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Cosmology Notes

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Warning: These are just notes I used for the lecture, not a well written text [it mixes english & french!]

for more details, see e.g.

- pdf articles and Reviews on the webpage
- P. Peter & J.P. Uzan, Cosmologie Primordiale
  Belin, 2005 (en french)
  (soon @ Oxford Univ. Press).
- V. Mukhanov, fundamental cosmology, CUP 2005
- S. Dodelson, Modern cosmology
- J. Peebles books
- Peacock, CUP
- ... and many others!

COURS N° I et II COSMOLOGICAL MODELS

Cosmology is the study of the large-scale structure and properties of the universe.

0- INTRODUCTION :

In the framework of general relativity spacetime is a 4-dimensional manifold on which is defined a metric $g_{\mu\nu}$. This metric is related to the matter distribution in spacetime by Einstein's equation

$$\boxed{G_{\mu\nu} = 8\pi G\, T_{\mu\nu}}$$

question: which solution of Einstein's equation describes the spacetime we observe ?

i.e. which solution corresponds to our universe ?

or, more modestly, is an idealized (good) model of our universe

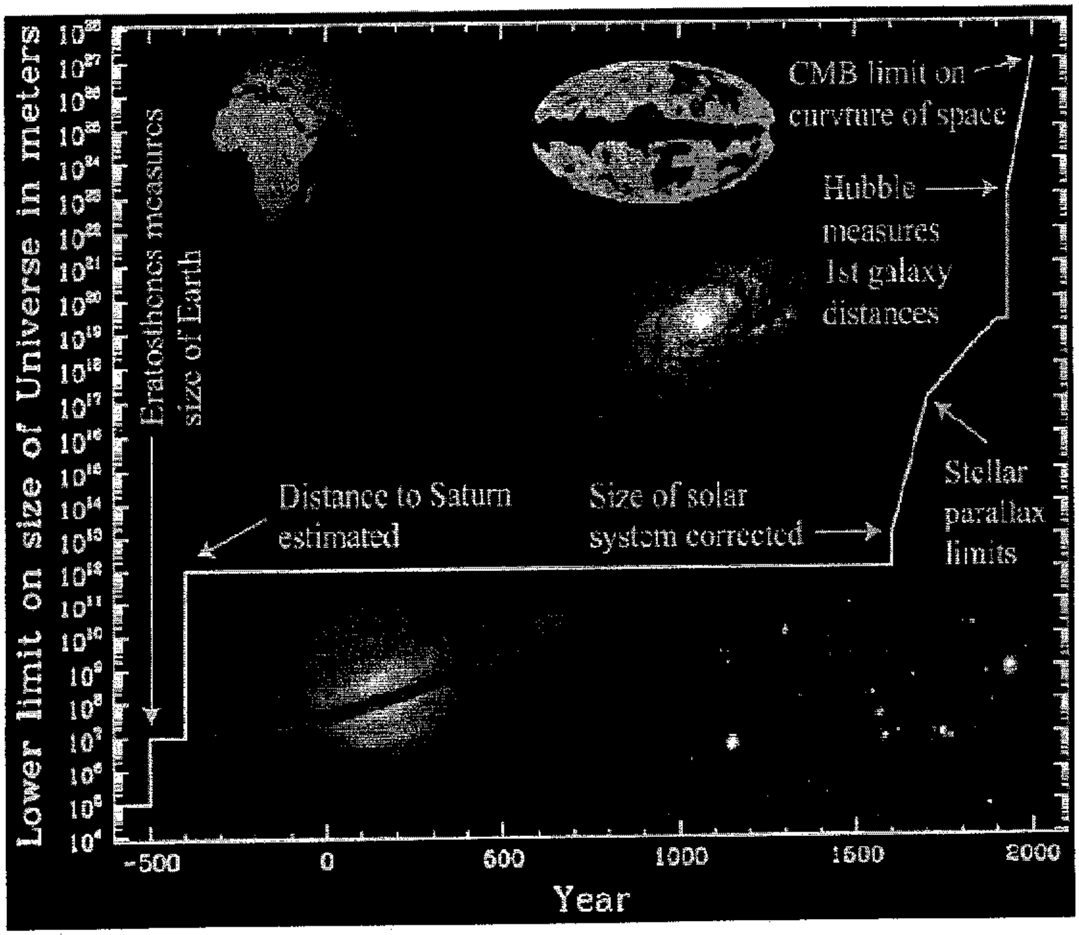
- This problem is indeed very difficult to answer. In particular it will depend on the sum of astrophysical data.
- Even with ideal data, there are some limitation typical to cosmology :

① we observe only 1 universe (it is a unique object and we cannot infer its probable nature by comparing it to similar object) → HISTORICITY DIMENSION

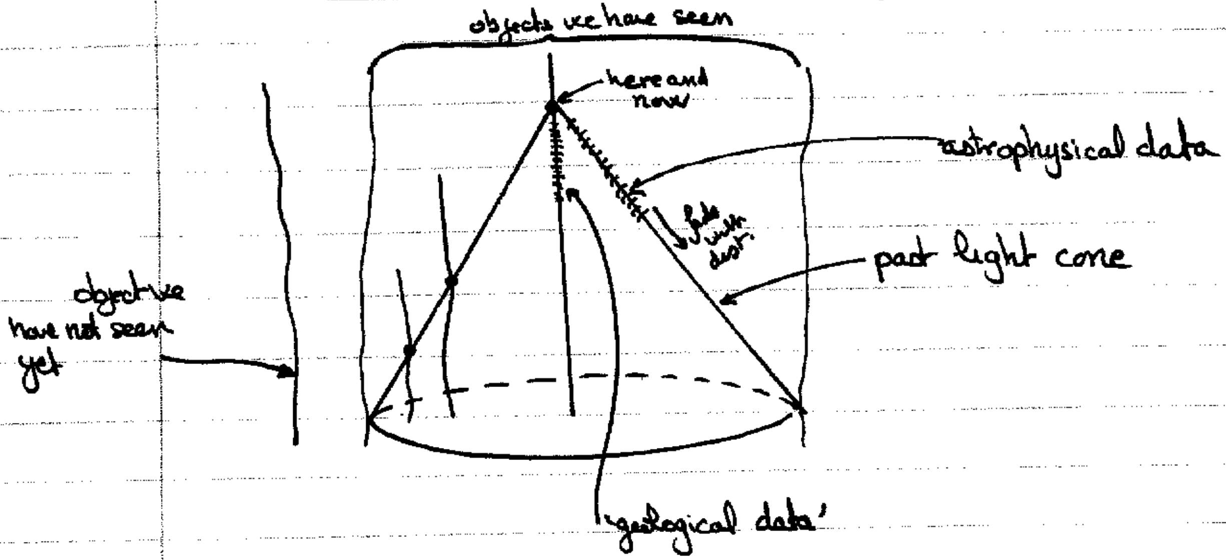
② we observe it from a 1 spacetime point (we are unable to choose this point !)

analogy: analogous to a premaritime man living on a small island in an ocean, who observes around him a host of other small islands apparently scattered at random on a seemingly limitless sea. Unable to move from his island, his

WHAT IS THE SIZE OF THE UNIVERSE?



- can we reconstruct the spacetime geometry from observation?



- Astrophysical data are mostly localized on our past light cone → It is a 3-dimensional hypersurface (just a portion of...)

1. There are various spacetimes compatible with the same 3-d hypersurface (we shall see examples)

2. Interpretation of data is not independent from the spacetime structure

⇒ we will look for a compatibility between universe model and observation.

③ we have to distinguish:

Universe > observable Universe

↑ includes regions we cannot directly influence or experiment

↑ astrophysical data

⇒ to construct a universe model (that is to find a solution of EFE) we need to have some philosophical prejudices

By this, I mean we must make some hypothesis (such as uniformity hypothesis) that cannot be verified.

I- UNIFORMITY PRINCIPLES :

(H1) hypothesis 1 : Gravitation is described by General Relativity.

+ General relativity is well tested on local scales (solar system / galaxy scales) see Will-Esposito farèse lecture.

+ Einstein equivalence principles implies that the laws of physics derived locally can be extrapolated (constant of nature are the same everywhere and at all time)

This justifies the local predictability assumption, that is the fact that whenever normal physics laws can be applied, they correctly predict the structure of the universe.

⇒ we will keep applying these laws as long as it is possible

+ we cannot exclude that General Relativity does not describe gravitation on large scale.

In particular, it is possible that it is described by a theory (e.g. scalar-tensor) that is attracted toward GR today.

→ we will need to design tests on GR

reasons to worry:
- Galaxy rotation curves : DM
- Pioneer
- acceleration of universe : DE
- physical constants

They will be discussed later....

Ⓗ2 hypothesis 2: Nature of matter

① we do not examine the spacetime itself or the distribution of matter in it. Rather we observe particular objects (stars, galaxies ....)

② how do we deal with the variation in the properties of individual object and of evolution effects.

⇒ we must determine how much the intrinsic properties of these objects influence our inference on spacetime (e.g. luminosity / we shall see examples later)

② on large scale, we will assume that matter can be described by a mixture of

- radiation $[P = \frac{1}{3}\rho]$
- pressureless matter $[P = 0]$

i.e. no matter outside the standard model of particle physics.

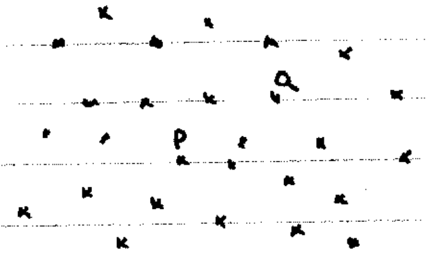
(H3) hypotheses 3: symmetry hypothesis

+ the observations of the universe around us indicate that the universe appears to be isotropic around us

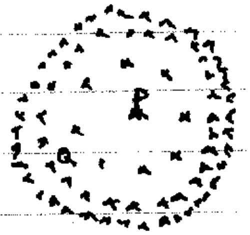
+ we can construct all spacetimes that would give isotropic observations about one particular galaxy. They will fall into 2 classes:

  a spatially homogeneous (that is isotropic around all galaxies)

  b centered on one particular galaxy: this galaxy is the center of the universe



ⓐ



ⓑ

Copernician Principles: we do not seat in a particular position in the universe (we are not at the center of the universe)

Cosmological Principle: The universe is homogeneous and isotropic.

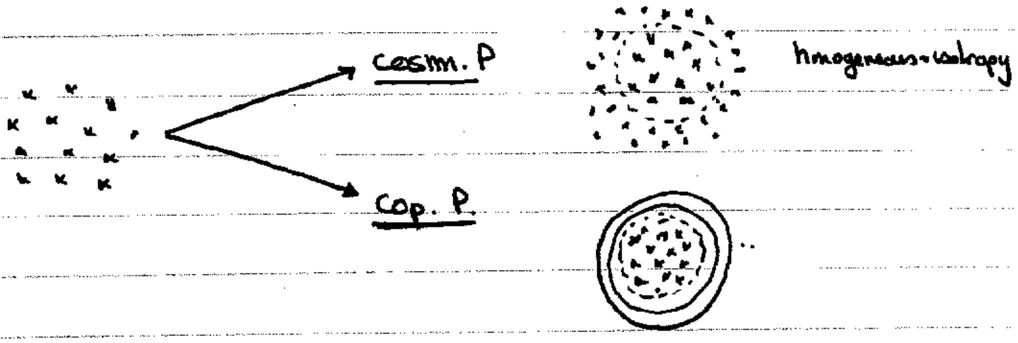
Cop + isotropy $\Rightarrow$ Cosm Princ.

But they are not equivalent:

- cosm Princ makes assumption about regions that cannot be accessed. It allows to infer the geometry of the whole spacetime from local observations

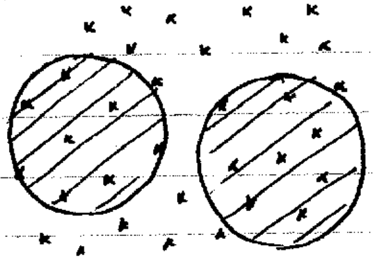
- Cop. Principle infers homogeneity only on scales where isotropic has been established.

- example of difference:



[The cosmological principle makes definite prediction about all unobservable regions beyond the observable universe.
The copernician principle makes no particular prediction about these hidden region. It can be tested.

Note: These notions of homogeneity and isotropy have to be taken in a statistical sense.
They implicitely involve a smoothing scale typically larger than the mean separation between galaxies.



⇒ Averaging issue:

- we will solve $G_{\mu\nu}[\bar{g}] = \bar{T} \longrightarrow \bar{g}$
- we should have solved $G_{\mu\nu}[g] = T \longrightarrow g$

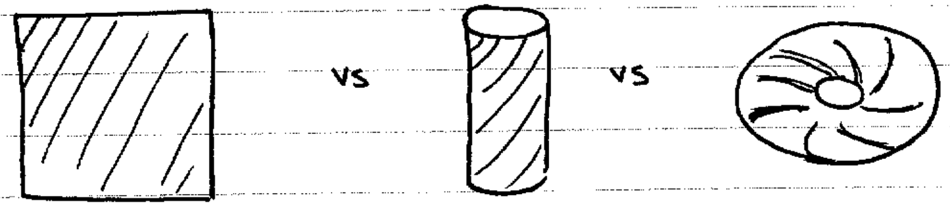
A priori $\langle g \rangle \neq \bar{g}$ because NL equations

Ⓗ4 hypothesis 4: Global structure

under hypothesis H1, H2, H3 ve can solve Einstein's equation to get the <u>local structure of spacetime</u>

There exist various spacetime with same local structure which correspond to different boundary condition

<u>ex</u>: In 2 dimension



ve have 4+1 topologically differt Euclidean structures.

→ <u>what is the topology of spacetime ?</u>

- $\pi_1(X)$: first fundamental groupe
- As ve shall see cosm. Princ $\Longrightarrow$ $\mathcal{M} = \mathbb{R} \times \underset{\substack{\uparrow \\ \text{3Dim-space}}}{\Sigma}$

more generally, it is a consequence of the <u>global hyperbolicity</u>

In conclusion, we have to make the following hypothesis

- H1 General relativity describes gravity on large scales
- H2 matter = rad. + pressureless matter
- H3 cosm. Principle
- H4 No topological structure.

we will start by constructing the spacetimes satisfying these hypothesis and then discuss the effects of relaxing them.

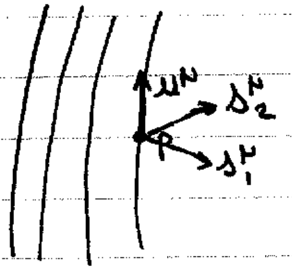
ISOTROPY:

+ Note that at each point, at most one observer can see the universe isotropic

+ a spacetime is said spatially isotropic at each point if there exists a congruence of timelike curves with tangent 4-velocities u^μ satisfying:

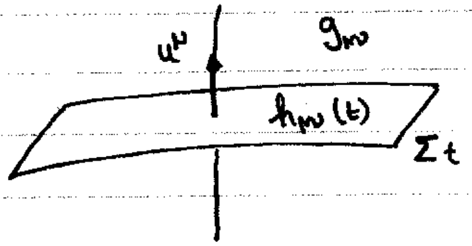
  Given any point p and any two unit spatial vectors s_1^μ and s_2^μ $[s_i^\mu u_\mu = 0]$ there exists an isometry of $g_{\mu\nu}$ which leaves p and u^μ at p fixed but rotates s_1^μ and s_2^μ.

Thus it is impossible to construct a geometrically constructed preferred spatial direction



observer with u^μ is called <u>isotropic observer</u>

For a homogeneous and isotropic spacetime, it is clear that Σ_t (hypersurfaces of homogeneity) must be orthogonal to u^μ



$h_{\mu\nu}(t) = g_{\mu\nu}\big|_{p\in\Sigma_t}$, vector tangent to

<u>Proof</u>: Assume Not.

Then, assuming isotropic observers and homogeneous spaces are unique, the failure of the tangent subsp $\perp$ to u^μ to coincide with Σ_t would enable to construct a geometrically preferred spatial vector, in contradic with isotropy.

[Now, Σ_t is a 3-dim space with induced metric $h_{\mu\nu}(t)$ and is a homogeneous and isotropic space.

This is fairly restrictive:

+ consider the Riemann tensor ${}^{3}R_{abc}{}^{d}$ constructed for h_{ab} on Σ_t and raise the third index with h_{ab} to get ${}^{(3)}R_{\mu\nu}{}^{\rho\sigma}$

$$ {}^{(3)}R_{\mu\nu\rho}{}^{\sigma} \longrightarrow {}^{(3)}R_{\mu\nu}{}^{\rho\sigma} $$ symmetric by exchange of pairs of indices

It may be viewed as a linear map, L, of the vector space of 2-forms [antisym. tensors of rank 2] at p into itself. L is symmetric $\Longrightarrow$ can be diagonalized and there exists an orthonormal basis of eigenvectors.

If the eigenvalues of these eigenvectors were different then one would be able to able to construct a prescription for picking up a preferred 2-form and thus a preferred vector.

That will violate isotropy.

Thus
$$\boxed{\begin{gathered} L = K I \quad \text{, that is} \quad {}^{(3)}R_{\mu\nu}{}^{\rho\sigma} = 2K\,\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]} \\ \Longrightarrow \quad {}^{(3)}R_{\mu\nu\rho\sigma} = 2K\, h_{\rho[\mu}h_{\nu]\sigma} \end{gathered}}$$

Then homogeneity implies that K is constant, that is

$$\boxed{K = K(t)}$$

At any $t \in \mathbb{R}$, Σ_t is a space of constant curvature

one needs to enumerate all 3-dimensional space of constant curvature

(* voir p. I.16b : unicité) ⇐

$\boxed{K>0}$: 3-sphere and its equation in a 4-dim Euclidea space is

$$x^2 + y^2 + z^2 + w^2 = R^2$$

and the metric in spherical coordinates is

$$\begin{cases} x = R\cos\chi \\ y = R\sin\chi\cos\theta \\ z = R\sin\chi\sin\theta\cos\varphi \\ w = R\sin\chi\sin\theta\sin\varphi \end{cases}$$

the metric $d\sigma^2 = dx^2 + dy^2 + dz^2 + dw^2$ induces

$$\boxed{d\sigma^2_{(3)} = R^2\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\,d\varphi^2)\right]}$$

$\boxed{K=0}$ 3-Euclidean space

$$\boxed{\begin{aligned} d\sigma^2_{(3)} &= dx^2 + dy^2 + dz^2 \\ &= \chi^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2) \end{aligned}}$$

$\boxed{K<0}$ 3-hyperboloïd and its equation in a 4-dim Minkowski spa

$$w^2 - x^2 - y^2 - z^2 = R^2.$$

Using hyperbolic coordinates

$$\begin{aligned} w &= R\,\mathrm{ch}\chi \\ x &= R\,\mathrm{sh}\chi\cos\theta \\ y &= R\,\mathrm{sh}\chi\sin\theta\cos\varphi \\ z &= R\,\mathrm{sh}\chi\sin\theta\sin\varphi . \end{aligned}$$

we obtain

$$\boxed{d\sigma^2_{(3)} = R^2\left[d\chi^2 + \mathrm{sh}^2\chi\,(d\theta^2 + \sin^2\theta\,d\varphi^2)\right]}$$

⚠ $R = R(t)$

In conclusion, we have 3 possibilities including a compact space without boundary.

→ topology!!

SPACETIME METRIC :

The world line of isotropic observers are orthogonals to the hypersurfaces of homogeneity.

This implies that

$$\boxed{g_{\mu\nu} = -u_\mu u_\nu + h_{\mu\nu}(t)}$$

In particular $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is a projector perpendicular to u^μ:

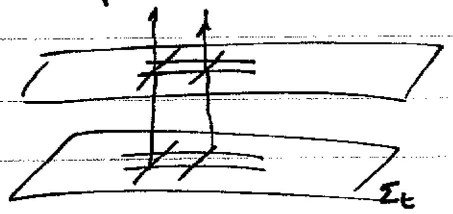
$$h_{\mu\nu} u^\mu = 0$$

$$h_\mu{}^\nu h_\nu{}^\rho = h_\mu{}^\rho$$

it allows to split any 4-vector into a 'time' and 'space' parts for the isotropic observers:

$$p^\mu = -(p^\nu u_\nu) u^\mu + \underbrace{h^\mu{}_\nu p^\nu} \longrightarrow \tilde{p}^\mu = h^\mu{}_\nu p^\nu : \quad \tilde{p}^\mu u_\mu = 0$$

we construct a coordinate system on one hypersurface Σ_t and then carry it to the other hypersurfaces by mean of the isotropic observers



the observer are thus comoving

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -(u_\mu dx^\mu) + h_{\mu\nu}(t) dx^\mu dx^\nu$$

$dt = +u_\mu dx^\mu$ is the proper time measured by a comoving observer.

The world line of the fundamental observer is given by

$$\{x^0 = t, \quad x^i = c^{t.}\}$$

so that, in these coordinates $u^\mu = \delta^\mu{}_0$

In conclusion the most general form of the spacetime metric is

$$ds^2 = -dt^2 + \overset{\nearrow\ \text{scale factor}}{a^2}(\overset{\nearrow\ \text{cosmic time}}{t})\, \gamma_{ij}\, dx^i dx^j$$

$$d\sigma^2 = \gamma_{ij}\, dx^i dx^j = d\chi^2 + f_K^2(\chi)\, d\Omega^2 \qquad \begin{matrix} \varphi = 0 .. 2\pi \\ \theta = -\pi .. \pi \end{matrix}$$

$$f_K^2(\chi) \doteq \begin{cases} \frac{1}{\sqrt{K}} \sin(\sqrt{K}\chi) & K > 0 \qquad \sqrt{K}\chi = 0 .. \pi \\ \chi & K < 0 \qquad\qquad\quad = 0 .. \infty \\ \frac{1}{\sqrt{-K}} \mathrm{sh}(\sqrt{-K}\chi) & K < 0 \end{cases}$$

Here K is a pure number (independent of t)

Exercise: show that the metric can be written as

$$ds^2 = -dt^2 + a^2\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right]$$

with $r = f_K(\chi)$

$$\{\ dr = f_K'(\chi) \qquad d\chi^2 = \frac{1}{f_K'^2} dr^2 \qquad f_K'^2 = 1 - Kr^2$$

The cosmological principle has reduced the metric to a single function of time $a(t)$ and a number K

CONFORMAL TIME

one can introduce the conformal time by

$$dt = a\,d\eta$$

so that

$$ds^2 = a^2(\eta)\left[-d\eta^2 + f_K^2(X)\,d\Omega^2\right]$$

in particular, it is easy to show that

$$\boxed{X' = \frac{dX}{d\eta} = a\dot{X} = a\frac{dX}{dt}}$$

Also, it is clear that for $n \neq 1$

$$\boxed{a(t) \propto t^n \iff a(\eta) \propto \eta^{\frac{n}{1-n}}}$$

4-velocity

from $u^\mu = \delta^\mu{}_0$ we get that

$$\boxed{\nabla_\mu u_\nu = H h_{\mu\nu} \equiv \frac{1}{3}\theta\, h_{\mu\nu}}$$

using that

$$\Gamma^0_{ij} = Ha^2\gamma_{ij} \qquad \Gamma^i_{j0} = H\delta^i_j \qquad \Gamma^i_{jk} = {}^{(3)}\Gamma^i_{jk}$$

Note that the form of Γ^i_{jk} requires to set coordinates of Σ but it is not required to express it, as we shall see later, mainly because we know ${}^{(3)}R_{ijkl}$.

KILLING VECTORS

During a change of coordinate $x^\mu \to x'^\mu - \xi^\mu$, $T^\mu{}_\nu$ transforms as

$$T^\mu{}_\nu \to T^\mu{}_\nu - \mathcal{L}_\xi T^\mu{}_\nu$$

$$\mathcal{L}_\xi T^\mu{}_\nu = \xi^\alpha \partial_\alpha T^\mu{}_\nu - T^\alpha_\nu \partial_\alpha \xi^\mu + T^\mu_\beta \partial_\nu \xi^\beta$$

A vector field satisfying $\mathcal{L}_\xi g_{\mu\nu} = 0$ is a Killing vector and reflects a symmetry of spacetime (it lets the metric invariant).
Besides it satisfies $(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\xi^\mu = R^\mu{}_{\nu\alpha\beta} T^\nu$ as any vector field.

The set of equation $\begin{cases} \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \\ (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\xi^\mu = R^\mu{}_{\nu\alpha\beta}\xi^\nu \end{cases}$ is very restrictive.

The symmetry $R_{\alpha[\mu\nu\sigma]} = 0$ leads to

$$\nabla_\alpha(\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu) + \nabla_\mu(\nabla_\nu \xi_\alpha - \nabla_\alpha \xi_\nu) + \nabla_\mu(\nabla_\alpha \xi_\nu - \nabla_\nu \xi_\alpha) = 0$$

which reduces to

$$\nabla_\alpha \nabla_\beta \xi_\nu + \nabla_\beta \nabla_\mu \xi_\alpha + \nabla_\mu \nabla_\alpha \xi_\beta = 0 \quad \text{for KV}$$

and

$$\nabla_\alpha \nabla_\beta \xi_\nu = R^\mu{}_{\alpha\beta\nu} \xi_\mu$$

This shows that given ξ_α and $\xi_{\alpha;\beta}$ all higher derivatives can be calculated. We thus need to specify these values on one point to determine the whole field

$$\left.\begin{array}{l} \xi_\alpha \to N \\ \xi_{\alpha;\beta} \to \frac{N(N-1)}{2} \text{ (antisym)} \end{array}\right\} \quad \boxed{\frac{N(N+1)}{2} \text{ KV}}$$

The maximum number cannot be reached in any space.

From the above equations it can be shown that

$$\left(R^{\mu}{}_{\alpha\beta\nu;\lambda} - R^{\mu}{}_{\lambda\beta\nu;\alpha}\right)\xi_{\mu} + \left(R^{\mu}{}_{\alpha\rho\nu}\, g^{\kappa}_{\lambda} - R^{\mu}{}_{\lambda\rho\nu}\, g^{\kappa}_{\alpha} + R^{\mu}{}_{\beta\alpha\lambda}\, g^{\kappa}_{\nu} - R^{\mu}{}_{\nu\alpha\lambda}\, g^{\kappa}_{\rho}\right)\xi_{\mu}$$

In ~~A~~ spacetime with $\frac{N(N+1)}{2}$ KV, this eq. should not set any restrictions on the values of $(\xi_{\mu}, \xi_{\mu;\nu})$ so that the 2 combinations of Riemann tensor must vanish.

By contracting them, it can be shown that

$$(N-1)\, R^{\mu}{}_{\alpha\beta\lambda} = R_{\alpha\lambda}\, g^{\mu}_{\beta} - R_{\alpha\beta}\, g^{\mu}_{\lambda}$$

$$N R^{\lambda}{}_{\nu} = R\, g^{\lambda}_{\nu}$$

So that $R_{\mu\alpha\beta\lambda} = \frac{R}{N(N-1)}\left[g_{\alpha\lambda}\, g_{\mu\beta} - g_{\alpha\beta}\, g_{\mu\lambda}\right]$

$\Rightarrow$ $\boxed{\text{A space with max number of KV is a constant curvature space}}$

isotropic: ∃ isometries that exchge the basis vector of the tangent space $\rightarrow \boxed{\frac{N(N-1)}{2}}$

homogeneous: ∃ isometries that translates space $\boxed{N}$

as we have seen before

Eisenhart (1949): maximally sym. space are cpletely determined By N and signature that is if
g and g' / $R = K[gg - gg]$ & $R' = K(g'g' - g'g']$
then ∃ chge of coordinate transfing g in g'

- killing vector of FLRW

Defining the coordinates

$$x^1 = f_k \cos\theta \; ; \quad x^2 = f_k \sin\theta \cos\phi \; ; \quad x^3 = f_k \sin\theta \sin\phi$$

we obtain the metric under the form

$$ds^2 = -dt^2 + a^2(t) f_{ij} dx^i dx^j \; ; \quad f_{ij} = \delta_{ij} + \frac{K \delta_{im} \delta_{in} x^m x^n}{1 - K \delta_{mn} x^m x^n}$$

$$f^{ij} = \delta^{ij} - k x^i x^l$$

one can check that there are 6 independant solution to $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \quad [\, g_{\mu\alpha} \nabla_\nu \xi^\alpha + g_{\nu\alpha} \nabla_\mu \xi^\alpha = 0 \,]$

$$\begin{cases} (0\text{-}0) \rightarrow \partial_0 \xi_0 = 0 \Leftrightarrow \dot{\xi}^0 = 0 \Leftrightarrow \xi^0 = F(x^k) \\ (0\text{-}k) \rightarrow \dot{\xi}^l = g^{kl} \partial_k \xi^0 \\ (i\text{-}j) \rightarrow f_{i\rho} \nabla_j \xi^\rho + f_{j\rho} \nabla_i \xi^\rho + 2H f_{ij} \xi^0 = 0 \end{cases}$$

$\partial_t (i\text{-}j)$ puis $(0\text{-}k)$ $\quad \nabla_i \nabla_j F + \dot{H} a^2 f_{ij} F = 0$

we need $F = 0$

$$\boxed{\begin{array}{l} P^\mu_{[r]} : \; P^0 = 0 \; ; \quad P^k = \delta^k_r \sqrt{1 - kr^2} \\ R^\mu_{[rs]} : \; R^0 = 0 \; ; \quad R^k = \delta^{kr} x^s - \delta^{ks} x^r \end{array}}$$

in de Sitter $\quad \dot{H} = \frac{k}{a^2} \quad$ and we have 4 more solutions

$$\boxed{\begin{array}{lll} T^\mu : T^0 = \sqrt{1 - kr^2} & T^k = -H x^k \sqrt{1 - kr^2} & \\ \mathcal{L}^\mu_{[r]} : \; \mathcal{L}_0^\mu = x^r & \mathcal{L}^k = H \, \frac{1}{2} \left[\delta^{kr} (r^2 - \eta^2) - x^k x^r \right] & k= \\ & \quad\; = H \left[k \delta^{kr} - x^k x^r \right] & k = \pm 1 \end{array}}$$

$6 + 4 = 10 \quad$ dS is maximally symetric.

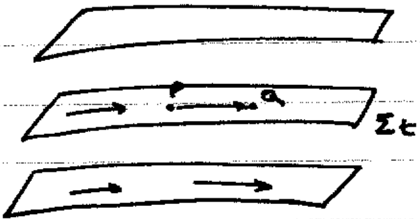
Because of expansion these 4 extra KV are not more KV of FLRW but they satisfy $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \frac{1}{2} (\nabla_\alpha \xi^\alpha) g_{\mu\nu}$: <u>conformal KV</u>

$$\boxed{\text{FLRW has 6 KV } (P^\mu ; R^\mu) \text{ and 4 conformal KV } (T^\mu ; \mathcal{L}^\mu)}$$

II. HOMOGENEOUS AND ISOTROPIC UNIVERSE MODELS

HOMOGENEITY: + At 'any instant of time' each point of space look like any other point

[+ a spacetime is said spatially homogeneous if there exists a one-parameter family of spacelike hypersurfaces Σ_t foliating the spacetime such that for each t and
$\forall\, p, q \in \Sigma_t$ $\exists$ isometry of the spacetime metric which takes p into q



$$\mathcal{M} = \mathbb{R} \times \Sigma_t$$

(III) KINEMATICS OF FLRW models :

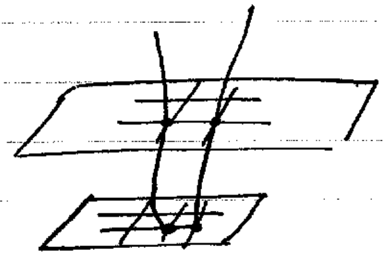
Various properties of these models can be infered without solving Einstein's equation, i.e. without knowing the specific form of a(t).

HUBBLE LAW

- consider 2 comoving observers defined by $x = x_1$ & $x = x_2$
  Their physical separation is

$$r_{12} = a(t)\,[\overbrace{x_1 - x_2}^{d(x_1,x_2)}]$$

It follows $\boxed{\dot r_{12} = \frac{\dot a}{a} r_{12}}$



The recession velocity increases with the separation of the objects.

Radial velocity (by symmetry]

$H = \frac{\dot a}{a}$ is the Hubble parameter

- If the galaxies are moving.

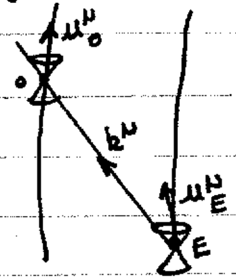
$$\boxed{\dot r = Hr + a(t)\,\dot x = v_{rec} + v_{propre}}$$

- observation of H(t) or H(a) allows to reconstruct a(t)

$$\int_{t_0}^{t} dt = t - t_0 = \int_{a_0}^{a} \frac{da}{a\,H(a)}$$

H(a): Hubble diagram.

consider a signal emitted by a comoving observer and observed by another observer



photon geodesic : $x^\alpha(\lambda)$

$$k^\alpha = \frac{dx^\alpha}{d\lambda}$$

. k^μ satisfies $k^\mu k_\mu = 0$ $k^\mu \nabla_\mu k^\nu = 0$

. It can be decomposed as $k^\mu = E u^\mu + p^\mu$
with $E = -k^\mu u_\mu$ being the energy measured by the observer comoving with u^μ
p^μ its impulsion

• $k_\mu k^\mu = 0$ implies $\boxed{-E^2 + a^2 \gamma_{ij} p^i p^j = 0}$
(which is nothing but the generalisation of $E = p$ for γ in M_4)

. the geodesic equation can be used under the form
$$u_\nu k^\mu \nabla_\mu k^\nu = 0$$

$$k^\mu \nabla_\mu (\underbrace{k^\nu u_\nu}_{-E}) = 0 + \overbrace{k^\mu k^\nu \nabla_\mu u_\nu}^{\frac{1}{3}\Theta h_{\mu\nu} k^\mu k^\nu = a^2 H \gamma_{ij} p^i p^j}$$

$$k^\mu \nabla_\mu E + a^2 H \gamma_{ij} p^i p^j = 0$$

$$E(\underbrace{u^\mu \nabla_\mu E}_{\dot{E}} + \underbrace{p^\mu \nabla_\mu E}_{0 \text{ car } E(t)}) + a^2 H \gamma_{ij} p^i p^j = 0$$

$$\boxed{E\dot{E} + a^2 H \gamma_{ij} p^i p^j = 0}$$

we deduce that $\dfrac{\dot{E}}{E} + H = 0 \Rightarrow \boxed{E = \dfrac{k_0}{a}}$

The energy (ie. ~~wavelength~~ frequency) of any photon behaves as a^{-1}. It follows that $\lambda \propto a$

All wavelengths are redshifted <u>achromatically</u>.

We define the redshift as

$$\boxed{1+z = \frac{(u^\mu k_\mu)_E}{(u^\mu k_\mu)_0} = \frac{a_0}{a}}$$

comparaison of spectra with lab. spectra $\Rightarrow z$ and gives an information about the time at which the photons where emitted

data (ayω) : z, θ, φ

χ (radial distance) is not measurable but infered though $dt = a\,d\chi$ it depends on $a(t)$

TEST PARTICLE

consider a massive test particle with 4-velocity v^μ

$$v^\mu v_\mu = -1 \qquad v^\mu \nabla_\mu v^\nu = 0$$

we decompose v^μ as

$$v^\mu = \alpha u^\mu + p^\mu \quad ; \quad u^\mu p_\mu = 0$$

It follows fr. $v^2 = u^2 = -1$ that $p^2 = -1 + \alpha^2$

The geodesic equation implies

$$\underset{(\alpha u^\mu + p^\mu)}{v^\mu} \nabla_\mu \big(\underbrace{u_\nu v^\nu}_{-\alpha} \big) = \underbrace{v^\mu u_\nu \nabla_\mu v^\nu}_{0} + v^\mu v^\nu \nabla_\mu u_\nu$$

$$-\alpha\dot{\alpha} - \alpha \underbrace{p^\mu \nabla_\mu \alpha}_{0} = \frac{1}{3}\theta h_{\mu\nu} p^\mu p^\nu = H p^2$$

It follows that

$$\alpha\dot{\alpha} + H(-1+\alpha^2) = 0$$

$$\hookrightarrow \frac{d\alpha^2}{-1+\alpha^2} = -2\frac{da}{a} \; ; \; \ln(-1+\alpha^2) \sim \ln a^{-2} \; ; \; -1+\alpha^2 = p^2 \propto a^{-2}$$

$$\boxed{p^2 \propto a^{-2} \; ; \; \alpha \to 1 \; \Rightarrow \; v^\mu \to u^\mu}$$

The proper motion of any test particle is damped as a^{-2} and it adjusts to the Hubble flow.

IV- DYNAMICS OF FLRW UNIVERSES

To determine the dynamics of spacetime, that is the law $a(t)$, we need to write the Einstein's equations.

we need : - compute the geometrical quantities

- describe the matter contents

in order to express:

$$\boxed{\begin{array}{l} G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \\ \nabla_\mu T^{\mu\nu} = 0 \end{array}}$$

Note that these 2 equations are not independent.

QUANTITES GEOMETRIQUES

we start from the FLRW metric

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j$$

$$\boxed{\underline{\text{christoffel symbols:}} \; \Gamma^0_{ij} = Ha^2\gamma_{ij} \; ; \; \Gamma^i_{j0} = H\delta^i_j \; ; \; \Gamma^i_{jk} = \gamma^i_{jk}}$$

γ^i_{jk} are the christoffel symbols of the metric γ_{ij}

We will not need to compute γ^i_{jk} because we have to remember that γ_{ij} satisfies:

$$^{(3)}R_{ij}{}^{kl} = K\left(\delta_i^k\delta_j^l - \delta_i^l\delta_j^k\right)$$

so that $\boxed{^{(3)}R_i{}^j = 2K\,\delta_i^j \qquad {}^{(3)}R = 6K}$

⚠ <u>Note</u>: here indices are lower with γ_{ij} not g_{ij}

<u>Ricci tensor</u>:

$$R_{\nu\beta} = \partial_\alpha\Gamma^\alpha_{\nu\beta} - \partial_\beta\Gamma^\alpha_{\nu\alpha} + \Gamma^\alpha_{\delta\alpha}\Gamma^\delta_{\nu\beta} - \Gamma^\alpha_{\delta\beta}\Gamma^\delta_{\nu\alpha}$$

- $R_{00} = \partial_\alpha\Gamma^\alpha_{00} - \partial_0\Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{\delta\alpha}\underbrace{\Gamma^\delta_{00}}_{0} - \Gamma^\alpha_{\delta 0}\Gamma^\delta_{0\alpha}$

$$= -\partial_0\Gamma^i_{0i} - \Gamma^i_{j0}\Gamma^j_{0i} = -3\,(\dot H + H^2) = -3\,\frac{\ddot a}{a}$$

- $R_{0i} = 0$

- $R_{ij} = \partial_\alpha\Gamma^\alpha_{ij} - \partial_j\Gamma^\alpha_{i\alpha} + \Gamma^\alpha_{\delta\alpha}\Gamma^\delta_{ij} - \Gamma^\alpha_{\delta j}\Gamma^\delta_{i\alpha}$

$$= \partial_0\Gamma^0_{ij} \quad\downarrow\quad + \Gamma^\rho_{0\rho}\Gamma^0_{ij} - \Gamma^0_{\rho j}\Gamma^\rho_{i0} - \Gamma^\rho_{0j}\Gamma^0_{i\rho}$$
$$\underbrace{+\,\partial_k\Gamma^k_{ij} - \partial_j\Gamma^k_{ik} + \Gamma^\rho_{k\rho}\Gamma^k_{ij} - \Gamma^q_{\rho j}\Gamma^\rho_{iq}}_{^{(3)}R_{ij}}$$

$^{(3)}R_{ij}$ [illegible]

$$= \left[(Ha^2)^{\cdot} + 3H^2a^2 - 2H^2a^2\right]\gamma_{ij} + {}^{(3)}R_{ij}$$
$$= \left(\frac{\ddot a}{a} + 2H^2 + 2\frac{K}{a^2}\right)a^2\gamma_{ij}$$

$$\boxed{\begin{aligned} &R_{00} = -3\,\frac{\ddot a}{a}\ ; \qquad R_{ij} = \left(\frac{\ddot a}{a} + 2H^2 + 2\frac{K}{a^2}\right)a^2\gamma_{ij} \\ &R = 6\left(H^2 + \frac{\ddot a}{a} + \frac{K}{a^2}\right)\end{aligned}}$$

Einstein tensor (Exercice)

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

$$\boxed{G_{00} = 3\left(H^2 + \frac{K}{a^2}\right) \qquad G_{ij} = -\left(H^2 + 2\frac{\ddot{a}}{a} + \frac{K}{a^2}\right) a^2 \gamma_{ij}}$$

Matter stress energy tensor

$T_{\mu\nu}$ is a symmetric rank-2 tensor. Its most general form compatible with the cosm. Princ is

$$T_{\mu\nu} = A u_\mu u_\nu + B h_{\mu\nu}$$

$A = T_{\mu\nu} u^\mu u^\nu \equiv \rho$ density measured by a fondamental obs

$B = \frac{1}{3} T_{\mu\nu} h^{\mu\nu} \equiv P$ pressure (isotropic)

$$\Longrightarrow \quad \boxed{T_{\mu\nu} = \rho u_\mu u_\nu + P h_{\mu\nu}}$$

ρ & P are functions of t only

This form is in fact completely fixed by the symmetry imposed!

Friedmann equations

Einstein equations imply:

$$\boxed{\begin{aligned} H^2 &= \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3} \end{aligned}}$$

conservation equation:

Starting from $\nabla_\mu T^{\mu\nu} = \nabla_\mu\left[(\rho+P)u^\mu u^\nu + P g^{\mu\nu}\right] = 0$ and projecting on u_ν we get:

$$u_\nu \nabla_\mu\left[(\rho+P)u^\mu u^\nu\right] + u^\nu \nabla_\nu P = 0$$

$$-u^\mu\nabla_\mu(\rho+P) + (\rho+P)\left\{-1\times \underbrace{\nabla_\mu u^\mu}_{\theta=3H} + \underbrace{u_\nu u^\mu \nabla_\mu u^\nu}_{0}\right\} + u^\nu\nabla_\nu P = 0$$

$$-\dot{\rho} - 3H(\rho+P) = 0$$

$$\boxed{\dot{\rho} + 3H(\rho+P) = 0}$$

one can check that it is not an independent equation and that it can be deduced from the Friedmann equations (because Bianchi identities).

conformal time expressions (exercise)

defining $\mathcal{H} = aH$ show that $a\ddot{a} = \frac{a''}{a} - \mathcal{H}^2$ & $a^2\dot{H} = \mathcal{H}' - \mathcal{H}^2$ so that

$$\mathcal{H}^2 = \frac{8\pi G}{3}\rho a^2 - K + \frac{\Lambda}{3}a^2$$
$$\mathcal{H}' = -\frac{4\pi G}{3}a^2(\rho+3P) + \frac{\Lambda}{3}a^2$$
$$\rho' + 3\mathcal{H}(\rho+P) = 0$$

Equation of state

we have 2 equations for 3 variables (ρ, P, a). one needs to describe the matter further to solve the systems. conveniently we use an equation of state of the form

$$\boxed{P = w\rho}$$

for radiation, matter, Λ
$$w = \begin{cases} 1/3 \\ 0 \\ -1 \end{cases}$$

and it can be check that curvature $\Leftrightarrow$ fluid $w = -1/3$

Using the conservation equation, and assuming $w = c^{t}$, it can be obtained that

$$\boxed{\rho \propto a^{-3(1+w)} \propto (1+z)^{3(1+w)}}$$

<u>Exercise</u>: using the Friedmann equation and defining the sound speed $c_s^2 \equiv P'/\rho'$, show that

$$w' = -3\mathcal{H}(1+w)(c_s^2 - w)$$

Quelques solutions (Exercice)

find the law $a(t)$ in the following cases

<u>a- $K=0$, $w \neq -1$</u>

$$a \propto t^{\frac{2}{3(1+w)}}$$

<u>b- $K=0$, $w=-1$</u>

$$a \propto e^{Ht}$$ de Sitter space

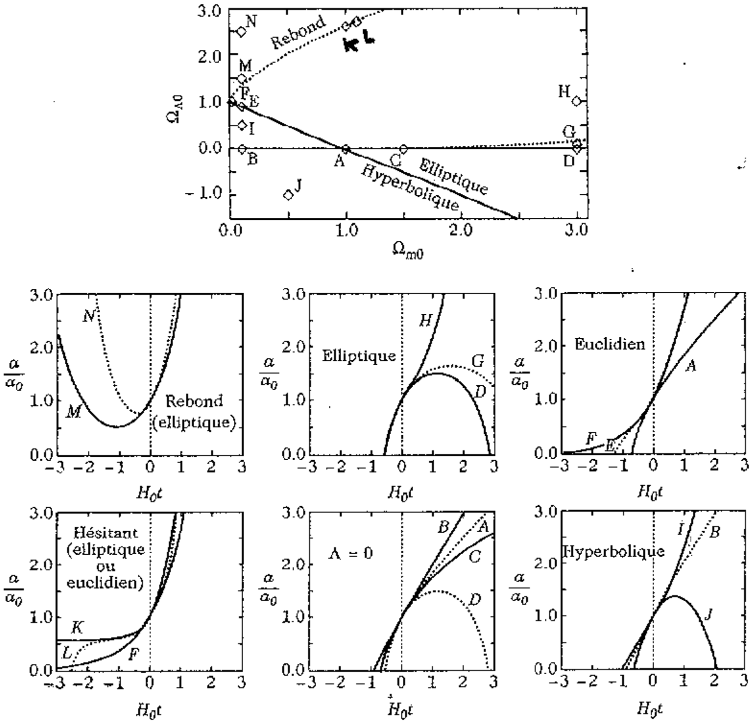
<u>c- $K=\pm 1$, $w \neq -1/3$</u>

$$a \propto (\mathrm{sh}\, \alpha\eta)^{1/\alpha} \quad K=-1 \qquad 2\alpha = 1+3w$$
$$(\sin \alpha\eta)^{1/\alpha} \quad K=+1$$

It is convenient to define Λ_E by $a^2 = \frac{1}{\Lambda_E}$ $\kappa\rho = 2\Lambda_E$
so that $\dot{a} = 0$
This is the Einstein static universe.

The friedmann eq. can be solved.

Here: $\begin{cases} w=0 \\ \Omega_{\Lambda_0} = \dfrac{\Lambda}{3H_0^2} \\ \Omega_{m_0} = \dfrac{8\pi G \rho_{m0}}{3H_0^2} \end{cases}$



Reduced form:

It is convenient to rewrite the Friedmann equations under a adimensional form.

Define $\Omega = \dfrac{8\pi G\rho}{3H^2}$; $\Omega_\Lambda = \dfrac{\Lambda}{3H^2}$; $\Omega_K = -\dfrac{K}{H^2a^2}$

Then, the first equation becomes

$$\sum_x \Omega_x + \Omega_\Lambda + \Omega_K = 0$$

Now, the conservation equations imply that $\rho_x = \rho_{x,0}(1+z)^{3(1+w_x)}$ so that

$$\Omega_x = \frac{8\pi G\rho_{x,0}}{3H_0^2}(1+z)^{3(1+w_x)}\left(\frac{H_0}{H}\right)^2 = \Omega_{x,0}\left(\frac{H_0}{H}\right)^2$$

Thus

$$\boxed{E^2(z) = \left(\frac{H}{H_0}\right)^2 = \sum_x \Omega_{x_0}(1+z)^{3(1+w)} + \Omega_{K_0}(1+z)^2 + \Omega_{\Lambda_0}}$$

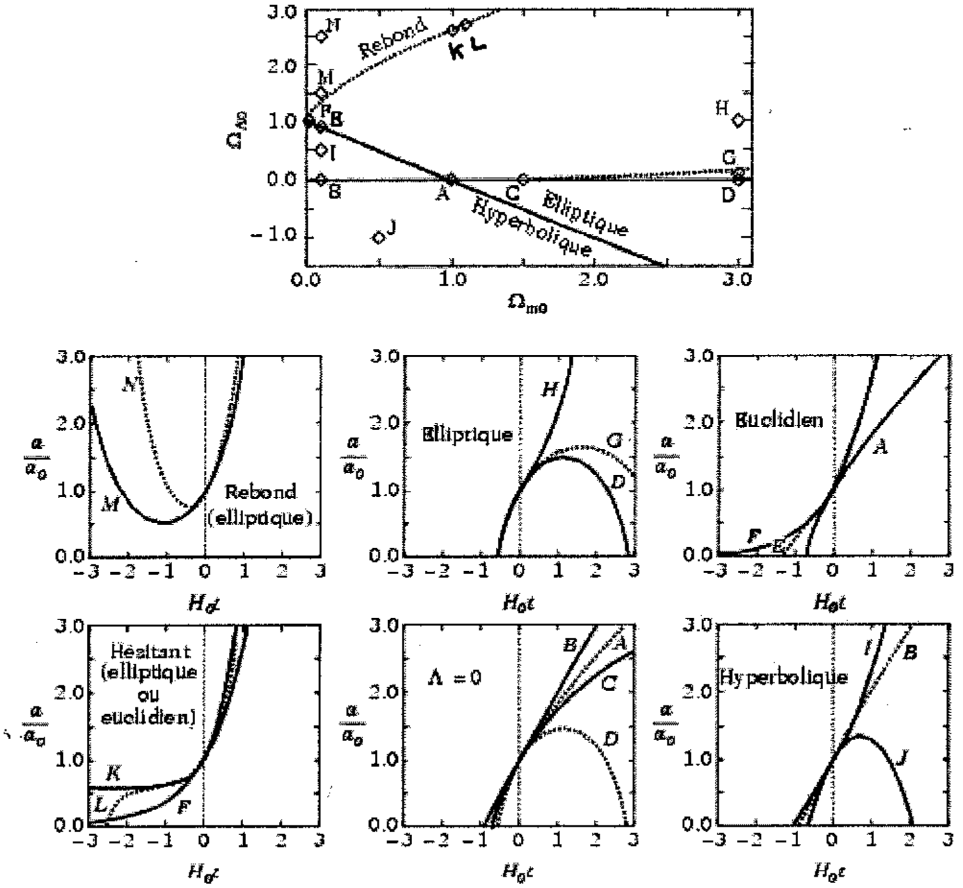
10. Describe (*in not more than 25 lines*), your scientific background and your responsibilities in the network.

I am a phD student in mathematics, my main research interests are : interacting particle systems, equilibrium fluctuations, hydrodynamic limits.

During my stay at IHP I participated in the "Geometry and Statistics of Random Growth" research program, I attended various courses, workshops and seminars and interacted with other attending mathematicians.

11. Provide other comments on your experience as a Young Researcher and make suggestions for further improvements to the programme.

FRIEDMANN EQUATIONS: DYNAMICS



This form encodes all the information about the dynamics.

In the standard FLRW models

$$\boxed{\begin{array}{ccc} \underline{\text{matter}} & \underline{\text{curvature}} & \underline{\Lambda} \\ \underbrace{\underbrace{\Omega_b \quad \Omega_c}_{\Omega_m} \qquad \underbrace{\Omega_\gamma \quad \Omega_\nu}_{\Omega_r}}_{\Omega} & \Omega_K = 1-\Omega-\Omega_\Lambda & \Omega_\Lambda \end{array}}$$

$\Rightarrow$ <u>4 free parameters:</u> $(\Omega_m, \Omega_r, \Omega_\Lambda, H_0)$

Friedmann equation as a dynamical system

I set $\gamma = w+1$

From $\dot{H} = \frac{\ddot{a}}{a} - H^2$, using Fried-2 and Fried-1, we get that

$$\frac{\dot{H}}{H^2} = -(1+q) \qquad \text{with} \qquad 2q \equiv (3\gamma-2)(1-\Omega_K) - 3\gamma\Omega_\Lambda$$

Use the new variable $p = \ln a$, it is clear that $\partial X/\partial p = \dot{X}/H$ so that [Note that $a' = a$!]

$$H' = -(1+q)H$$

Now derive Ω, Ω_Λ, Ω_K and use the previous expression to express H' and the conservation equation as $\rho' = -3\gamma\rho$

$$\Omega' = \frac{8\pi G}{3}\left(\frac{\rho}{H^2}\right)' = \frac{8\pi G}{3H^2}\left(\rho' - 2\rho\frac{H'}{H}\right) = \frac{8\pi G\rho}{3H^2}\left(-3\gamma + 2(1+q)\right)$$

etc...

$$\boxed{\begin{cases} \Omega' = (2q+2-3\gamma)\Omega \\ \Omega'_\Lambda = 2(1+q)\Omega_\Lambda \\ \Omega'_K = 2q\,\Omega_K \end{cases}}$$

we only need to keep 2 of these equations so that the dynamical system can be chosen as

$$(*)\begin{cases}\Omega'_\Lambda = 2(1+q)\Omega_\Lambda \\ \Omega'_K = 2q\,\Omega_K\end{cases}$$

Fixed points:

They are defined by $(\Omega_\Lambda, \Omega_K)' = (0,0)$ and thus are solutions of

$$(1+q)\,\Omega_\Lambda = 0\ ;\quad q\,\Omega_K = 0$$

They are (stable or unstable) points of equilibrium and we have 3 solutions:

$$(\Omega_K, \Omega_\Lambda) = \quad \underset{\substack{\downarrow \\ \text{Einstein} \\ \text{de sitter} \\ (\text{EdS})}}{(0,0)} \quad \underset{\substack{\text{desitter} \\ (\text{dS})}}{(0,1)} \quad \underset{\substack{\text{Milne} \\ (\text{M}) \\ \text{vide } K<0}}{(1,0)}$$

stability:

we set $\Omega_K = \bar\Omega_K + \omega_K$; $\Omega_\Lambda = \bar\Omega_\Lambda + \omega_\Lambda$ and we linearize the system $(*)$

$$\begin{pmatrix}\Omega_K \\ \Omega_\Lambda\end{pmatrix}' = \begin{pmatrix}F_K(\Omega_\Lambda, \Omega_K) \\ F_\Lambda(\Omega_\Lambda, \Omega_K)\end{pmatrix} \Longrightarrow \begin{pmatrix}\omega_K \\ \omega_\Lambda\end{pmatrix}' = \underbrace{\begin{pmatrix}\frac{\partial F_K}{\partial \Omega_K} & \frac{\partial F_K}{\partial \Omega_\Lambda} \\ \frac{\partial F_\Lambda}{\partial \Omega_K} & \frac{\partial F_\Lambda}{\partial \Omega_\Lambda}\end{pmatrix}_{(\bar\Omega_K, \bar\Omega_\Lambda)}}_{P} \begin{pmatrix}\omega_K \\ \omega_\Lambda\end{pmatrix}$$

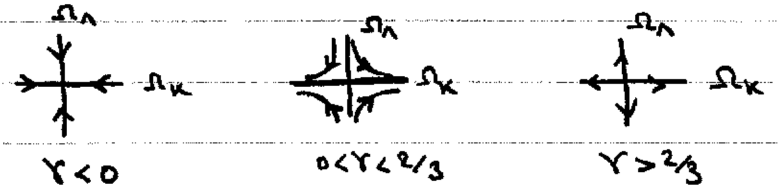
$\boxed{\text{EdS}}$ $\quad P = \begin{pmatrix}2q & 0 \\ 0 & 2(1+q)\end{pmatrix}_{0,0}$ and $2q = 3\gamma - 2$

so that $P = \begin{pmatrix}3\gamma-2 & 0 \\ 0 & 3\gamma\end{pmatrix}$ so that the 2 eigenvalues are $3\gamma-2$ and 3γ.

Thus EdS is an attractor for $\gamma \in]-\infty, 0[$; a saddle point for $\gamma \in]0, 2/3[$ and repulsor for $\gamma \in]2/3, +\infty[$

The eigendirections associated with these 2 eigenvalues are

$$u_{(3\gamma)} = \begin{pmatrix}0\\1\end{pmatrix} \quad \& \quad u_{(3\gamma-2)} = \begin{pmatrix}1\\0\end{pmatrix}$$



$\boxed{dS}$ we perform the same analysis

$$P = \begin{pmatrix}-2 & 0\\ 2-3\gamma & -3\gamma\end{pmatrix} \Rightarrow \lambda = -2 \,;\, -3\gamma$$

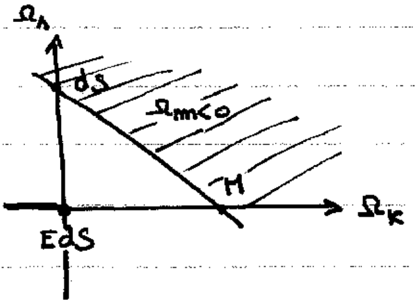
dS is a sadle point for $\gamma \in]-\infty;0[$ and an attracter for $\gamma \in]0;+\infty[$

eigendirections: $u_{(-3\gamma)} = \begin{pmatrix}0\\1\end{pmatrix}$ $u_{(-2)} = \begin{pmatrix}1\\-1\end{pmatrix}$

$\boxed{M}$ $$P = \begin{pmatrix}2-3\gamma & -3\gamma\\ 0 & 2\end{pmatrix} \Rightarrow \lambda = 2 \,;\, 2-3\gamma$$

$\gamma \in]-\infty; 2/3[$ repulsor ; $\gamma \in]2/3;+\infty[$ saddle

$$u_{(2-3\gamma)} = \begin{pmatrix}1\\0\end{pmatrix} \qquad u_{(2)} = \begin{pmatrix}1\\-1\end{pmatrix}$$



γ	$-\infty$		0		2/3		$+\infty$
EdS		A		S		S	
dS		S	A	——	——+——	A	
M		S	R		R	S	

$w \geq 0 \Rightarrow \gamma \geq 1 \Rightarrow$ EdS is a saddle point and dS is an attracter, as well as milne.

Even if $\Omega_\Lambda = 0$ initially and $\Omega_K = \varepsilon$ then $\longrightarrow$ M.

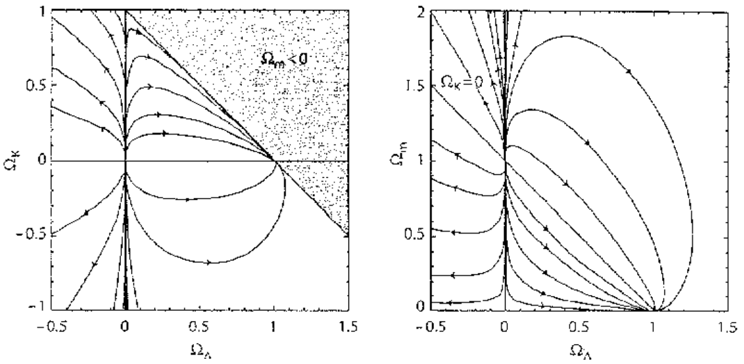


Figure 3.4. Évolution dynamique dans le plan (Ω_K,Ω_Λ) (gauche) et (Ω_m,Ω_Λ) (droite) pour $\gamma = 1$ ($w = 0$). Extrait de Ref. [12].

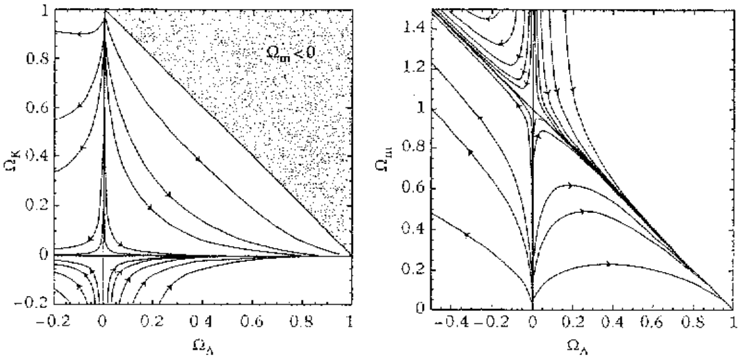


Figure 3.5. Évolution dynamique dans le plan (Ω_K,Ω_Λ) (gauche) et (Ω_m,Ω_Λ) (droite) pour $\gamma = 1/3$. Extrait de Ref. [12].

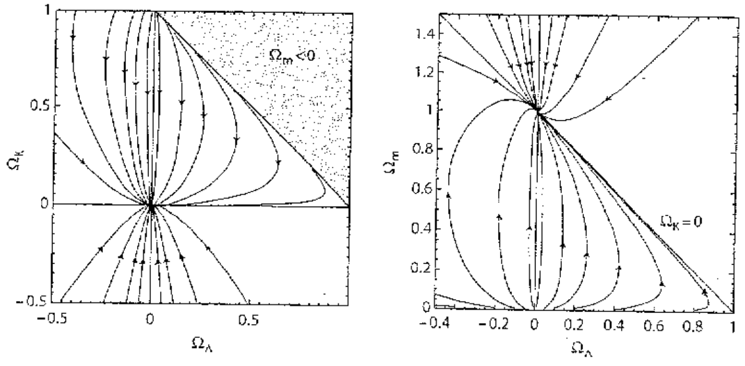


Figure 3.6. Évolution dynamique dans le plan (Ω_K,Ω_Λ) (gauche) et (Ω_m,Ω_Λ) (droite) pour $\gamma = -1$. Extrait de Ref. [12].

V. TIME AND DISTANCES

The Hubble parameter today fixes a natural time and length scale for the observable universe:

$$t_{H_0} = \frac{1}{H_0} \quad ; \quad D_{H_0} = \frac{c}{H_0}$$

Today $H_0 = 100\, h$ km.s^{-1}/Mpc and $h \sim 0.7$ so that

$$\boxed{\begin{aligned} D_{H_0} &= 9.26\, h^{-1}\, 10^{25}\, m \simeq 3000\, h^{-1}\, \mathrm{Mpc} \\ t_{H_0} &= 9.78\, h^{-1}\, 10^{9}\, yr. \end{aligned}}$$

- Age of the universe

$$H = \frac{\dot a}{a} \longrightarrow dt = t_{H_0} \frac{da}{a E(a)}$$

It follows that the age of the universe is obtained by integrating between $a=0$ and $a=a_0$ [$z=\infty$ & $z=0$]

$$\boxed{t_0 = t_{H_0} \int_0^{\infty} \frac{dz}{(1+z)E(z)} = t_{H_0} \int_0^{1} \frac{dx}{x\, E(x)}}$$

where $x = \frac{1}{1+z} = \frac{a}{a_0}$

Similarly the âge of the universe at a time a photon of redshift z_* was emitted is

$$t(z_*) = t_{H_0} \int_{z_*}^{\infty} \frac{dz}{(1+z)E(z)}$$

- Look-back time

It is the difference between the age of the universe and $t(z_*)$

$$\boxed{\Delta t(z_*) = t_0 - t(z_*) = t_{H_0} \int_0^{z_*} \frac{dz}{(1+z)\, E(z)}}$$

TIME AND DISTANCE SCALES

H_0 sets a typical time and distance scales for the universe

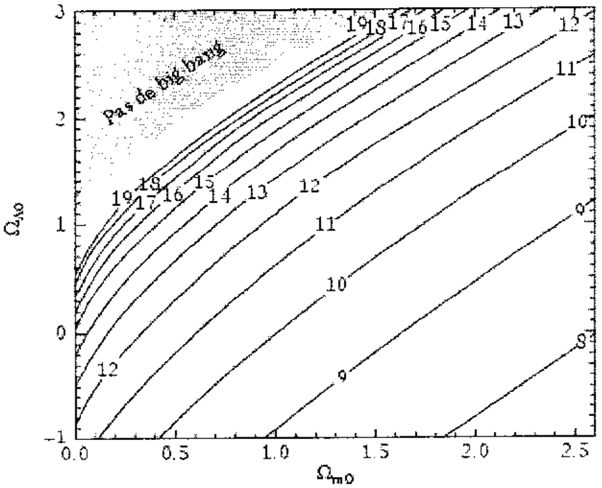
$$H_0 = 100\,h \text{ km/s/Mpc}$$

$$D_{H_0} = c/H_0 = 3000h^{-1}\,\text{Mpc} = 2.9h^{-1} \times 10^{25}\,\text{m}$$

$$t_{H_0} = 1/H_0 = 9.78\,h^{-1} \times 10^{9}\,\text{yr}$$

Dynamical age of the universe

$$\mathrm{d}t = \frac{\mathrm{d}a}{\dot{a}} = t_{H_0}\frac{\mathrm{d}a}{aE(a)}$$



It gives the time elapsed between a photon is emitted at z_* and observed today.

- comoving radial distance:

This is the distance (comoving) obtained by integrating along a radial null geodesic between $\chi=0$ and $\chi(z)$:

$$dt^2 = a^2 d\chi^2 \longrightarrow \underset{\substack{\downarrow \\ a\frac{da}{\dot a}}}{dt} = a\, d\chi$$

$$\frac{da}{H(a)} = a^2 d\chi \longrightarrow a_o d\chi = \frac{dx}{x^2 H(x)} = - \frac{dz}{H(z)}$$

so that

$$\boxed{a_o \chi(z_*) = D_{H_0} \int_0^{z_*} \frac{dz}{E(z)}}$$

For a flat ($K=0$) universe with no cosm. constant ($\Lambda=0$) one gets

$$a_o \chi(z) = 2 D_{H_0} \left(1 - \frac{1}{\sqrt{1+z}} \right)$$

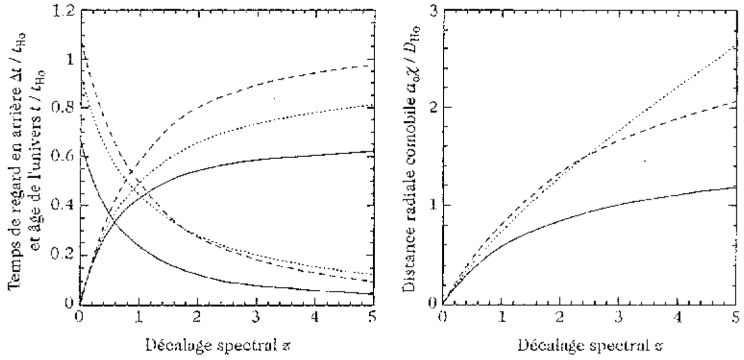
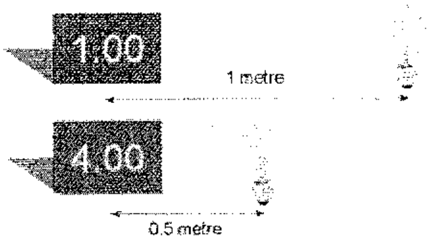


Figure 3.9. (gauche) Âge et temps de regard en arrière en unité du temps de Hubble, $t(z)/t_{H_0}$ et $\Delta t(z)/t_{H_0}$ en fonction de z pour trois modèles cosmologiques définis par $(\Omega_{m0}, \Omega_{\Lambda 0}) = (1,0), (0.05,0)$ et $(0.2, 0.8)$ respectivement en trait plein, en pointillés et en tirets. (droite) Distance radiale comobile en unité de la longueur de Hubble, $a_0\chi(z)/D_{H_0}$, en fonction de z pour les trois mêmes modèles cosmologiques.

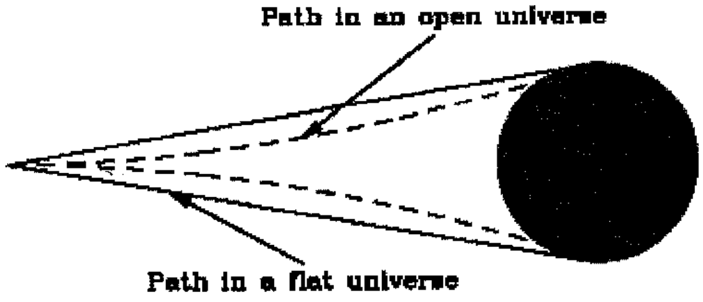
DISTANCES

Measuring Distances with Standard Light Bulbs



An Object becomes fainter by the square of its distance

$$\phi_{\rm obs} = \frac{L_{\rm source}}{4\pi D_L^2}$$



$$D_A^2 = \frac{{\rm d}S^{\rm phys}_{\rm source}}{{\rm d}\Omega_{\rm obs}}$$

- Angular distances

* comoving angular diameter

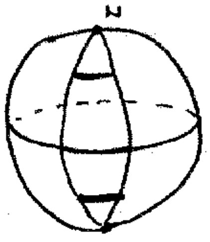
It relates the transverse comoving size of an object and the solid angle under which it is observed:

$$dS^{com}_{source} = R^2_{ang}(\chi)\, d\Omega^2_{obs}$$

Since the radius of a sphere of comoving radius χ around $\chi = 0$ is $S^{com} = 4\pi f_K^2(\chi)$, it is obvious that

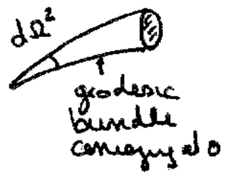
$$\boxed{R_{ang}(z) = f_K[\chi(z)]}$$

It is not always monotonous



* Angular distance

It generalizes the notion of parallax. It relates the transverse physical size of an object to the solid angle under which it is observed



$$\underset{\displaystyle\overset{\big\downarrow}{a^2\, dS^{com}_{source}}}{dS^{phys}_{source}} = D_A^2\, d\Omega^2_{obs}$$

so that $D_A = a\, R_{ang}$

$$\boxed{D_A = \frac{a_0 R_{ang}}{(1+z)} = \frac{a_0}{1+z} f_K[\chi(z)]}$$

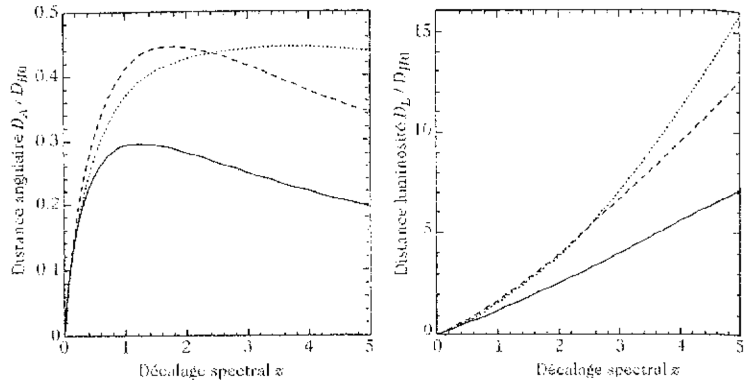


Figure 3.10. (gauche) La distance angulaire $D_A(z)/D_{H_0}$ en fonction du décalage spectral z pour trois modèles cosmologiques définis par $(\Omega_{m0}, \Omega_{\Lambda 0}) = (1,0), (0.05,0)$ et $(0.2, 0.8)$ respectivement en trait plein, en pointillés et en tirets. (droite) La distance luminosité $D_L(z)/D_{H_0}$ en fonction du décalage spectral z pour les trois mêmes modèles cosmologiques.

- Luminosity distance

It relates the luminosity of a source to the observed flux:

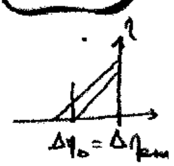
$$\phi_{obs} = \frac{L_{source}}{4\pi D_L^2}$$

To compute the observed luminosity to the intrinsic luminosity, note that $L_{em} = \frac{\Delta E_{em}}{\Delta t_{em}}$.

Because of the expansion $\Delta E_{obs} = \frac{a}{a_0} \Delta E_{em}$ $\boxed{aE = c^{t}}$

and

$\Delta t_{obs} = \frac{a_0}{a} \Delta t_{em}$



It follows that

$$L_{obs} = (1+z)^{-2} L_{source}$$

and thus

$$\phi_{obs} = \frac{L_{obs}}{S^{(phys)}} = \frac{L_{source}\,(1+z)^{-2}}{4\pi a_0^2 f_k^2(\chi)}$$ because $S^{(phys)} = a_0^2 S^{(c)}$

It follows that

$$\boxed{D_L = a_0 (1+z) f_k[\chi(z)]}$$

Note that the intrinsic luminosity of the source is unknown. By comparing to an object at z_*:

$$D_L(z) = D_L(z_*) \sqrt{\frac{L}{L_*}\frac{\phi_*}{\phi}}$$

If one can extract a family of object with same intrinsic luminosity (standard candles) the flux ratio allows to measure distance (construct a distance ladder).

- distance modulus (argue !!)

The luminosity distance is often compared to a source located at 10 pc:

> (to match magnitude def by Hipparque (1-6) for stars)

$$m - M = -\bigcirc\!\!\!\!2.5 \log\left[\phi(z)/\phi(10\text{pc})\right]$$

Noting that $D_L(10\text{pc}) = 10\text{pc} = \frac{D_{H_0}}{3.10^8 h}$, we get

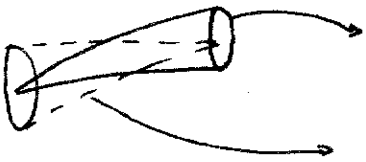
$$\boxed{m - M = 25 + 5\log\left[3000\,\frac{D_L}{D_{H_0}}\right] - 5\log h}$$

↳ absolute magnitude defined by comparing to the sun

$$M = -2.5\,\text{Log}\left(\frac{L}{L_\odot}\right) + 4.76$$

Fainter objects have higher magnitude.

- Reciprocity theorem



$$dS^{phys}_{source} = r_0^2\, d\Omega_0^2$$

$$dS^{phys}_{obs} = r_s^2\, d\Omega^2_{source}$$

It can be shown that if

- γ follows null geodesics
- geodesic deviation equation holds

then $r_s = r_o (1+z)$

unfortunately r_s is not measurable

If the number of photons is conserved then $D_L = r_s(1+z)$ so that

$$\boxed{D_L = (1+z)^2 D_A}$$

This is indeed what we have obtained in FLRW, but the relation is much more general!

- Small redshift behaviour

$$a(t) = a_0 \left[1 + H_0 (t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 + \dots \right]$$

with $\boxed{q_0 \equiv - \left.\frac{\ddot{a}}{a H^2}\right|_0 \quad \text{being the deceleration parameter}}$

exercice: check from the Friedmann equations that

$q_0 = \frac{1}{2} \Omega_{m_0} - \Omega_{\Lambda_0}$ for a ΛCDM

Now starting from $E(z)$, we get that

$$E(z) = 1 + \frac{1}{2} \left(\sum_x 3(1+w) \Omega_{x_0} + 2 \Omega_{K_0} \right) z + \mathcal{O}(z^2)$$
$$= 1 + (q_0 + 1) z + \mathcal{O}(z^2)$$

one can deduce (left as exercices) that

$$\begin{cases} a_0\chi(z) = D_{H_0}\left[1 - \frac{1}{2}(q_0+1)z\right]z + \mathcal{O}(z^3) \\ D_A(z) = D_{H_0}\left[1 - \frac{1}{2}(q_0+3)z\right]z + \mathcal{O}(z^3) \\ D_L(z) = D_{H_0}\left[1 - \frac{1}{2}(q_0-1)z\right]z + \mathcal{O}(z^3) \end{cases}$$

- Units :

When decomposing $l^{(phys)} = a(t)\, l^{(com)}$ we have two possibilities :

1. $[a]=1$ and $[l^{co}]=L$
2. $[a]=L$ and $[l^{co}]=1$

- When $[a]=L$, χ is dimensionless and we can normalize K to $0, \pm 1$. The Friedmann equations evaluated today give a_0:

$$\boxed{K=0,\pm1 \qquad a_0 = \frac{1}{H_0}\frac{1}{\sqrt{|1-\Omega_0|}}}$$

It is related to the curvature scale. For $K=0$ a_0 is arbitrary because of the invariance by dilatation of the Euclidean space

- when $[a]=1$, χ has dimension of length so that $[K]=L^{-2}$

setting $a_0 = 1$

$$\boxed{K = \frac{(0,\pm1)}{R_c^2} \qquad R_c = \frac{1}{a_0H_0}\,\frac{1}{\sqrt{|1-\Omega_0|}}}$$

$R_c(K=0) = \infty$.