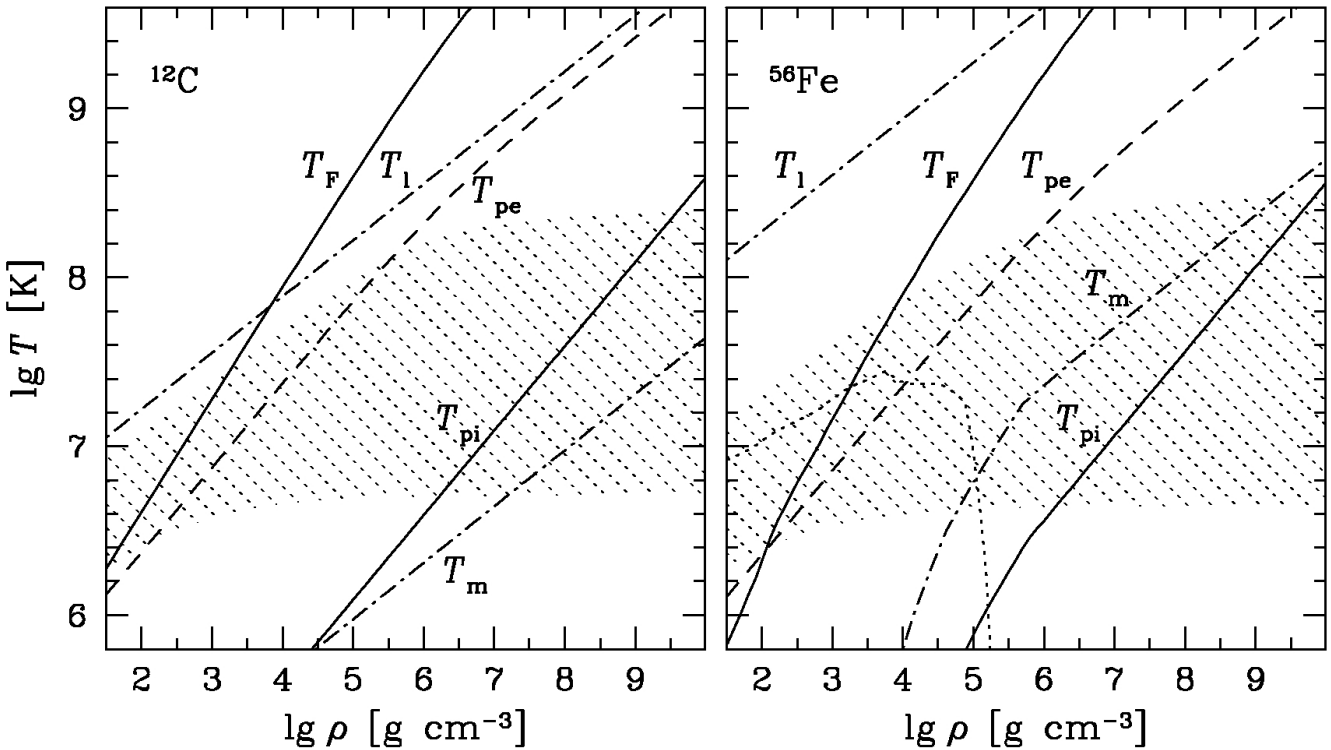
$$T_{\rm pi} \equiv \frac{\hbar \omega_{\rm pi}}{k_{\rm B}} \approx 7.832 \times 10^6 \left(\frac{\rho_6}{A'} \frac{Z^2}{A}\right)^{1/2} \ {\rm K} \tag{16}$$

is the plasma temperature of ions determined by the ion plasma frequency

$$\omega_{\rm pi} = \left(4\pi e^2 \, n_{\rm i} Z^2 / m_{\rm i}\right)^{1/2}. \tag{17}$$

If $T_{\rm pi} \gg T_{\rm m}$, huge zero-point ion vibrations suppress crystallization and reduce the melting temperature. Under these conditions, the actual melting temperature decreases with growing ρ and drops to zero at a certain $\rho = \rho_{\rm m}$.

$\rho - T$ diagram for the outer envelope



Density-temperature diagram for the outer envelope composed of carbon (*left*) or iron (*right*). Electron Fermi temperature ($T_{\rm F}$), electron and ion plasma temperatures ($T_{\rm pe}$ and $T_{\rm pi}$), temperature of the gradual gas-liquid transition ($T_{\rm l}$), and the temperature of the sharp liquid-solid phase transition ($T_{\rm m}$). Shaded: typical temperatures in the outer envelopes of middle-aged cooling neutron stars (age $\sim 10^4 - 10^6$). Corner bounded by the dotted line: strong electron non-uniformity or bound-state formation (atoms!) in the iron plasma. Difficult region!

Pressure

Decomposition

$$P = P_{\rm id}^{\rm (i)} + P_{\rm id}^{(e)} + P_{\rm xc} + P_{\rm ii} + P_{\rm ie}. \tag{18}$$

$P_{\rm id}^{\rm (i)}$ - ideal Boltzmann gas of ions

$P_{\rm id}^{(e)}$ - ideal Fermi gas of electrons

$P_{\rm xc}$ - exchange corrections in electron gas, negligible for $\rho \gg 1$ g cm^{-3}

$P_{\rm ii}$ - Coulomb corrections calculated assuming an uniform ("rigid") electron background

$P_{\rm ie}$ - Coulomb corrections owing to electron polarization (inhomogeneities)

Pressure - electrons - main term

(id)e pressure and the number density are functions of μ_e and T:

$$\begin{aligned} P_{\rm id}^{(e)} &= 2k_{\rm B}T \int \ln\left[1+\exp\left(\frac{\mu_e-\epsilon}{k_{\rm B}T}\right)\right]\frac{{\rm d}^3p}{(2\pi\hbar)^3} \\ &= \frac{8}{3\sqrt{\pi}}\frac{k_{\rm B}T}{\lambda_e^3}\left[I_{3/2}(\chi,t_{\rm r})+\frac{t_{\rm r}}{2}I_{5/2}(\chi,t_{\rm r})\right], \end{aligned} \tag{19}$$

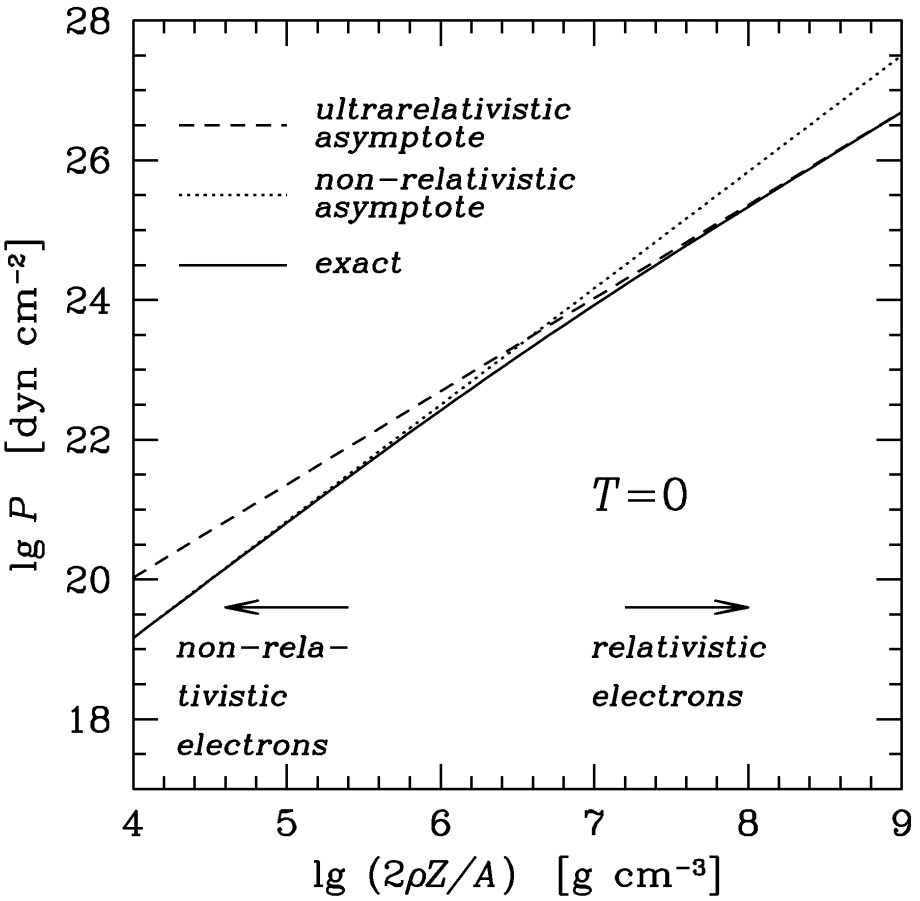
$$n_e = 2\int f^{(0)}(\epsilon-\mu_e,T)\frac{{\rm d}^3p}{(2\pi\hbar)^3} = \frac{4}{\sqrt{\pi}\,\lambda_e^3}\left[I_{1/2}(\chi,t_{\rm r})+t_{\rm r}I_{3/2}(\chi,t_{\rm r})\right], \tag{20}$$

where λ_e is the electron thermal wavelength, $\chi=(\mu_e-m_ec^2)/(k_{\rm B}T)$, and

definitions of λ_e, $t_{\rm r}$ - in previous slides

$$I_\nu(\chi,\tau) \equiv \int_0^\infty \frac{x^\nu\,(1+\tau x/2)^{1/2}}{\exp(x-\chi)+1}\,{\rm d}x \tag{21}$$

is a Fermi-Dirac integral.



The electron pressure $P = P_{\rm id}^{(e)}$ versus density for an ideal strongly degenerate electron gas (the solid line) together with non-relativistic and ultrarelativistic asymptotes (the dotted and dashed lines).

The internal energy is given by

V=volume of plasma

$$\frac{U_{\rm id}^{(e)}}{V} = \frac{4}{\sqrt{\pi}} \frac{k_{\rm B}T}{\lambda_e^3} \left[I_{3/2}(\chi, t_{\rm r}) + t_{\rm r}\, I_{5/2}(\chi, t_{\rm r}) \right] . \qquad (22)$$

Coulomb gas of ions

(id)i *Ideal classical gas*. The free energy of $N = N_{\mathcal{N}} = n_{\mathcal{N}} V \qquad (n_{\mathcal{N}} = n_{\rm i})$ - non-relativistic classical ions is

$$F_{\rm id}^{\rm (i)} = N_{\mathcal{N}} k_{\rm B} T \left[\ln(n_{\mathcal{N}} \lambda_{\rm i}^3 / g_{\rm i}) - 1 \right], \tag{23}$$

where $g_{\rm i}$ is the spin degeneracy (the rest-mass energy is not included here). The internal energy and pressure contributions from the ideal gas of ions are simply

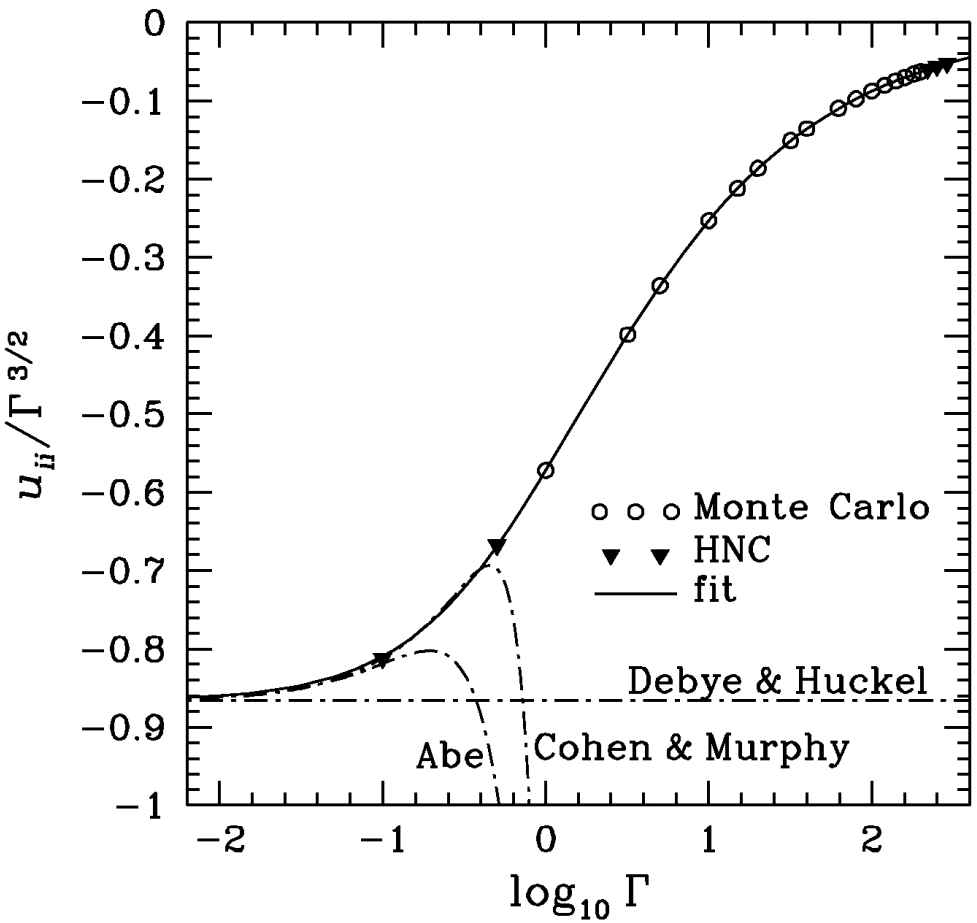
$$U_{\rm id}^{\rm (i)} = \frac{3}{2} N_{\mathcal{N}} k_{\rm B} T, \quad P_{\rm id}^{\rm (i)} = n_{\mathcal{N}} k_{\rm B} T. \tag{24}$$

(ii) *Ion correlations* Additional terms from Coulomb interactions. In a classical OCP the quantity $F_{\rm ii}/N_{\mathcal{N}} k_{\rm B} T$ is a function of a single argument Γ. This function determines all other "excess" quantities which can also be expressed via some universal functions of Γ. Methods for calculation of the "ii" term - Monte Carlo simulations and hypernetted chain (HNC) method. Very precise fitting formula to numerical results of DeWitt et al. (1996) is

$$u_{\rm ii} \equiv \frac{U_{\rm ii}}{k_{\rm B} T N_{\mathcal{N}}} = \Gamma^{3/2} \left(\frac{A_1}{\sqrt{A_2 + \Gamma}} + \frac{A_3}{1 + \Gamma} \right) + \frac{B_1 \, \Gamma^2}{B_2 + \Gamma} + \frac{B_3 \, \Gamma^2}{B_4 + \Gamma^2}, \tag{25}$$

where A_i and B_i are constants.

Instructive figure



Comparison of the fit (25) with the Debye-Hückel (see, e.g., Landau & Lifshitz *Stat. Phys. 1*, Abe (1959), and Cohen-Murphy (1969) approximate expressions valid at small $\Gamma \ll 1$ and with very precise extensive Monte Carlo and HNC calculations. It is clear that Coulomb contribution to the EOS of strongly-coupled OCP with $\Gamma > 1$ is difficult to calculate. One has to be very careful, and not use approximate low-Γ expressions if $\Gamma > 1$!

Strongly coupled Coulomb liquid

$N = N_{\mathcal{N}} = N_{\rm i} = n_{\mathcal{N}} V$

In the gaseous regime the Coulomb energy is small, $|U_{\rm ii}| \ll N k_{\rm B} T$, so that the Coulomb binding is lower than the thermal energy of ions, and the ions are nearly free. In the strongly coupled liquid ($\Gamma \gtrsim 1$) one has $|U_{\rm ii}| \gtrsim N k_{\rm B} T$ and the Coulomb energy exceeds the thermal one. In this regime the ions are bound in Coulomb potential wells (whose depth is $\sim Z^2 e^2 / a_{\rm i}$) and slightly oscillate there due to thermal motion. In a Coulomb liquid there is no exact order and the wells slowly migrate within the liquid. In the limit of $\Gamma \gg 1$ from Eq. (25) one obtains the linear Γ dependence, $U_{\rm ii}/N k_{\rm B} T \approx A_1\, \Gamma$. In this limit $U_{\rm ii} \approx N A_1 Z^2 e^2 / a_{\rm i}$ and $P_{\rm ii} \approx A_1\, n_{\mathcal{N}} Z^2 e^2 / (3 a_{\rm i})$ are nearly independent of temperature.

Numerical factor A_1 is very close to the value $A_1^{\rm IS} = -0.9$ predicted by the **ion-sphere model**. In that model $n_e = const.$, Coulomb energy of one ion sphere is equal to $-0.9 Z^2 e^2 / a_{\rm i}$: it is a sum of (repulsive) electron-electron contribution $0.6 Z^2 e^2 / a_{\rm i}$, and the attractive electron-ion term $-1.5 Z^2 e^2 / a_{\rm i}$ (see, e.g., Shapiro & Teukolsky 1983).

Crystallization of ions

At $T < T_{\rm m}$, an infinite (unlimited, unbound) ion motion is replaced by oscillations near equilibrium positions: a crystal is formed.
Internal energy at $T = 0$ consists of two parts,

$$U_0 = U_{\rm M} + U_{\rm q},, \tag{26}$$

where

$$U_{\rm M} = -N_{\mathcal{N}} C_{\rm M} (Ze)^2 / a_{\rm i}. \tag{27}$$

Here, $U_{\rm M}$ is the classical static-lattice energy, $C_{\rm M}$ is the *Madelung constant*. $U_{\rm q}$ accounts for the zero-point quantum vibrations.

$C_{\rm M}({\rm bcc}) = 0.895929$ (Fuchs 1935) bcc - body-centered-cubic

Klaus Fuchs - famous for his role in the Soviet A-bomb program in 1940s

$C_{\rm M}({\rm fcc}) = 0.895874,$ fcc - face-centered-cubic

$C_{\rm M}({\rm hcp}) = 0.895838$ fcc - hexagonal closest packing, see, e.g., Kittel 1986.

The ion-sphere expression $U_{\rm M} = -0.9\, N_{\mathcal{N}}\, Z^2 e^2 / a_{\rm i}$ is sufficiently accurate for the ion liquid and solid, as long as the ions are strongly coupled ($\Gamma \gg 1$)

Melting - phase transition

The solid-liquid phase transition occurs at $\Gamma = \Gamma_{\rm m}$, where the free energies $F(\Gamma)$ of liquid and solid phases intersect. It is difficult to find $\Gamma_{\rm m}$ with high precision, because the intersection of the free energy curves is strongly affected by the thermal corrections which are small at $\Gamma \sim \Gamma_{\rm m}$ (the free energy curves are nearly parallel). For instance, a 0.1% error in $F_{\rm ii}$ shifts the intersection by $\Delta\Gamma \approx 15$ (Pollock & Hansen 1973)!

The phase transition is 1st order but very weak. Fix the number density of ions and decrease the temperature $\Longrightarrow$ discontinuous changes of the ion energy $U_{\rm i}$ and the ion pressure $P_{\rm i}$ at the melting point. One can easily show that both quantities undergo discontinuous drops at $\Gamma = \Gamma_{\rm m}$. The drop of the ion energy density $U_{\rm i}/V$, associated with the crystallization to the bcc lattice at $\Gamma_{\rm m} = 175$, is $\Delta U_{\rm i}/V \approx 0.76\, n_{\mathcal{N}} k_{\rm B} T_{\rm m}$, with $\Delta U_{\rm i}/U_{\rm i} \approx 0.5\%$, and the drop of the ion pressure $\Delta P_{\rm i}/P_{\rm i}$ is approximately three times smaller. Because the ion pressure is only a small part of the total pressure P (the leading part $P_{\rm e}$ is provided by the electrons), the fractional drop of the total pressure $\Delta P/P$ is much smaller than the fractional drop of the ion pressure $\Delta P_{\rm i}/P_{\rm i}$.

Latent heat at freezing

The latent heat released at freezing equals

$$Q_{\rm latent} = U_{\rm m,liq} - U_{\rm m,solid} \approx \Delta U_{\rm i}, \tag{28}$$

where and $U_{\rm m,liq}$ and $U_{\rm m,solid}$ are the corresponding values of the internal energy at $T = T_{\rm m}$.

$Q_{\rm latent}$ is only $\approx 0.5\%$ of the ion contribution $U_{\rm i}$, which is only a tiny fraction of the total internal energy U (mainly determined by degenerate electrons at the conditions mentioned above).

Nevertheless, the absolute value of $Q_{\rm latent}$ ($\approx 0.76 k_{\rm B} T$ per ion) is a substantial fraction of thermal thermal energy of the ions. It is sufficient, for instance, to delay the cooling of old white dwarfs with crystallizing cores (Chabrier 1999, Hansen 2004). $\Longrightarrow$ important for the white dwarf chronology.

Note: *dense matter may form a supercooled liquid and freeze at* $T < T_{\rm m}$.

Quantum zero-point vibrations and melting

At high densities the amplitude of zero-point quantum vibrations of ions may become comparable to the lattice constant and make lattice ordering difficult: $T_{\rm m}(\rho)$ becomes then significantly lower than the classical melting temperature and at some critical density $\rho_{\rm m}$ zero-point vibrations destroy the lattice even at $T=0$ (i.e., $T_{\rm m}(\rho_{\rm m})=0$). Hence at $\rho>\rho_{\rm m}$ the liquid does not solidify at all, and one obtains the *quantum liquid* domain at $T\lesssim T_{\rm pi}$ and $\rho\gtrsim\rho_{\rm m}$. In that domain spin statistics of ions (fermion or boson) is important.

In practice the appearance of quantum liquid is actually important only for hydrogen and helium in the outer neutron star envelope, where zero-point motion drastically suppresses the crystallization of these ions.

Electrostatic corrections to electron presure

Take the pressure $P_{\rm id}^{(e)}$ of the ideal fully degenerate electron gas ($T=0$ approximation is valid) and supplement it with the Coulomb correction calculated within the ion-sphere model, $P_{\rm ii} \approx -0.3\, n_{\mathcal{N}}\, Z^2 e^2/a_{\rm i}$. This gives

$$P = \frac{P_{\rm r}}{8\pi^2}\left[x_{\rm r}\left(\frac{2}{3}\,{x_{\rm r}}^2 - 1\right)\gamma_{\rm r} + \ln(x_{\rm r}+\gamma_{\rm r})\right] - 0.3\, n_{\mathcal{N}}\,\frac{Z^2 e^2}{a_{\rm i}}. \qquad (29)$$

This EOS can be used in the outer envelope of NS not too close to the surface. It is also widely used in white dwarf cores. It is temperature independent and applies to any composition of the matter. Its derivation implies that the electron contribution $P_{\rm id}^{(e)}$ should be much larger than the Coulomb correction $P_{\rm ii}$. In the ultrarelativistic electron gas ($x_{\rm r} \gg 1$) both pressures have the same (polytropic) density dependence, $P_{\rm id}^{(e)} \propto P_{\rm ii} \propto {x_{\rm r}}^4$. Therefore, their ratio is then density independent being determined only by the ion charge number Z,

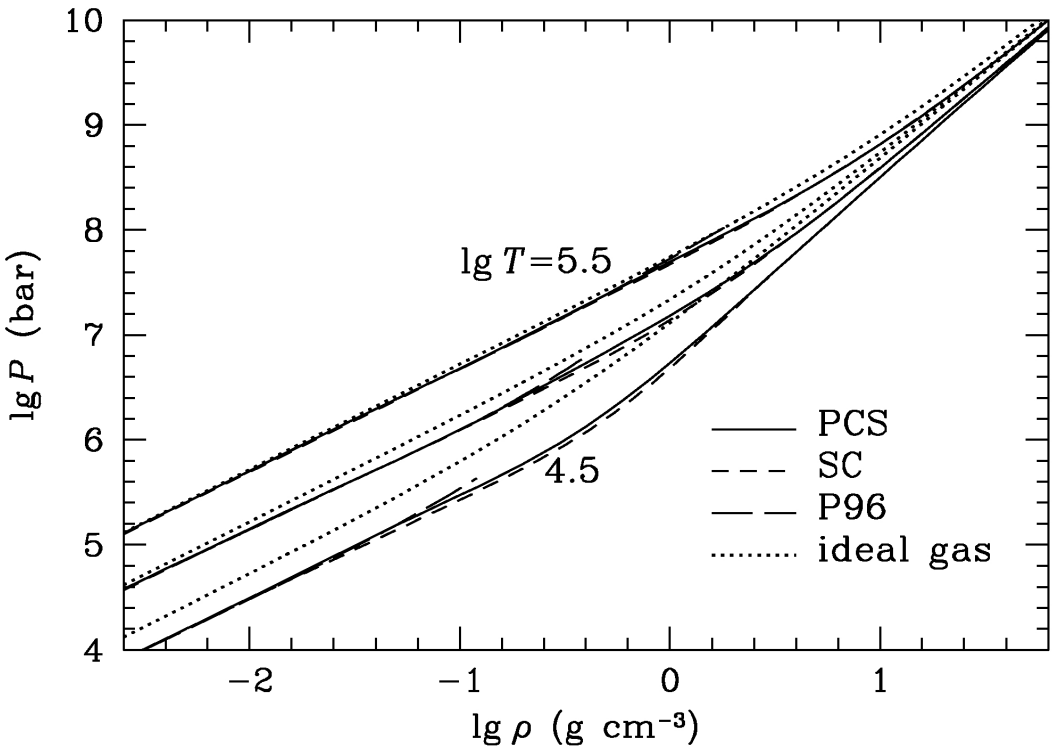
$$\frac{P_{\rm ii}}{P_{\rm id}^{(e)}} = -\frac{6}{5}\left(\frac{4}{9\pi}\right)^{1/3}\frac{Z^{2/3}e^2}{\hbar c} \approx -0.0046\, Z^{2/3}, \qquad (30)$$

which gives $P_{\rm ii}/P_{\rm id}^{(e)} \approx -0.04$ for the matter composed of iron.

Other corrections relevant to electron contribution

Electron exchange and correlation, electron polarization in ion liquid and ion solid, partial ionization.

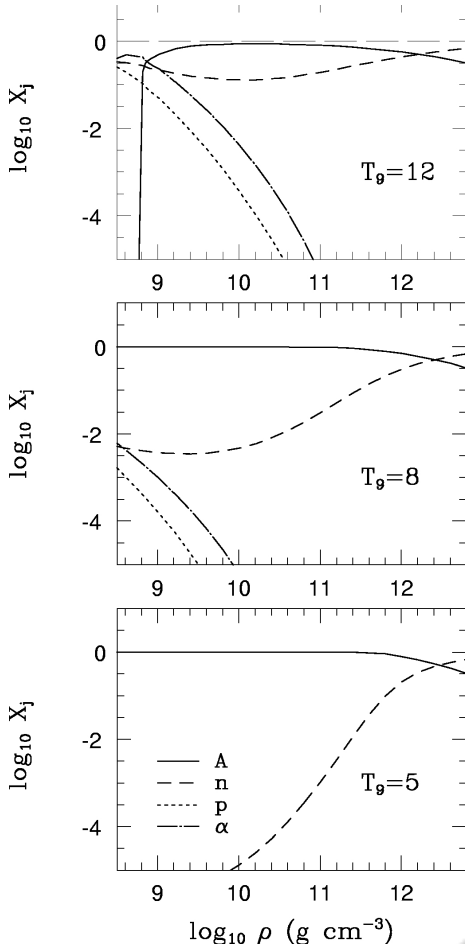
An example: EOS of hydrogen plasma accreted onto neutron star



Pressure isotherms ($\lg T$ [K] $= 4.5$, 5.0, and 5.5) of partially ionized hydrogen given by three theoretical models, compared with the EOS of ideal fully ionized gas (shown by dotted lines). Models: PCS - Potekhin et al. 1999b; P96 - Potekhin 1996b; SC - Saumon et al. 1995

Formation of the crust in newly born neutron stars

NS is born as a very hot object - $T \sim 10^{11}$ K. Cools rapidly due to neutrino emission, and within hours the bulk of outer 1% of mass has $T < 10^{10}$ K, and within a year it gets cooled to $\sim 10^9$ K.



Mass fractions of different constituents of the outer envelope of a newly born neutron star versus matter density in beta equilibrium at different temperatures $T_9 = T/(10^9 \text{ K})$ (after Haensel et al. 1996). Calculations are performed for the Lattimer & Swesty (1991) model of nuclear matter with a specific choice ($K_0 = 220$ MeV) of the incompressibility of cold symmetric nuclear matter at the saturation density.

After one year the thermal effects for $\rho > 10^6$ g cm^{-3} are negligible $T = 0$ approximation for the EOS is fine. Simplifying assumptions: single A, Z-nucleus; crust in the ground state.

Ground state crust - $T = 0$

Notation: enthalpy per nucleon $= h$; energy per nucleon$=E$; energy density - $\mathcal{E}$.

Neglect envelope of NS with $\rho < 10^8$ g cm^{-3} (it has mass $< 10^{-8} M_\odot$) - it could have been accreted, and the T-effects there might be significant. Consider crust at $T = 0$ and $P = 0$, i.e., $h = E = \mathcal{E}/n_{\rm b}$. In this case the minimum energy per nucleon (ground state) is reached for a body-centered-cubic (bcc) lattice of ^{56}Fe, and is $E(^{56}\mathrm{Fe}) = 930.4$ MeV. It corresponds to $\rho = 7.86$ g cm^{-3} and $n_{\rm b} = 4.73 \times 10^{24}$ cm$^{-3} = 4.73 \times 10^{-15}$ fm^{-3}.

Remark It is worth to mention that ^{56}Fe is not the most tightly bound free atomic nucleus. Binding energy per nucleon $b \equiv [(A-Z)m_n c^2 + Z m_p c^2 - M(A,Z)c^2]/A$ in a nucleus with the ground-state mass $M(A,Z)$ reaches maximum for ^{62}Ni, $b(^{62}\mathrm{Ni}) = 8.7945$ MeV, to be compared with $b(^{56}\mathrm{Fe}) = 8.7902$ MeV. Let us notice that $b(^{58}\mathrm{Fe}) = 8.7921$ MeV is also higher than $b(^{56}\mathrm{Fe})$.

Ground state of matter at a given P or $n_{\rm b}$ is called **cold catalyzed matter**. Cold - because $T = 0$. Catalyzed - because all reaction have been completed so that the ground state has been reached.

Ground state crust - $T = 0$

Ion-sphere model used - $\Gamma \gg 1$. Terminology: ion sphere $\equiv$ (spherical) unit cell, (spherical) **Wigner-Seitz cell**.
Charge neutrality of the unit cell, under pressure P,

$$n_e = Z n_{\mathcal{N}} \ , \qquad P = P_e(n_e, Z) + P_{\rm L}(n_{\mathcal{N}}, Z) \ , \tag{31}$$

where P_e is the electron pressure and $P_{\rm L}$ is the "lattice" contribution (also called the electrostatic correction) resulting from the Coulomb interactions ($P_{\rm L} = P_{\rm ii}$). Spherical "unit cell" per one nucleus in a OCP. At $T = 0$ enthalpy of the cell=Gibbs free energy ($G = H - TS$) of the cell $= G_{\rm cell}$.

$$G_{\rm cell}(A, Z) = W_{\mathcal{N}}(A, Z) + W_{\rm L}(Z, n_{\mathcal{N}}) + [\mathcal{E}_e(n_e, Z) + P]/n_{\mathcal{N}} \ , \tag{32}$$

where $W_{\mathcal{N}}$ is the energy of the nucleus (including rest energy of nucleons), W_L is the lattice (Coulomb) energy per cell, and $\mathcal{E}_e$ is the mean electron energy density. Neglecting quantum and thermal corrections and the nonuniformity of the electron gas, we have

$$W_{\rm L} = -C_{\rm M}\ Z^2 e^2 / r_{\rm c}, \quad C_{\rm M} \approx 0.9. \tag{33}$$

Ground state crust - $T = 0$

The lattice (Coulomb) contribution to the pressure, Eq.(31), is thus $P_{\rm L} = P_{\rm ii} = \frac{1}{3} W_{\rm L} n_{\mathcal{N}}$.
The Gibbs free energy per nucleon $h = G_{\rm cell}/A$ is just the **baryon (=nucleon) chemical potential** $\mu_{\rm b}(A, Z)$ for a given nuclide. To determine the ground state at a given P, one has to minimize $\mu_{\rm b}(A, Z)$ with respect to A and Z.
To a very good approximation, a density jump, at which optimal values (A, Z) change into (A', Z'), is given by

$$\frac{\Delta\rho}{\rho} \approx \frac{\Delta n_{\rm b}}{n_{\rm b}} \approx \frac{Z}{A}\,\frac{A'}{Z'} - 1\ . \tag{34}$$

This equation follows from the continuity of the pressure $P \simeq P_e$.
A sharp discontinuity in ρ and $n_{\rm b}$ is a consequence of the assumed one-component plasma model. Detailed calculations of the ground state of dense matter by Jog & Smith (1982) show, that actually the transition between $(A,\ Z)$ and $(A',\ Z')$ shells takes places through a very thin layer of a *mixed state* of these two species. However, since the pressure interval, where this mixed phase exists, is $\sim 10^{-4} P$, the approximation of a sharp density jump is quite adequate.

A model of ground state crust

Haensel & Pichon (1994)(HP94) took experimental masses of nuclei from the tables of Audi (1992, 1993 – private communication).[1] Because of **pairing** effect (nucleons in nuclei are superfluid!) only even-even nuclei are relevant for the ground-state problem. For the remaining isotopes, up to the last one stable with respect to the emission of a neutron pair, HP94 used theoretical masses obtained from the mass formula of Möller (1992, private communication).).
The equilibrium nuclides present in the "cold catalyzed matter" are listed in Table 1. In the fifth column one finds the maximum density $\rho_{\max}$ at which a given nuclide is present. The sixth column gives the electron chemical potential μ_e at $\rho = \rho_{\max}$. The transition to the next nuclide has a character of a first-order phase transition. The corresponding fractional density jump $\Delta\rho/\rho$ is given in the last column. The last row above the horizontal line, which divides the table into two parts, corresponds to the maximum density, at which the ground state of dense matter contains a nucleus with mass measured in laboratory. The last row of Table 1 corresponds to the neutron drip point which is determined theoretically.

[1] Some masses of unstable nuclei in these tables are actually semi-empirical evaluations based on the knowledge of masses of neighboring isotopes. More recent evaluations of nuclear masses are given by Audi et al. (1997) and Audi et al. (2003).

Ground state crust - $T = 0$ - nuclei

Table: Nuclei in the ground state of cold dense matter (after Haensel & Pichon 1994), with slight modification. Upper part is obtained with experimentally measured nuclear masses. Lower part: from mass formula of Möller. The last line corresponds to the neutron drip point. For a recent re-calculation of the ground state of the crust - see Ruster et al. (2006).

element	Z	N	Z/A	$\rho_{\max}$ (g cm^{-3})	μ_e (MeV)	$\Delta\rho/\rho$ (%)
^{56}Fe	26	30	0.4643	7.96×10^{6}	0.95	2.9
^{62}Ni	28	34	0.4516	2.71×10^{8}	2.61	3.1
^{64}Ni	28	36	0.4375	1.30×10^{9}	4.31	3.1
^{66}Ni	28	38	0.4242	1.48×10^{9}	4.45	2.0
^{86}Kr	36	50	0.4186	3.12×10^{9}	5.66	3.3
^{84}Se	34	50	0.4048	1.10×10^{10}	8.49	3.6
^{82}Ge	32	50	0.3902	2.80×10^{10}	11.4	3.9
^{80}Zn	30	50	0.3750	5.44×10^{10}	14.1	4.3
^{78}Ni	28	50	0.3590	9.64×10^{10}	16.8	4.0
^{126}Ru	44	82	0.3492	1.29×10^{11}	18.3	3.0
^{124}Mo	42	82	0.3387	1.88×10^{11}	20.6	3.2
^{122}Zr	40	82	0.3279	2.67×10^{11}	22.9	3.4
^{120}Sr	38	82	0.3167	3.79×10^{11}	25.4	3.6
^{118}Kr	36	82	0.3051	(4.32×10^{11})	(26.2)	

Approximate calculation of neutron drip point

A rough estimate of the neutron drip density can be obtained using a simplified version of the nuclear mass formula. Neglecting Coulomb, surface and all other finite-size terms and keeping only quadratic term in $\delta = (N - Z)/A$, one can write the energy per nucleon in an atomic nucleus (subtracting the rest energy and neglecting neutron-proton mass difference) as $E_{\mathcal{N}}(A, Z)/A \simeq E_0 + S_0\, \delta^2$, where E_0 is the energy per nucleon in the **symmetric nuclear matter** and S_0 is the **symmetry energy**, both calculated at saturation density (see section on nuclear matter in Lecture 2). In this approximation, the neutron and proton chemical potentials (without rest energy contribution) are

$$\mu_n' = \partial E_{\mathcal{N}}/\partial N = E_0 + (2\delta + \delta^2)\, S_0 \ , \quad \mu_p' = \partial E_{\mathcal{N}}/\partial Z = E_0 + (-2\delta + \delta^2)\, S_0 \ . \tag{35}$$

The value of δ corresponding to $\rho_{\rm ND}$ can be calculated from the condition $\mu_n' = 0$,

$$\delta_{\rm ND} = \sqrt{1 - (E_0/S_0)} - 1 \ . \tag{36}$$

Putting the experimental values $E_0 = -16$ MeV and $S_0 = 32$ MeV we get $\delta_{\rm ND} = 0.225$.

Matter should be in equilibrium with respect to beta decay and electron capture

$$(A, Z) + e^- \longrightarrow (A, Z-1) + \nu_e$$

$$(A, Z) \longrightarrow (A, Z+1) + e^- + \overline{\nu}_e \ , \tag{37}$$

which results in relation between the chemical potentials

$$\mu_n = \mu_p + \mu_e \ . \tag{38}$$

Therefore

$$\mu_e = \mu_n - \mu_p \simeq 4 S_0 \, \delta \ . \tag{39}$$

Using the formula $\mu_e = 0.516 \ (\rho_6 Z/A)^{1/3}$ MeV, we get a rough estimate

$$\text{bulk approximation :} \qquad \rho_{\rm ND} \simeq 2.2 \times 10^{11} \text{ g cm}^{-3}, \tag{40}$$

which is quite close to the refined theoretical value.

Hartree-Fock calculation of ground state crust

Put $A = N + Z$ nucleons and Z electrons in a box of volume $V_{\rm cell}$ (=unit cell) and calculate ground state of the system at fixed $n_{\rm b} = A/V_{\rm cell}$. The $A - body$ problem - solved using nuclear many-body theory. $A \sim 50 - 1000$.
"Effective nuclear" Hamiltonian

$$\hat{H}_{\rm N}^{\rm eff} = \sum_{j=1}^{A} \hat{t}_j + \sum_{k<j\leq A} \hat{v}_{jk}^{\rm eff} \ , \tag{41}$$

where $\hat{t}_j$ is the kinetic energy operator of j-th nucleon, while $\hat{v}_{jk}^{\rm eff}$ is an operator of an effective two-body interaction between a jk nucleon pair.
The effective nuclear Hamiltonian $\hat{H}_{\rm N}^{\rm eff}$ has to reproduce – as accurate as possible within the Hartree-Fock approximation – relevant properties of the ground state of a many-nucleon system, particularly, the ground state energy E_0. This last condition can be written as $\langle\Phi_0|\hat{H}_{\rm N}^{\rm eff}|\Phi_0\rangle \simeq \langle\Psi_0|\hat{H}_{\rm N}|\Psi_0\rangle$, where Φ_0 and Ψ_0 are, respectively, the Hartree-Fock and exact wave functions, and $\hat{H}_{\rm N}$ is the exact nuclear Hamiltonian.

Effective nucleon-nucleon interaction

Most popular: **Skyrme interaction**. The basic assumption which justifies a Skyrme-type effective NN interaction (Skyrme 1956) is that its range is small as compared with the internucleon distances. This means, that in momentum representation the effective NN interaction $\hat{\tilde{v}}^{\,\mathrm{eff}}(\boldsymbol{k},\boldsymbol{k}')$ can be approximated by a momentum independent term plus terms quadratic in the initial and final relative momenta of an interacting nucleon pair, $\boldsymbol{k}$ and $\boldsymbol{k}'$, with the appropriate spin dependence.

Numerical values of the parameters of effective interaction are to be determined from fitting masses of laboratory nuclei in ground states and low-lying excited states. The complete Hamiltonian of the unit cell is $\hat{H}^{\mathrm{eff}}_{\mathrm{cell}} = \hat{H}^{\mathrm{eff}}_{\mathrm{N}} + V_{\mathrm{Coul}} + \hat{H}_e$, where V_{Coul} describes Coulomb interaction between protons and electrons, and $\hat{H}_e$ corresponds to a uniform electron gas.

Nucleon wave functions

Box=unit cell=ion sphere. The Hartree-Fock approximation for the many-body nucleon wave function is

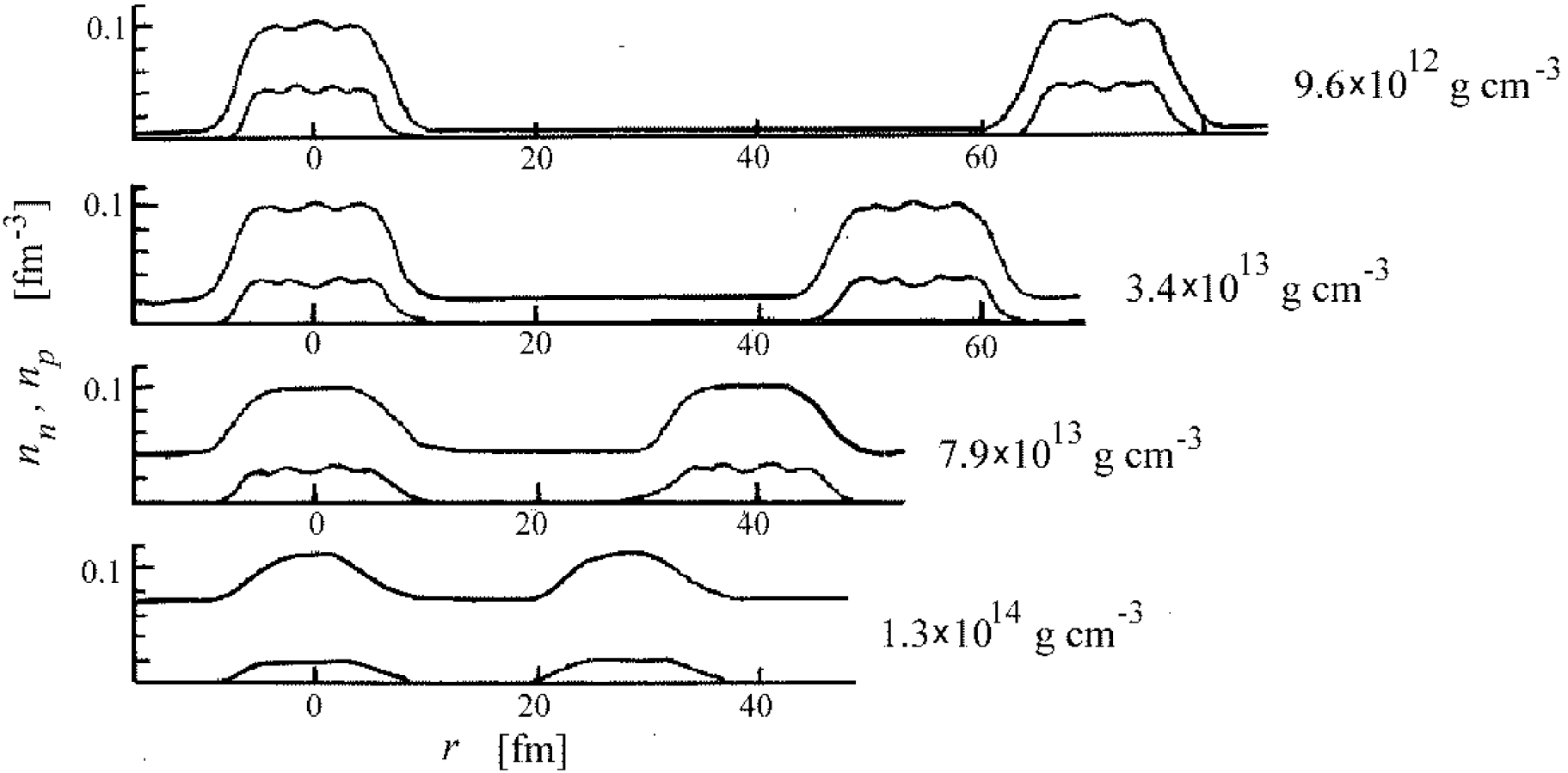
$$\Phi_{NZ} = \mathcal{C}_{NZ} \det\left[\varphi^{(p)}_{\alpha_i}(\xi_k)\right] \det\left[\varphi^{(n)}_{\beta_j}(\zeta_l)\right] \ , \tag{42}$$

where $\varphi^{(n)}_{\beta_j}(\zeta_l)$ and $\varphi^{(p)}_{\alpha_i}(\xi_k)$ are single-particle wave functions (orbitals) for neutrons $(j, l = 1, \ldots, N)$ and protons $(i, k = 1, \ldots, Z)$, respectively, and $\mathcal{C}_{NZ}$ is a normalization constant. The space and spin coordinates of a k-th proton are represented by ξ_k; ζ_l is the same for an l-th neutron; $\{\alpha_i\}$ and $\{\beta_j\}$ are sets of quantum numbers of occupied single-particle states for protons and neutrons, respectively.
The Hartree-Fock equations for $\varphi^{(p)}$ and $\varphi^{(n)}$ were derived by minimizing the Hartree-Fock energy functional at a fixed volume $V_{\rm c}$ of the unit cell,

$$E_{\rm cell}\left[\varphi^{(p)}_{\alpha}, \varphi^{(n)}_{\beta}\right] = \langle \Phi_{NZ}\Phi_e \,|\hat{H}^{\rm eff}_{\rm cell}|\, \Phi_{NZ}\Phi_e\rangle = \text{minimum} \ , \tag{43}$$

where Φ_e is the plane-wave Slater determinant for an ultra-relativistic electron gas of constant density $n_e = Z/V_{\rm c}$. The minimization was performed at fixed average neutron and proton densities, $\overline{n}_n = N/V_{\rm c}$, $\overline{n}_p = Z/V_{\rm c} = n_{\mathcal{N}} Z$.

Nucleon density distribution in the inner crust - HF



Density profiles of neutron and protons, at several average densities ρ, along a line joining the centers of two adjacent unit cells. Based on Fig. 3 of Negele & Vautherin (1973)

Determining the ground state of inner crust

unit cell = ion sphere

Once the Hartree-Fock orbitals $\varphi_\beta^{(n)}$ and $\varphi_\alpha^{(p)}$ are determined, one finds the minimum (ground state) value of $E_{\rm cell}(N,Z)$, filling the lowest N neutron and Z proton states. Then, the absolute ground state configuration is found by minimizing $E_{\rm cell}(N,Z)$ at a fixed $A=N+Z$.

Let us notice, that α_Z and β_N correspond to "Fermi levels" for protons and neutrons, respectively. In terms of the single-nucleon orbitals, the neutron drip point corresponds to the threshold density at which the neutron Fermi level becomes *unbound*, i.e., $\varphi_{\beta_N}^{(n)}$ extends over the entire unit cell.

Remark: In reality an unbound $\varphi_\beta^{(n)}$ extends over all volume of the crystal V, and not only over $V_{\rm c}$. Moreover, in an infinite crystal $\varphi_\beta^{(n)}$ should fulfill the same symmetry (periodicity) conditions as the crystal lattice itself.

Semi-classical - Thomas-Fermi method

$\sim 10^3$ nucleons in unit cell - full quantum description is "too detailed" for getting ground state energy - semiclassical approximation is quite precise

In HF method energy was a **functional of nucleon wave functions**. In TF the energy is a **functional of nucleon densities**

$$E_{\rm N} = \int_{\rm cell} \left\{ \mathcal{E}_{\rm N} \left[n_n(\boldsymbol{r}), n_p(\boldsymbol{r}), \nabla n_n(\boldsymbol{r}), \nabla n_p(\boldsymbol{r}) \right] + m_n c^2 n_n(\boldsymbol{r}) + m_p c^2 n_p(\boldsymbol{r}) \right\} \mathrm{d}^3 r \ . \tag{44}$$

$$E_{\rm Coul} = \frac{1}{2} \int_{\rm cell} e \left[n_p(\boldsymbol{r}) - n_e \right] \phi(\boldsymbol{r}) \, \mathrm{d}^3 r \ , \tag{45}$$

where $\phi(\boldsymbol{r})$ is the electrostatic potential to be calculated from the Poisson equation,

$$\nabla^2 \phi(\boldsymbol{r}) = 4\pi e \left[n_p(\boldsymbol{r}) - n_e \right] \ . \tag{46}$$

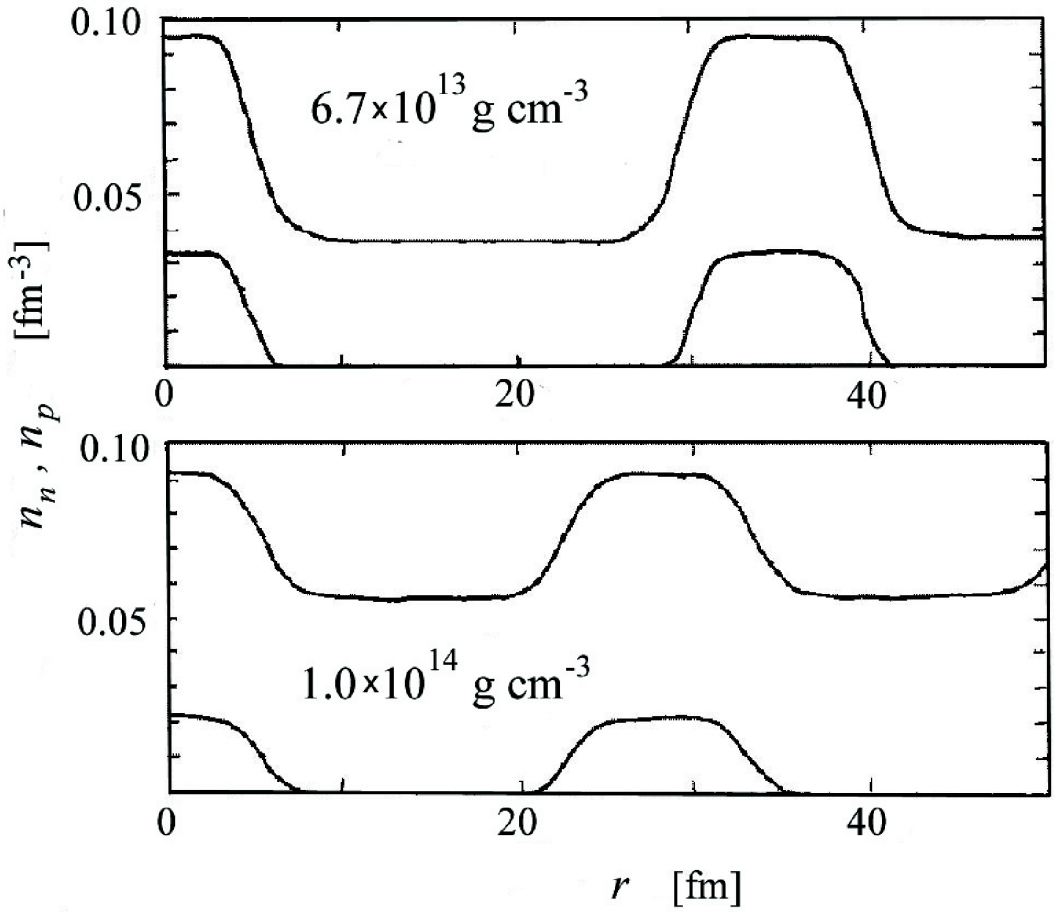
Thomas-Fermi method - continued

To determine the ground state at a given $n_{\rm b}$, one has to find $n_n(\boldsymbol{r})$ and $n_p(\boldsymbol{r})$, which minimize $E_{\rm cell}/V_{\rm c}$ under the constraints

$$V_{\rm c} n_{\rm b} = \int_{\rm cell} \left[n_n(\boldsymbol{r}) + n_p(\boldsymbol{r}) \right] \, {\rm d}^3 r \ , \qquad \int_{\rm cell} \left[n_p(\boldsymbol{r}) - n_e \right] \, {\rm d}^3 r = 0 \ . \tag{47}$$

The problem is simplified assuming spherical symmetry; in this case the unit cell is approximated by a sphere of the radius $r_{\rm c} = (3V_{\rm c}/4\pi)^{1/3}$. The boundary conditions are such that far from the nucleus surface the nucleon densities are uniform. This requires the nuclear radius to be significantly smaller than $r_{\rm c}$.

Thomas-Fermi method - continued



Neutron and proton density profiles at two average mass densities along a line joining the centers of adjacent unit cells. Based on Fig. 5 of Cheng et al. (1997). No wiggles in profiles - because no **quantum interference** between nucleon wave functions.

Compresible Liquid Drop Model

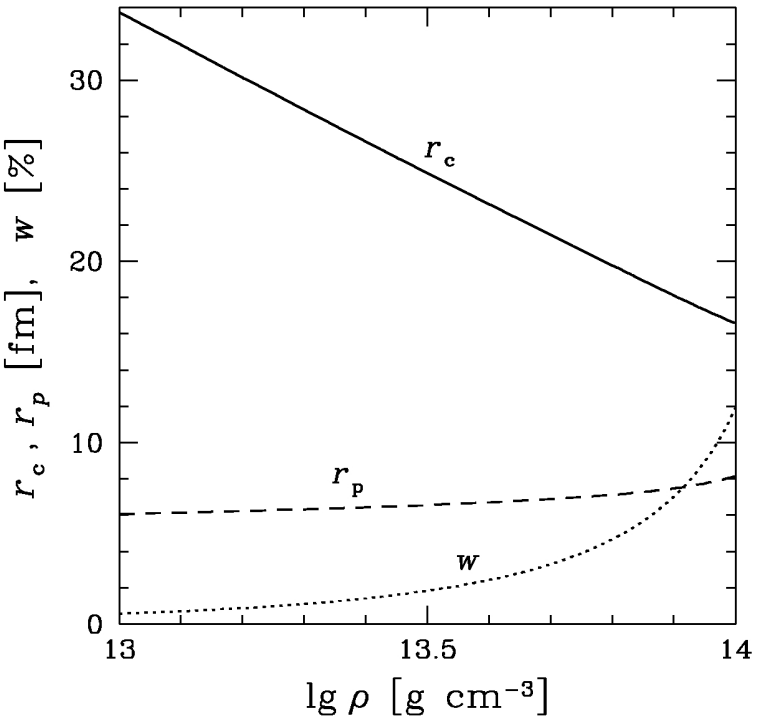
The model is classical (non-quantum) *par excellence*.

Nucleons divided between three thermodynamical subsystems: bulk nuclear matter inside nuclei - (i), neutron gas outside nuclei (o), nucleons in the transition layer between (i) and (o) - the surface (s) or (surf).

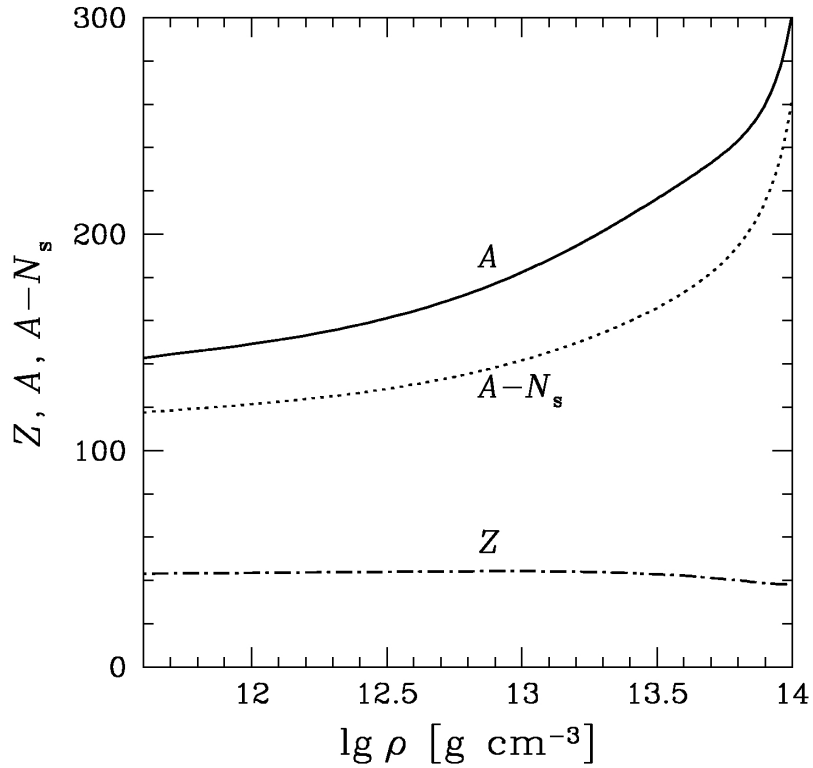
$$E_{\rm cell} = E_{\rm N,bulk} + E_{\rm N,surf} + E_{\rm Coul} + E_e \ . \tag{48}$$

Contrary to $E_{\rm N,bulk}$, $E_{\rm N,surf}$ and $E_{\rm Coul}$ depend on sizes and shapes of nuclear structures. All necessary parameters calculated using many-body microscopic theories. Unknown parameters: nuclear radii, fraction of neutrons in the gas outside, density of neutron and protons inside nuclei, surface tension at the nuclear surface, number of neutrons "adsorbed at the nuclear surface", radius of the unit cell .... Then thermodynamic equilibrium conditions imposed (mechanical and "chemical", charge neutrality of the unit cell) $\Longrightarrow$ structure of the ground state of the inner crust at a given $n_{\rm b}$.

Compressible Liquid Drop Model

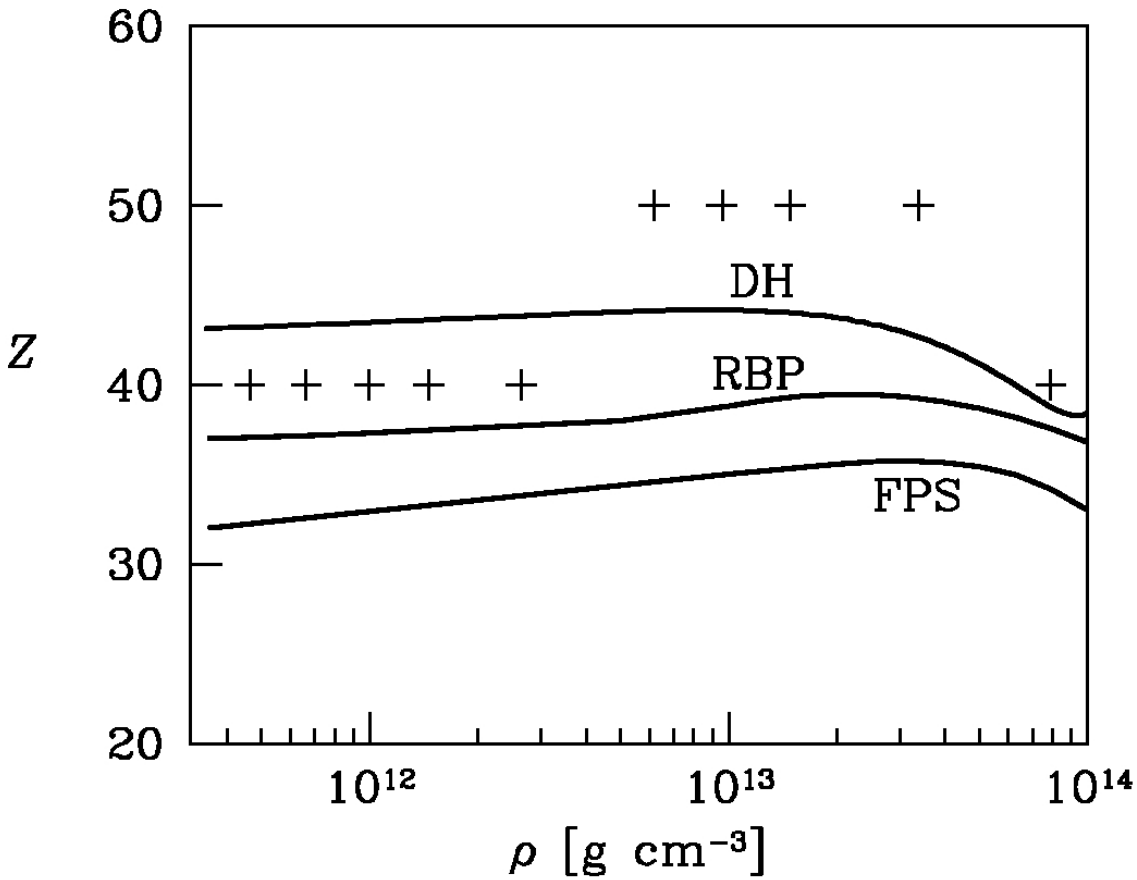


Spherical unit cell radius r_c, the proton radius of nuclei r_p, and the fraction of volume w filled by protons (in percent) vs. average matter density ρ. Douchin & Haensel (2000)



Nuclei: A and Z vs. ρ. The dotted line gives number of the nucleons in nuclei after subtracting the neutrons belonging to neutron skin. Douchin & Haensel (2000)

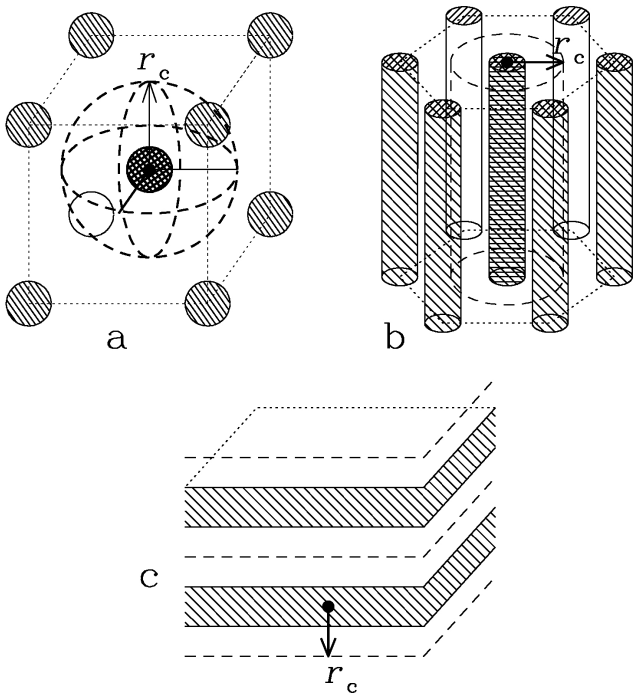
Z of nuclei in the inner crust



Numbers of protons per nucleus in the ground state of the inner crust obtained by various authors. Solid lines: RBP – Ravenhall et al. (1972); FPS – as quoted by Pethick & Ravenhall (1995); DH – Douchin & Haensel (2000,2001). Crosses – Negele & Vautherin (1973).

Funny nuclei - "nuclear pasta"

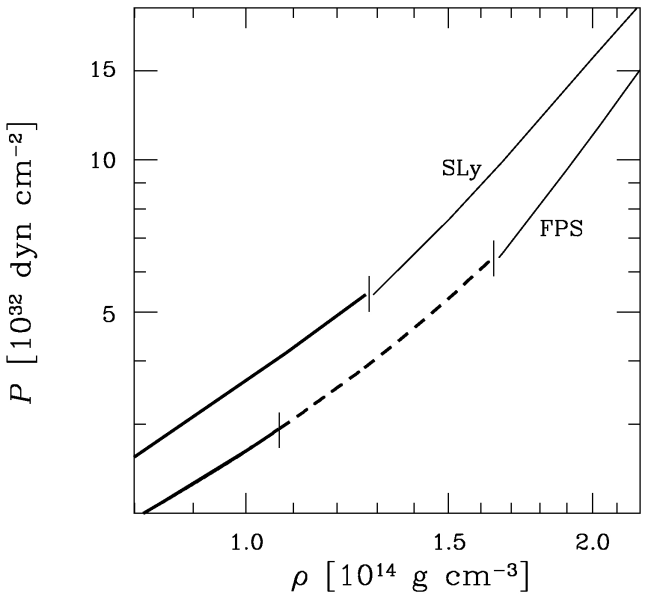
Generally, while minimizing the energy, the nuclear shape has to be treated as a thermodynamic variable. The actual shape of nuclei in the ground state corresponds to the minimum of $\mathcal{E}$ at a given $n_{\rm b}$.



Unit cells for a set of nuclear shapes (spheres, rods, plates) in the inner crust. The size of the unit cell is $r_{\rm c}$. Hatched regions show nuclear matter, while blank regions show neutron gas. In "bubbular phases" (tubes, spherical bubbles) one has to exchange the blank and hatched regions. Culinary names for "pasta phases": cylinders - spaghetti; plates - lasagna; spherical bubbles - Swiss cheese, ...

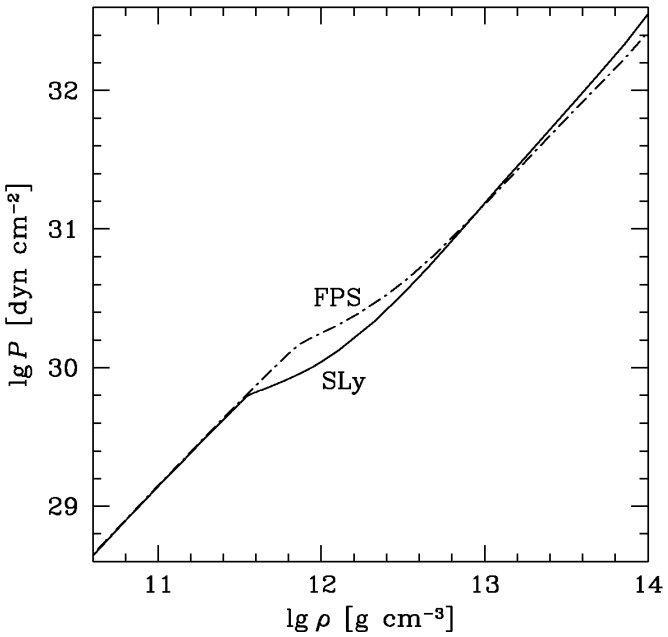
Funny nuclei - nuclear pasta and EOS

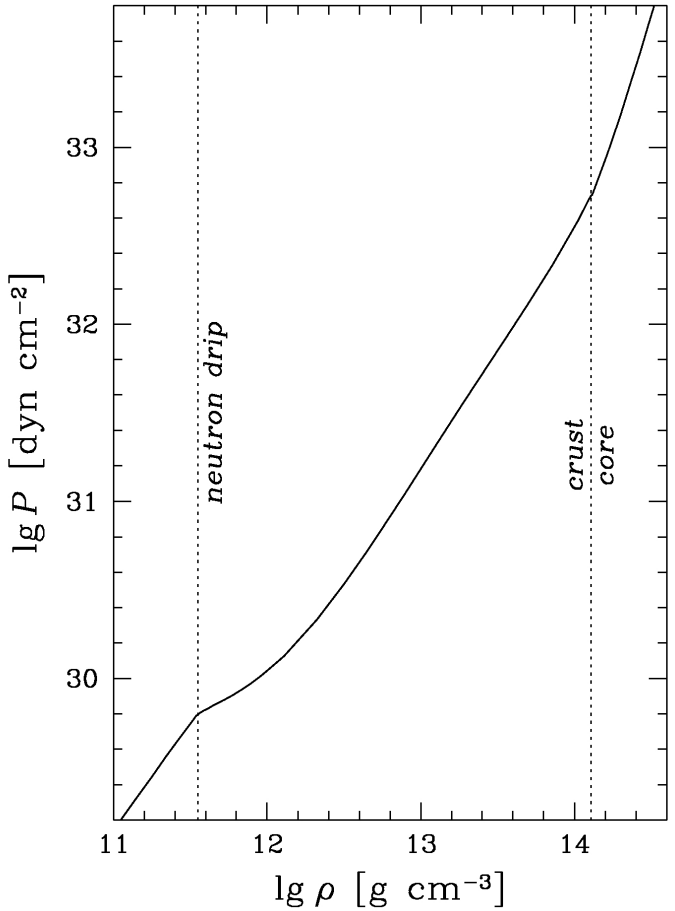
Presence or absence of "pasta" are model dependent. Only NS observations can tell us the actual situation With increasing density SLy (**S**kyrme **Ly**on – Douchin & Haensel 2000): spheres $\longrightarrow$ nuclear matter. FPS (**F**riedman - **P**andharipande - **S**kyrme – Lorenz et al. 1993): spheres $\longrightarrow$ columns $\longrightarrow$ plates $\longrightarrow$ tubes $\longrightarrow$ spherical bubbles $\longrightarrow$ nuclear matter. Large surface tension prevents pasta to appear before nuclei merge into homogeneous nuclear matter. Each transition is accompanied by a very small density jump $\Delta\rho/\rho \lesssim 1\%$. Effects of "pasta" on the EOS - some smoothing of the crust-core transition region. **But: huge effect on the transport and elastic properies !**



Comparison of the SLy and FPS EOSs near the crust-core transition. Thick solid lines refer to the inner crust with spherical nuclei. The dashed line is for "exotic nuclear shapes". Thin solid lines refer to the uniform npe matter.

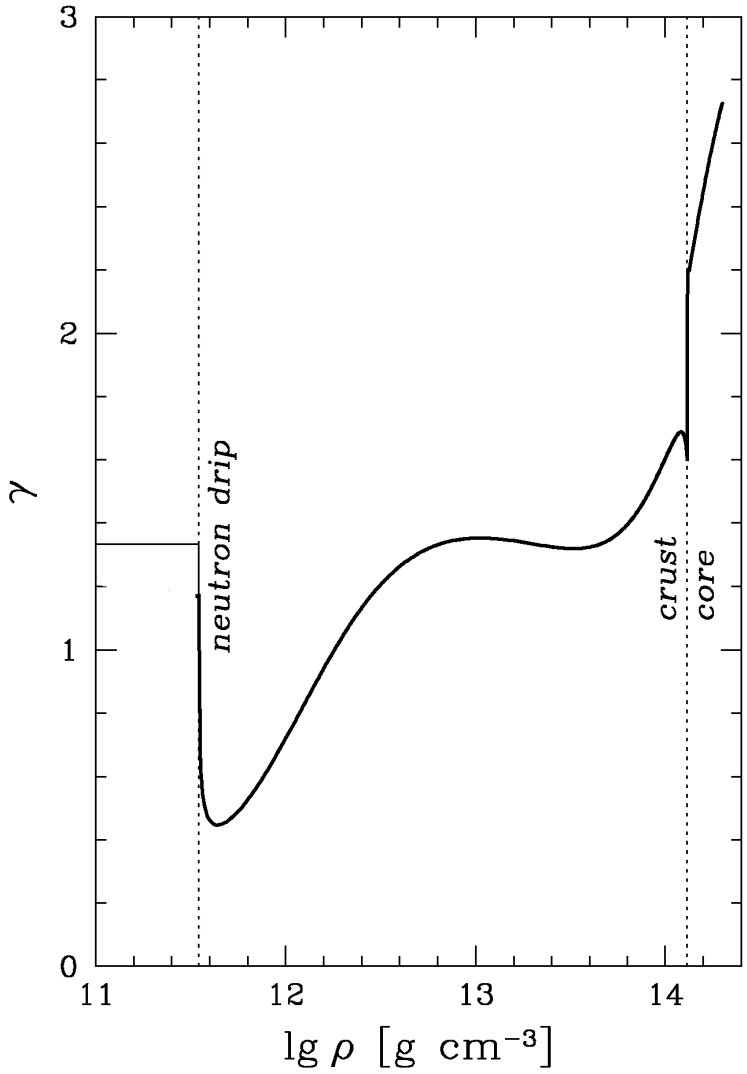
Overall inner crust EOSs - SLy and FPS





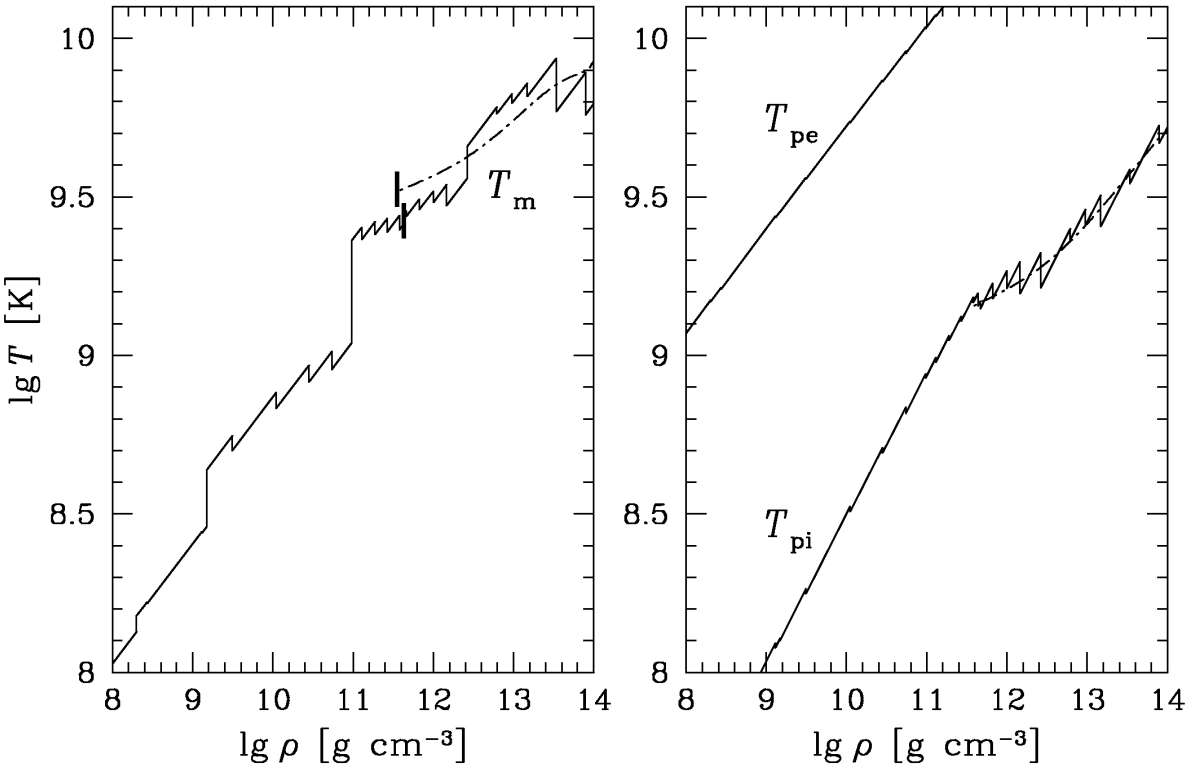
Left: Comparison of the SLy and FPS EOSs. These are **unified EOSs** - based on a one single model to describe both crust and core. *Right:* SLy EOS of the crust. Dotted lines: neutron drip and crust-core transition.

Overall crust EOSs - adiabatic index



Adiabatic index γ for the ground state model of the outer crust of Haensel & Pichon (1994) before neutron drip and Compressible Liquid Drop Model of Douchin & Haensel (2001) at higher ρ. Dotted vertical lines indicate the neutron drip and crust-core interface.

Melting temperature



Melting temperature (*left*) and electron and ion plasma temperatures (*right*) of the ground-state matter in the crust. Solid lines are based on the models of Haensel & Pichon (1994) and Negele & Vautherin (1973) for the outer and inner crusts, respectively.

Jumps are associated with changes of nuclides.
Dot-and-dashed lines are based on the CLDM of Douchin & Haensel (2000,2001); their smooth behavior is an approximation inherent to the model. Thick vertical dashes indicate the neutron drip.

Elastic properties of the crust

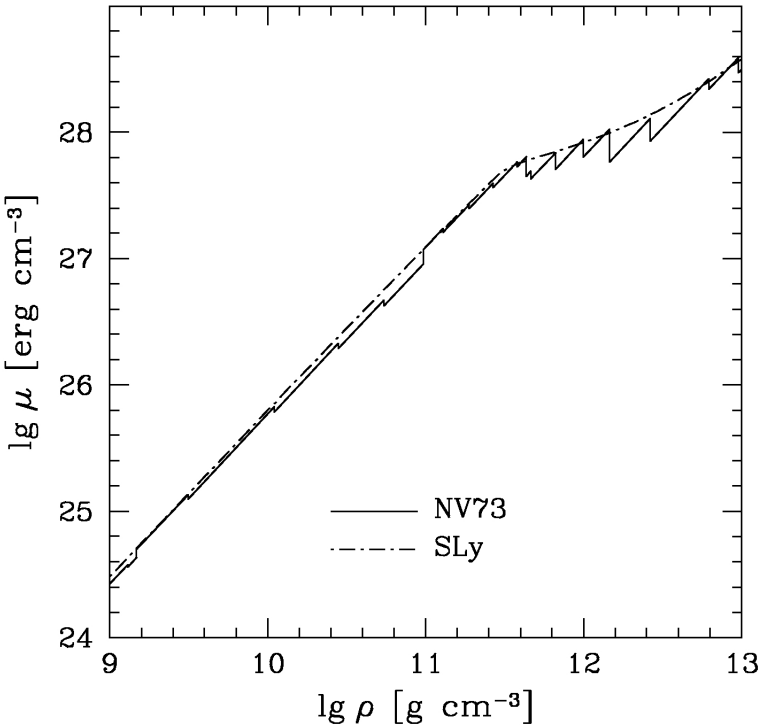
Isotropic solid = polycrystal of bcc crystals. Initially - all nuclei in the equilibrium positions. Consider *displacement* of the nuclei into their new positions $\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{u}$, where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{r})$ is the displacement vector. In the continuum-medium limit, relevant for macroscopic phenomena, both $\boldsymbol{r}$ and $\boldsymbol{u}$ are treated as continuous fields. The displacement $\boldsymbol{u}$ produces an elastic strain (i.e., a force which tends to return the matter element to the equilibrium state with the minimum energy $\mathcal{E}_0$) and determines the *deformation energy* $\mathcal{E}_{\rm def} = \mathcal{E}(\boldsymbol{u}) - \mathcal{E}_0$. A uniform translation does not contribute to $\mathcal{E}_{\rm def}$.

$$\mathcal{E}_{\rm def} = \frac{1}{2}\, K\, (\nabla \cdot \boldsymbol{u})^2 + \mu \left(u_{ik} - \frac{1}{3}\, \delta_{ik}\, \nabla \cdot \boldsymbol{u} \right)^2 . \tag{49}$$

Here, μ is the *shear modulus* and K is the *compression modulus*.

Calculation for bcc, after averaging over orientations gives $\mu = 0.0159\ (Z/26)^{2/3}\, P_e$, so that $\mu/K = 0.016\ (Z/26)^{2/3}\, (P_e/\gamma P) \ll 1$, where $\gamma = (n_{\rm b}/P)(\mathrm{d}P/\mathrm{d}n_{\rm b})$ is the **adiabatic index**.

Shear modulus of the crust

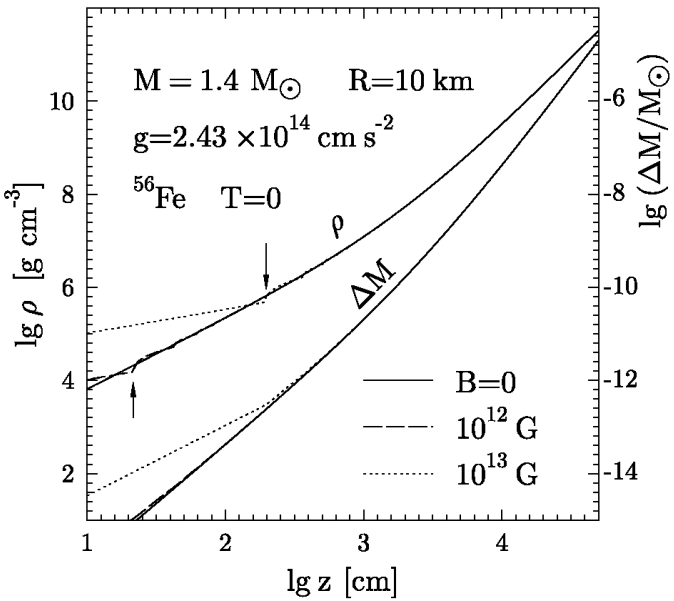


Effective shear modulus μ versus density at $T = 0$ for bcc lattice. The solid line is for the models of Haensel & Pichon (1994) and Negele & Vautherin (1973)(in the outer and inner crust, respectively. The dot-and-dashed line is for the model of Douchin & Haensel (2000).

Notice: The crust is much more susceptible to shear than to compression; its Poisson coefficient $\sigma \simeq 1/2$, while its Young modulus $E \simeq 3\mu$.
Strictly speaking, the above formulae hold for the outer crust, where the radius of nuclei $r_{\mathcal{N}} \ll r_{\rm c}$ and $P \simeq P_e$. In the inner crust they are only approximate.

Effect of B on the envelope & crust EOS

Effect of B on EOS significant when only a few lowest Landau levels (e.g., $n = 0, 1$) are populated by electrons



Matter density ρ at the depth z and the mass of the spherical outer shell of the thickness z for $B = 0$ and $B = 10^{12}$ G and $B = 10^{13}$ G.

electron states:

$B = 0: \quad (p_x, p_y, p_z, s)$

$B > 0 \quad (\widetilde{p}_x, p_z, n, s)$

Landau levels $n = 0: \quad s = -1$

$n = 1, 2, \ldots: \; s = \pm 1$

cyclotron frequency $\omega_c = eB/(m_e c)$

For $\rho > 10^7$ g cm^{-3} the effects of $\boldsymbol{B}$ on the EOS are negligible. However, in the outer envelope ~ 10 cm the effects may be huge (magnetized atmospheres!). Strong effect on the conductivity and opacity as long as collision frequency for electrons $\nu_{\rm coll} \ll \omega_c/(2\pi) \Longrightarrow$ anisotropy with opacity $\kappa_\perp \gg \kappa_\parallel$.