5. Adiabatic approximation to radiation reaction

Radiation reaction to the Carter constant

Schwarzschild "constants of motion" $E, L_i \Leftrightarrow$ Killing vector

Conserved current for GW corresponding to Killing vector exists.

$$E_{GW} = \int d\Sigma^{\mu} t_{\mu\nu}^{(GW)} \xi^{\nu}$$

$$\dot{E} = -\dot{E}_{gw}$$ In total, conservation law holds.

Kerr conserved quantities $E, L_z \Leftrightarrow$ Killing vector

$Q \not\Leftrightarrow$ Killing vector

How can we evaluate dQ/dt?

Adiabatic approximation

$$T << \tau_{RR}$$

T: orbital period, τ_{RR} : timescale of radiation reaction

- At lowest order, the trajectory is given by a geodesic specified by E, L_z, Q (Carter const.).

- Backreaction to E, L_z, Q can be evaluated by using radiative field rather than R field.

$$\left\langle \frac{D}{d\tau} X \right\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} d\tau \frac{\partial X}{\partial u^\alpha} F^\alpha \left[h_{\mu\nu}^{(rad)} \right]$$

$$X = E,\ L_z\ , Q \qquad h_{\mu\nu}^{(rad)} = \left[h_{\mu\nu}^{(ret)} - h_{\mu\nu}^{(adv)} \right] / 2$$

$$G^R_{\mu\nu\alpha\beta}(x,z) = 2G^{rad}_{\mu\nu\alpha\beta}(x,z) - \frac{1}{8\pi} v_{\mu\nu\alpha\beta}(x,z)$$

tail

·Radiative field does not have divergence at the location of the particle.

$$G^{(rad)} = \frac{1}{2}\left(G^{(ret)} - G^{(adv)}\right)$$

$$\Box G^{(adv/ret)} = \delta^4\left(x - x'\right)$$

$$\Rightarrow \Box G^{(rad)} = 0$$

- For E and L_z the results are consistent with the balance argument. (shown by Gal'tsov '82)

- For Q, it was proved that the use of the radiative field gives the correct long time average. (shown by Mino '03)
  - Key: under the transformation

$$\left(t, r, \theta, \phi\right) \rightarrow \left(-t, r, \theta, -\phi\right) \qquad a \rightarrow -a$$

every geodesic is transformed into itself.

Constants of motion for geodesics in Kerr

Kerr geometry:

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r}{\Sigma}\sin^2\theta\right)\sin^2\theta d\phi^2,$$

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2.$$

Two Killing vectors and One Killing tensor:

$$\xi^\mu_{(t)} = (1,0,0,0), \quad \xi^\mu_{(\phi)} = (0,0,0,1), \quad K_{\mu\nu} = 2\Sigma l_{(\mu} n_{\nu)} + r^2 g_{\mu\nu},$$

$$l^\mu = \left(\frac{r^2+a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right), \quad n^\mu = \frac{\Delta}{2\Sigma}\left(\frac{r^2+a^2}{\Delta}, -1, 0, \frac{a}{\Delta}\right),$$

$$m^\mu = \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left(ia\sin\theta, 0, 1, \frac{i}{\sin\theta}\right).$$

Constants of motion:

$$E = -u^\alpha \xi^{(t)}_\alpha \qquad L = u^\alpha \xi^{(\phi)}_\alpha \qquad Q = K_{\alpha\beta} u^\alpha u^\beta \qquad C \equiv Q - (aE - L)^2$$

$$\dot{Q} = K_{\alpha\beta;\sigma} u^\alpha u^\beta u^\sigma = 0$$

$$\because \quad K_{(\alpha\beta;\sigma)} = 0$$ ← definition of Killing tensor

dQ/dt formula

Sago, Tanaka, Hikida & Nakano ('05)

- Self-force f^a is explicitly expressed in terms of $h_{\mu\nu}$ as

$$\frac{dQ}{d\tau} = 2K^{\nu}{}_{\mu} u^{\mu} f_{\nu} \qquad f^{\alpha} = -\frac{1}{2}\left(g^{\alpha\beta} + u^{\alpha}u^{\beta}\right)\left(h_{\beta\gamma;\delta} + h_{\beta\delta;\gamma} - h_{\gamma\delta;\beta}\right)u^{\gamma}u^{\delta}$$

Introducing $\tilde{u}^{\mu}(x)$ which coincides with u^{μ} in the limit $x \to z(\tau)$

$$\tilde{u}_{\alpha;\beta} = \tilde{u}_{\beta;\alpha} \quad K_{(\mu\nu;\rho)} = 0 \quad \Rightarrow \quad \boxed{\frac{dQ}{d\tau} \approx \left[K^{\nu}{}_{\mu}\tilde{u}^{\mu}\partial_{\nu}\frac{\psi}{\Sigma}\right]_{x\to z(\lambda)}}$$

$$\psi \equiv h_{\alpha\beta}\tilde{u}^{\alpha}\tilde{u}^{\beta}\Sigma$$

Or explicitly,

$$\frac{dQ}{d\tau} \approx \int d\lambda \left[\left(-\frac{P(r)}{\Delta}\left(\left(r^2+a^2\right)\partial_t + a\partial_{\phi}\right) - \frac{dr}{d\lambda}\partial_r\right)\psi\right]_{x\to z(\lambda)}$$

$\exists$ expression as simple as dE/dt by the balance argument?

Similarity between expressions for dE/dt and dQ/dt

- Energy loss can be also evaluated from the self-force.

$$\frac{dE}{d\tau} \approx \left[-\xi^{\nu}_{(t)} \partial_\nu \frac{\psi}{\Sigma} \right]_{x \to z(\lambda)} \Leftrightarrow \frac{dQ}{d\tau} \approx \left[K^{\nu}{}_{\mu} \tilde{u}^{\mu} \partial_\nu \frac{\psi}{\Sigma} \right]_{x \to z(\lambda)}$$

- Formula obtained by the energy balance argument:

$$\frac{dE}{dt} \approx -\sum_{l,m,\omega} \left| Z_{l,m,\omega} \right|^2 \qquad \frac{dL}{dt} \approx -\sum_{l,m,\omega} \frac{m}{\omega} \left| Z_{l,m,\omega} \right|^2$$

$$Z_{l,m,\omega} \approx \int \left[\bar{\Pi}^{\mu\nu} T_{\mu\nu} \right]_{x \to z(\tau)} d\tau$$

- dQ/dt has a similar formula

$$\frac{dQ}{dt} \approx -\sum_{l,m,\omega} \hat{Z}_{l,m,\omega} \overline{Z_{l,m,\omega}}$$

$$\hat{Z}_{l,m,\omega} \approx \int \frac{d\tau}{-i\omega} K^{\rho}{}_{\sigma} u^{\sigma} \left[\partial_\rho \left(\Pi^{\mu\nu} T_{\mu\nu} \right) \right]_{x \to z(\tau)}$$

Final expression for dQ/dt

Sago, Tanaka, Hikida & Nakano ('05)

$$\left\langle \frac{dQ}{dt} \right\rangle = 2 \left\langle \frac{\left(r^2 + a^2\right) P(r)}{\Delta} \right\rangle \left\langle \frac{dE}{dt} \right\rangle - 2 \left\langle \frac{a P(r)}{\Delta} \right\rangle \left\langle \frac{dL}{dt} \right\rangle + 2 \sum_{l,m,n_r,n_\theta} \frac{n_r \Omega_r}{\omega} \left| Z_{l,m,\omega} \right|$$

$$\omega = \omega(m, n_r, n_\theta) \sim m\Omega_\phi + n_r \tilde{\Omega}_r + n_\theta \tilde{\Omega}_\theta$$

$$P(r) = E\left(r^2 + a^2\right) - aL$$

orbital freq. in r & θ directions

This expression is as easy to evaluate as dE/dt and dL/dt.

$$\left\langle \frac{dE}{dt} \right\rangle = - \sum_{l,m,n_r,n_\theta} \left| Z_{l,m,\omega} \right|^2 \qquad \left\langle \frac{dL}{dt} \right\rangle = - \sum_{l,m,n_r,n_\theta} \frac{m}{\omega} \left| Z_{l,m,\omega} \right|^2$$

Analytic evaluation of dE/dt, dL/dt and dQ/dt
for generic orbits has been done (Ganz et al., in prep.)

dQ/dt to $O(v^5 e^2 y)$

Sago et al. ('05)

$$
\begin{aligned}
\left\langle \frac{dQ}{dt} \right\rangle = & -\frac{64}{5}\mu^2 v^6 \\
& \times [1 - qv - \frac{743}{336}v^2 - (\frac{1637}{336}q - 4\pi)v^3 \\
& +(\frac{439}{48}q^2 - \frac{129193}{18144} - 4\pi q)v^4 + (\frac{151765}{18144}q - \frac{4159}{672}\pi - \frac{33}{16}q^3)v^5 \\
& +\{\frac{43}{8} - \frac{51}{8}qv - \frac{2425}{224}v^2 - (\frac{14869}{224}q - \frac{337}{8}\pi)v^3 \\
& -(\frac{453601}{4536} - \frac{3631}{32}q^2 + \frac{369}{8}\pi q)v^4 \\
& +(\frac{141049}{9072}q - \frac{38029}{672}\pi - \frac{929}{32}q^3)v^5\}e^2 \\
& +\{\frac{1}{2}qv + \frac{1637}{672}qv^3 - (\frac{1355}{96}q^2 - 2\pi q)v^4 \\
& -(\frac{151765}{36288}q - \frac{213}{32}q^3)v^5\}y \\
& +\{\frac{51}{16}qv + \frac{14869}{448}qv^3 + (\frac{369}{16}\pi q - \frac{33257}{192}q^2)v^4 \\
& +(-\frac{141049}{18144}q + \frac{5981}{64}q^3)v^5\}e^2 y]
\end{aligned}
$$

$v = \sqrt{M / r_0}$

r_0 ~ mean radius

e ~ eccentricity

$$q = \frac{a}{M}$$

$$y = \frac{C}{L_z^2} = \frac{Q - (aE - L_z)^2}{L_z^2} \quad \left(\sim \frac{L_x^2 + L_y^2}{L_z^2} \right)$$

Summary

- BH perturbation is a useful tool to investigate EMRI.
- For Schwarzschild BH, the metric perturbation can be solved by RWZ formalism.
- For Kerr BH, there exists no metric perturbation theory, but NP-formalism can be used to solve the Weyl curvature perturbation.
- QNMs dominates GW at late times.
- MiSaTaQuWa force describes local gravitational reaction to an orbiting particle.
- Under adiabatic approximation, rate of change of energy, angular momentum and Carter constant can be evaluated by GW amplitudes at infinity and horizon.

Selected references

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