Gravitation: Theories and Experiments

Part I: Clifford M. Will, WUGRAV, Washington U., St. Louis, USA

Phenomenological approach

Part II: Gilles Esposito-Farese, GRεCO / IAP, Paris, France, gef@iap.fr

Field-theoretical approach

- A: Scalar-tensor gravity (October 10th & 11th)
- B: Binary-pulsar tests (October 11th)
- C: Modified Newtonian dynamics (October 12th)

Gravitation: Theories and Experiments

Part II: Field-theoretical approach
(Gilles Esposito-Farese)

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Gravitation: Theories and Experiment

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Ⓐ Scalar-tensor gravity

October 10th, 2006

A.1: General relativistic action

* GR is based on two independent hypotheses, which are most conveniently described by decomposing its action as

$$S = S_{gravity} + S_{matter}$$

(Imposing that this action is at an extremum, $\delta S = 0$, implies the field equations, both for gravity and matter.)

* The first assumption is that all matter fields are minimally coupled to a single symmetric tensor $g_{\mu\nu}$, the "metric".

$$S_{matter}[\psi \, ; \, g_{\mu\nu}]$$

↑ ψ: any matter field, including gauge bosons

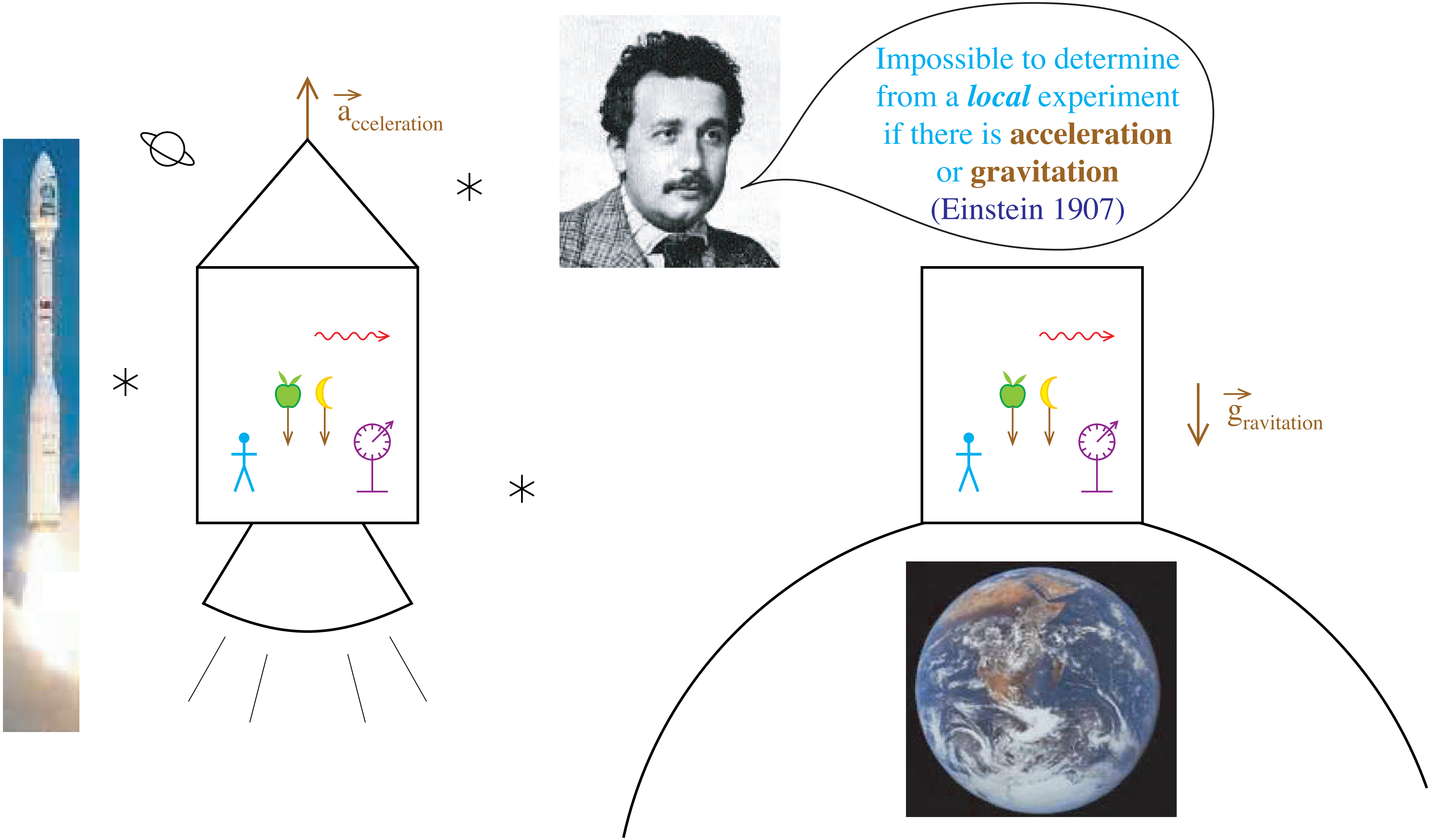
↑ [ ] = functional dependence, i.e. depends on $g_{\mu\nu}$ and its derivatives via the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$.

(This metric defines the lengths and times measured by laboratory rods and clocks, since they are made of matter. It is thus often called the "physical metric".)

MATTER–GRAVITY COUPLING

$S_{\text{matter}}[\,\text{matter}\,,\, g_{\mu\nu}]$

Metric coupling chosen to satisfy the (weak) **equivalence principle**



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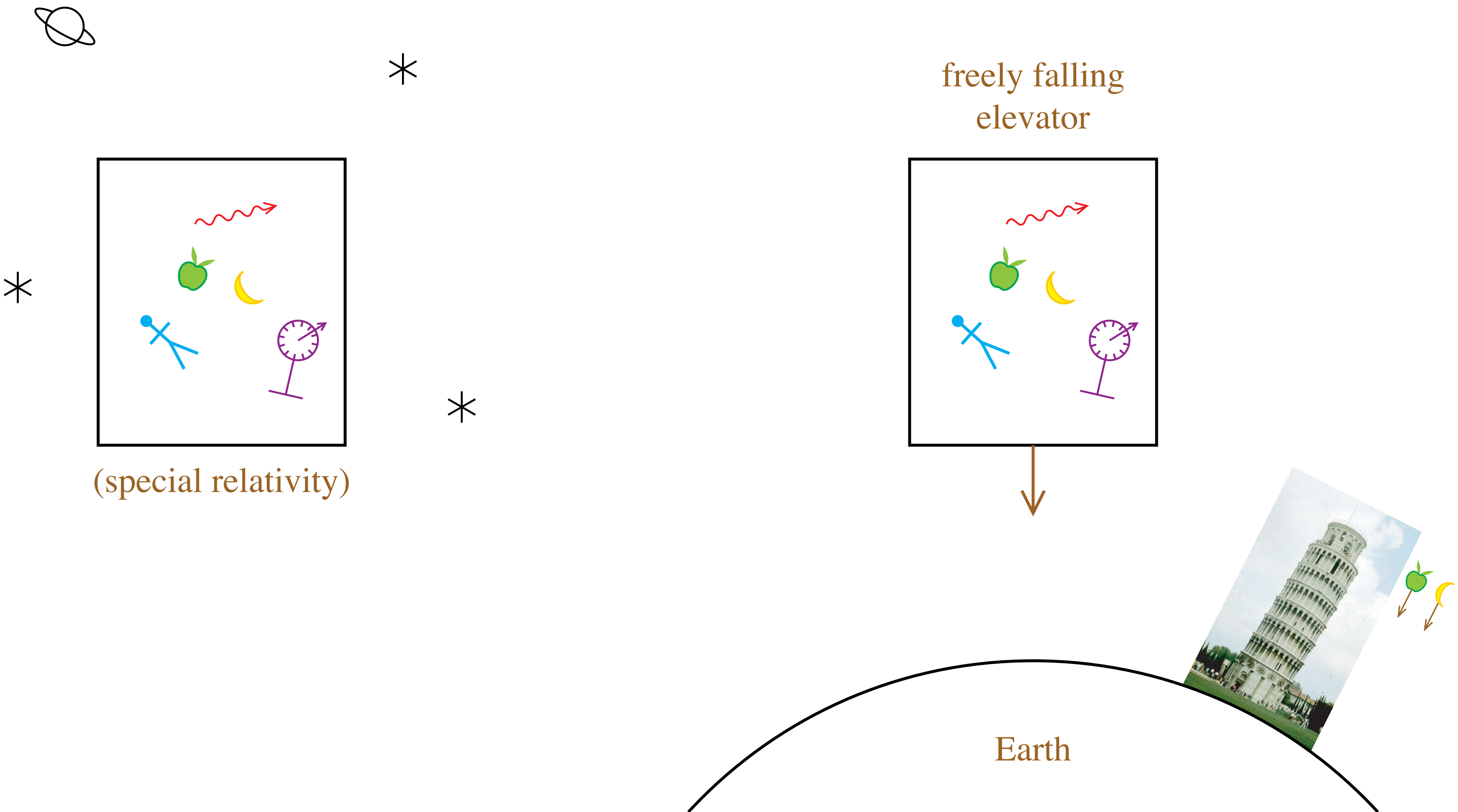


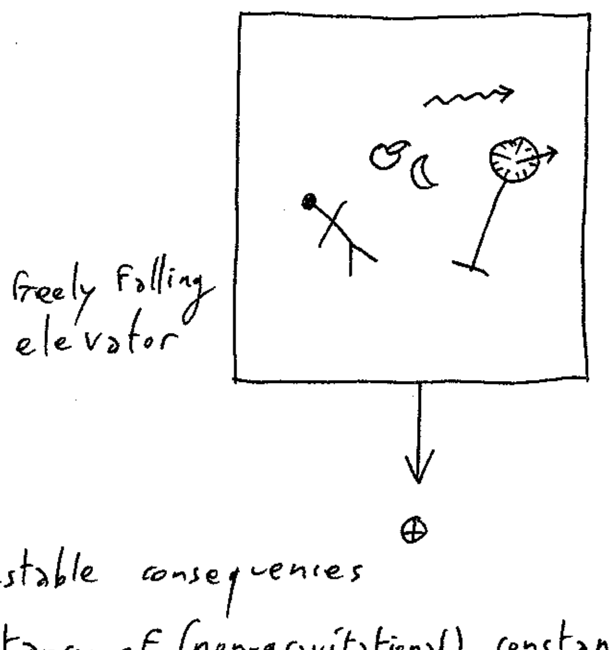
Such a metric coupling implies the Einstein Equivalence principle (cf. C. Will's lectures). Indeed, one may consider a freely falling elevator, in which locally

$$g_{\mu\nu} = \mathrm{diag}(-1,1,1,1) + O(X^2) \qquad \text{and} \quad \Gamma^{\lambda}_{\mu\nu} = 0 + O(X)$$

↑ size of the elevator

[In mathematical language, this is called a Fermi coordinate system: all along a (one-dimensional) worldline, one can always choose the coordinates such that $g_{\mu\nu}=0$ and $\Gamma^{\lambda}_{\mu\nu}=0$. This is the same derivation as C. Will showed at a given spacetime point, and one then proves that the same conditions may be integrated along a line.]

Intuitively



gravity is effaced within the elevator (up to tidal effects $\propto \frac{\text{size of elevator}}{\text{distance to grav. source}}$), so that special relativity may be used to describe physics.

⇒ 4 testable consequences

① Constancy of (non-gravitational) constants, cf. $\left|\frac{\dot{\alpha}}{\alpha}\right| < 7\times10^{-17}\,\mathrm{yr}^{-1}$

② Local Lorentz invariance, cf. isotropy of space tested at 10^{-27} level.

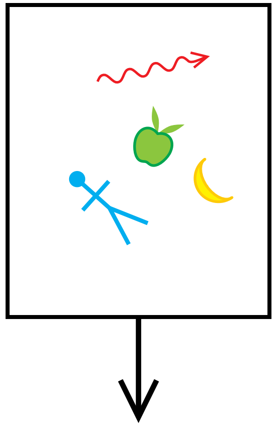
③ Universality of Free Fall, tested at the 4×10^{-13} level

④ Universality of gravitational redshift, tested at the 2×10^{-4} level

MATTER–GRAVITY COUPLING

Metric coupling: $S_{\text{matter}}\,[\,\text{matter}\,,\,g_{\mu\nu}\,]$

Freely falling elevator
(= Fermi coordinate system)



$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\Gamma^{\lambda}_{\mu\nu} = 0$$

$\Rightarrow$

1 Constancy of the constants

Space & time independence of coupling constants and mass scales of the Standard Model

Oklo natural fission reactor
$|\dot{\alpha}/\alpha| < 7\times10^{-17}\,\text{yr}^{-1} \ll 10^{-10}\,\text{yr}^{-1}$ (cosmo)
[Shlyakhter 76, Damour & Dyson 96]

2 Local Lorentz invariance

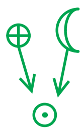
Local non-gravitational experiments are Lorentz invariant

Isotropy of space verified at the 10^{-27} level
[Prestage et al. 85, Lamoreaux et al. 86, Chupp et al. 89]

3 Universality of free fall

Non self-gravitating bodies fall with the same acceleration in an external gravitational field

Laboratory: 4×10^{-13} level [Baessler et al. 99]



: 2×10^{-13} level [Williams et al. 04]

4 Universality of gravitational redshift

In a static Newtonian potential

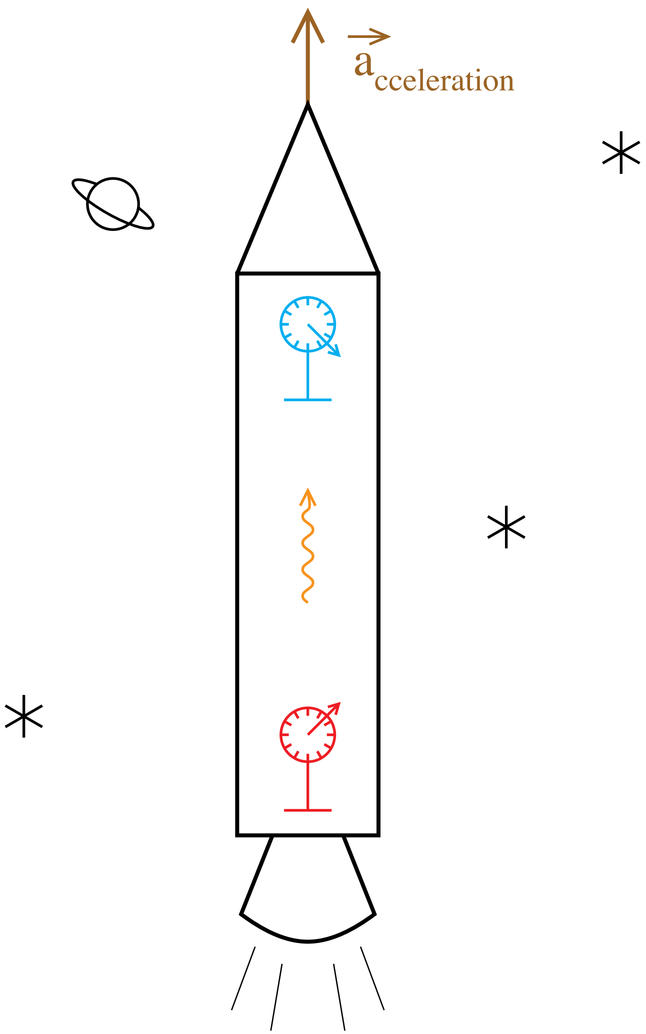
$$g_{00} = -1 + 2\,U(x)/c^2 + O(1/c^4)$$

the time measured by two clocks is

$$\tau_1/\tau_2 = 1 + [U(x_1) - U(x_2)]/c^2 + O(1/c^4)$$

Flying hydrogen maser clock: 2×10^{-4} level
[Vessot et al. 79–80, Pharao/Aces will give 5×10^{-6}]

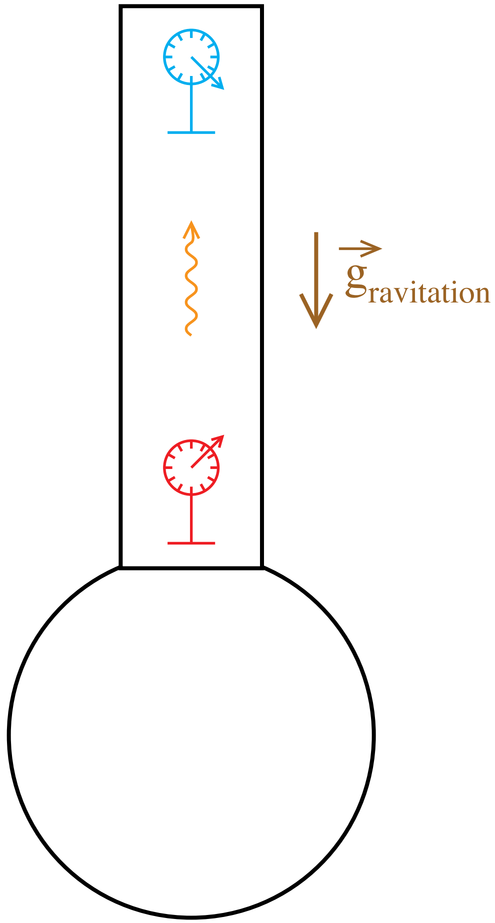
(4) Universality of gravitational redshift (time dilation)



Doppler effect
(cf. fire-truck siren)

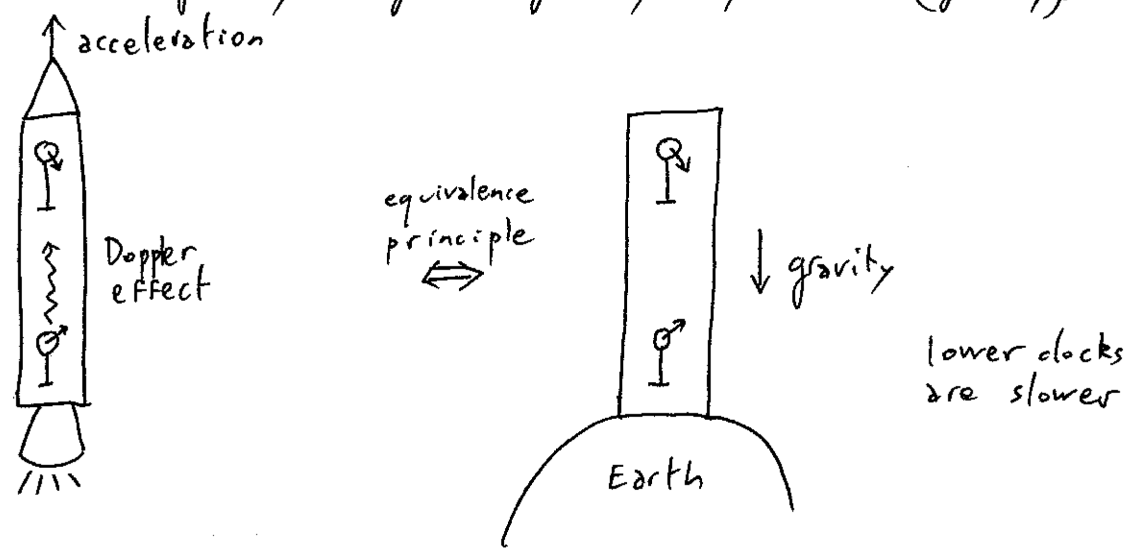
$\Rightarrow$ Whatever their composition,
lower clocks are slower

($\Rightarrow$ impossible to
synchronize
even static clocks)



①, ② and ③ are obvious consequences of the fact that special relativity is assumed to be valid within the freely falling elevator. ④ = the "Einstein effect" has been shown in C. Will's lectures to depend only on this hypothesis of a metric coupling $S_m[\psi; g_{\mu\nu}]$, without assuming anything on gravity's dynamics ($S_{gravity}$).

cf.



Although one may consider theories violating this equivalence principle (cf. superstrings, which do predict that ≠ kinds of matter couple to ≠ tensors $g^{(i)}_{\mu\nu}$!), it is so precisely tested that we will focus in these lectures on metrically-coupled theories, as C. Will did.

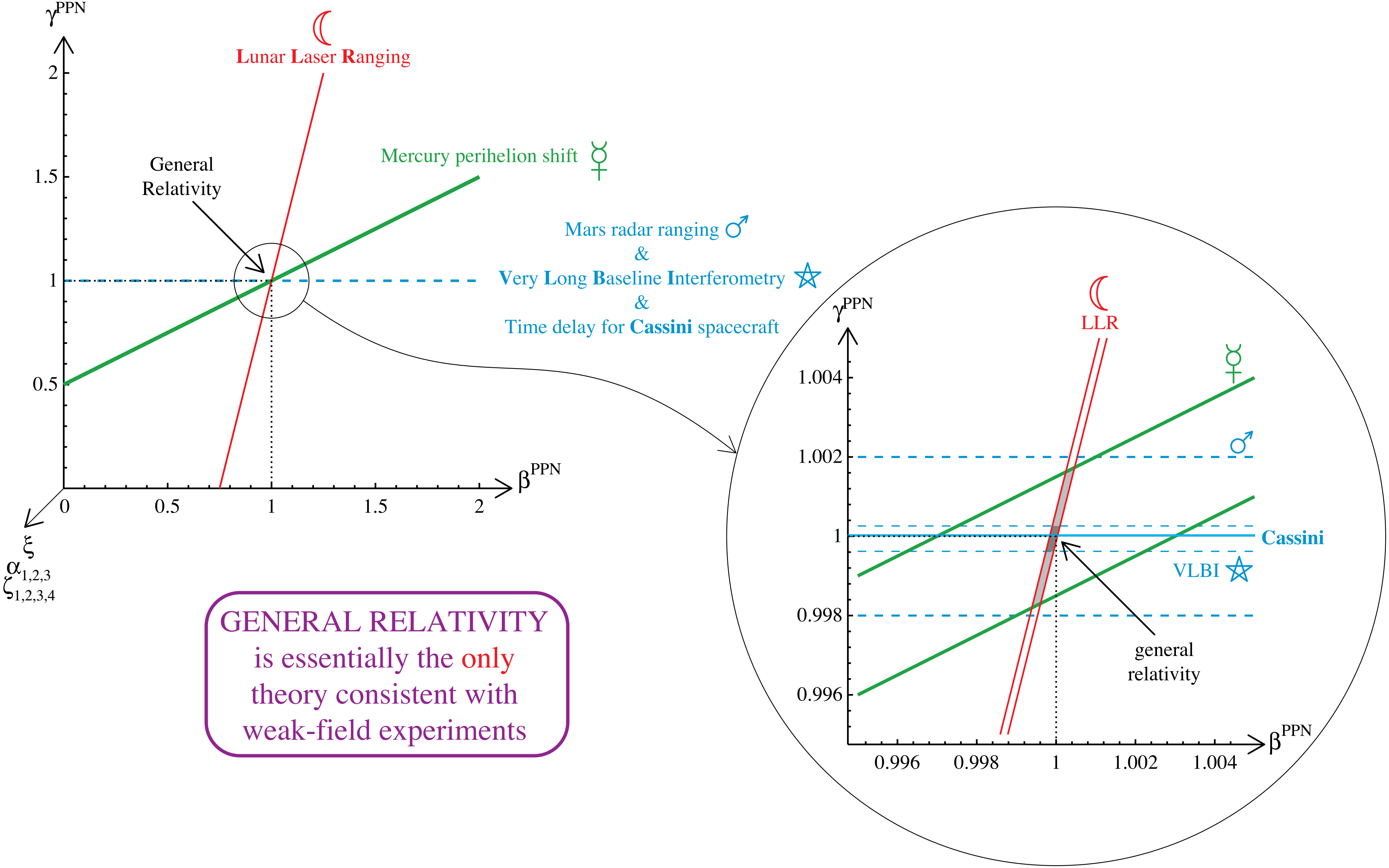
* GR's second assumption is that gravity is mediated by a spin-2 field, described by the Einstein-Hilbert action

$$\boxed{S_{gravity} = \frac{c^4}{16\pi G} \int \frac{d^4x}{c} \sqrt{-g}\, R}$$

(Same sign conventions as C. Will, notably signature $-+++$, but we keep $G \neq 1$ and $c \neq 1$ to clarify.)

{N. Deruelle showed in her lectures that the variation of $S_{grav.} + S_{matter}$ with respect to $g_{\mu\nu}$ gives Einstein's field equations

Conclusion of experimental tests in the **P**arametrized **P**ost-**N**ewtonian formalism



GENERAL RELATIVITY
is essentially the only
theory consistent with
weak-field experiments

– In C. Will's lectures, a phenomenological viewpoint was adopted: nothing precise was assumed about $S_{gravity}$, but $g_{\mu\nu}$ was supposed to depend on all possible potentials that one may define from the matter distribution, at the 1st post-Newtonian (1PN) order, i.e. $1/c^2 \times$ Newton.

– Now, we will adopt a field-theoretical viewpoint: we will assume that $g_{\mu\nu}$ is a combination of various fields, described by a consistent action $\boxed{S_{gravity}}$.
This will allow us to study the predictions even in the strong-field regime (where the PPN framework would need a priori an infinite number of parameters).

A.2 : Higher-order gravity

* A natural way to consider extensions to GR is to take into account the higher-order terms predicted by quantum loops.
* For instance, 't Hooft & Veltman computed in 1974 that the divergence of  needs a counterterm

$$\Delta\mathcal{L} = \frac{\sqrt{-g}}{8\pi^2(d-4)}\left(\frac{53}{90}R_{\mu\nu\rho\sigma}^2 - \frac{361}{180}R_{\mu\nu}^2 + \frac{43}{72}R^2\right)$$

$$= \frac{\sqrt{-g}}{8\pi^2(d-4)}\left[\frac{149}{360}\,G.B. + \frac{7}{40}C_{\mu\nu\rho\sigma}^2 + \frac{1}{8}R^2\right]$$

where $G.B. = R_{\mu\nu\rho\sigma}^2 - 4R_{\mu\nu}^2 + R^2$ is the Gauss-Bonnet topological invariant (in dim 4), which does not contribute to the local field equations

and $C_{\mu\nu\rho\sigma}^2 = G.B. + 2R_{\mu\nu}^2 - \frac{2}{3}R^2$ is the square of the Weyl (fully trace-free) tensor

* In 1977, K. Stelle proved in his thesis that the theory

$$\boxed{S_{gravity} = \frac{c^4}{16\pi G} \int \frac{d^4x}{c} \sqrt{-g} \left(R + \alpha\, C_{\mu\nu\rho\sigma}^2 + \beta R^2 + \gamma\, G.B. \right)}$$

is renormalizable (to all orders) provided both $\underline{\alpha \neq 0}$ and $\underline{\beta \neq 0}$ (γ does not play any role).

* However, he also underlined that such a theory always involve a "ghost", i.e. a negative (kinetic) energy degree of freedom. Intuitively, the propagator reads

$$\frac{1}{p^2 + \alpha p^4} = \frac{1}{p^2} \ominus \frac{1}{p^2 + \frac{1}{\alpha}}$$

(p^2 comes from R; αp^4 comes from $\alpha\, C_{\mu\nu\rho\sigma}^2$; $\frac{1}{p^2}$: usual massless graviton; $\frac{1}{p^2+\frac{1}{\alpha}}$: extra degree of freedom, with mass $m^2 = \frac{1}{\alpha}$)

Therefore, the extra d.o.f may be a tachyon ($\alpha < 0$, $m^2 < 0$) or not ($\alpha > 0$, $m^2 > 0$) depending on the sign of α, but it is anyway a <u>ghost</u>.

The theory is thus violently unstable, because the vacuum can disintegrate into an arbitrary amount of (positive-energy) usual gravitons and a compensating amount of (negative-energy) ghosts.

[N.B.: The only way to make this ghost disappear is to impose $\alpha = 0$, in which case the theory is no longer renormalizable!]

[N.B.2: Superstring theory does generically predict terms like $\alpha\, C_{\mu\nu\rho\sigma}^2$ in the effective 4-dim action, but also an infinite series depending on even higher derivatives of $g_{\mu\nu}$, because it is a nonlocal (although causal) theory. Such higher derivatives start having an observable influence at the same order as $\alpha\, C_{\mu\nu\rho\sigma}^2$ $\Rightarrow$ no meaning to truncate the series.]

* Actually, the above reasoning fails for the βR^2 term: it does generate an extra degree of freedom, but its kinetic energy is positive.

[Intuitive reason: it corresponds to a perturbation of the scalar mode of gravity, i.e. to the already negative Newtonian potential, so that the above calculation is correct provided a minus sign multiplies it globally
$\Rightarrow$ the extra d.o.f has positive energy.
Serious reason: the full calculation, involving all contracted indices [cf. K. Stelle's thesis] shows that the extra d.o.f caused by the βR^2 term is a positive-energy spin-0 mode.]

* More generally, let us prove that

$$S = \frac{c^4}{16\pi G}\int \frac{d^4x}{c}\sqrt{-g}\, f(R) + S_{\text{matter}}[\psi ; g_{\mu\nu}]$$

is a positive-energy scalar-tensor theory.

[cf. Teyssandier & Tourrenc 1983 for the $R+\beta R^2$ case, and many people for the $f(R)$ case, for instance D. Wands, Class. Quantum. Grav. **11**, 269 (1994).]

Forgetting for a while the global constant factor, one may introduce a Lagrange parameter ϕ and write

$$S_{\text{gravity}} = \int d^4x \sqrt{-g}\,\left[f(\phi) + (R-\phi)\, f'(\phi)\right]$$

– Varying this action with respect to ϕ gives $(R-\phi)f''(\phi)=0$, so that $\phi = R$ within each spacetime domain where $f''(\phi)\neq 0$.

– The variation with respect to $g_{\mu\nu}$ gives a field equation with reduces to the original one [deriving from $\int\sqrt{-g}\,f(R)$] when $\phi = R$.
(as can be seen without rigor by replacing ϕ by R in the above action).

Therefore, the $f(R)$ theory is equivalent to

$$\boxed{S_{gravity} = \int d^4x \sqrt{-g} \left\{ f'(\phi) R - 0 (\partial_\mu \phi)^2 - [\phi f'(\phi) - f(\phi)] \right\}}$$

- $f'(\phi)$: nonstandard factor
- $0(\partial_\mu\phi)^2$: no explicit kinetic term
- $[\phi f'(\phi) - f(\phi)]$: potential for ϕ

* We will see in § A.5 that Brans-Dicke theory's action reads

$$S_{gravity} = \frac{c^4}{16\pi G} \int \frac{d^4x}{c} \sqrt{-g} \left\{ \Phi R - \frac{\omega}{\Phi} (\partial_\mu \Phi)^2 - 2U(\Phi) \right\}$$

- $\Phi = f'(\phi)$ here
- $\omega = 0$ here
- $U(\Phi)$: potential $= 0$ in Brans-Dicke theory

[N.B.: Cassini bound on $|\gamma^{PPN} - 1| \Rightarrow \omega > 4000$ if $U(\Phi) = 0$, but here $U(\Phi) \neq 0$.]

The $f(R)$ theory is thus of the "Brans-Dicke" type, but with no explicit kinetic term for Φ.

Does this degree of freedom propagate?

A.3: Einstein and Jordan Frames

* Let us assume (for a short while) that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ is nearly flat. The kinetic term of the above model reads schematically

$$\Phi\left(\partial^2 h + \partial h \partial h\right) \quad + 0 \, \partial\Phi \partial\Phi$$

$$\Phi_{background} \, \partial h \partial h$$ gives the usual spin-2 graviton

but there remains $\int \Phi \partial^2 h = -\int \partial\Phi \partial h$ by partial integr.

We have thus a schematic kinetic term of the form

$$\begin{pmatrix} \partial_\mu h & \partial_\mu \Phi \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & 0 \end{pmatrix} \begin{pmatrix} \partial^\mu h \\ \partial^\mu \Phi \end{pmatrix}$$

One can diagonalize this 2×2 matrix by defining

$$h^*_{\mu\nu} = h_{\mu\nu} + \frac{\beta}{\alpha} \Phi \, \eta_{\mu\nu}$$

so that the kinetic term reads now

$$\begin{pmatrix} \partial_\mu h^* & \partial_\mu \Phi \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & -\beta^2/\alpha \end{pmatrix} \begin{pmatrix} \partial^\mu h^* \\ \partial^\mu \Phi \end{pmatrix}$$

and the degrees of freedom are well separated.

* Let us show how such a field redefinition works without assuming an almost flat metric, and without writing $h_{\mu\nu}$ as if it were a scalar.

* Exercice:

Define $g^*_{\mu\nu} = \Phi\, g_{\mu\nu}$, and show that

$$R^* = \frac{1}{\Phi}\left[R - 3\,\Box \ln\Phi - \frac{3}{2}\left(\partial_\mu \ln\Phi\right)^2\right]$$

Since $\sqrt{-g^*} = \Phi^2\sqrt{-g}$, one thus gets

$$\int \sqrt{-g}\,\Phi R = \int \sqrt{-g^*}\left[R^* + 3\,\Box^* \ln\Phi - \frac{3}{2} g_*^{\mu\nu}(\partial_\mu \ln\Phi)(\partial_\nu \ln\Phi)\right]$$

($\Box^*$: d'Alembertian with respect to metric $g^*_{\mu\nu}$; $g_*^{\mu\nu}$: inverse of $g^*_{\mu\nu}$)

* Therefore, if one sets

$$\begin{cases} g^*_{\mu\nu} \equiv f'(\phi)\, g_{\mu\nu} \\ \varphi \equiv \dfrac{\sqrt{3}}{2} \ln f'(\phi) \\ V(\varphi) \equiv \dfrac{\phi f'(\phi) - f(\phi)}{4 f'^2(\phi)} \\ A(\varphi) = e^{\varphi/\sqrt{3}} \end{cases}$$

"conformal (or Weyl) transformation"

(for $A(\varphi)$: ← imposed by our $f(R)$ form.)

the $f(R)$ theory above can be written as

$$S = \frac{c^4}{4\pi G}\int \frac{d^4x}{c}\sqrt{-g^*}\left\{\frac{R^*}{4} - \frac{1}{2} g_*^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - V(\varphi)\right\} + S_{\text{matter}}\left[\psi\,;\, g_{\mu\nu} = A^2(\varphi)\, g^*_{\mu\nu}\right]$$

* This form, where the kinetic terms of the spin-2 $\left[\int \sqrt{-g^*}\, R^* = \text{Einstein-Hilbert}\right]$ (gravitons) and the spin-0 $\left[-\frac{1}{2}\int \sqrt{-g^*}\, g^{*\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi\right]$ (scalar) degrees of freedom are separated is called the "Einstein frame"

Note that matter is directly coupled to φ via the coupling function $A(\varphi)$.

This frame is useful to analyze the mathematical consistency of the theory and the solutions.

* On the other hand, the choice of variables where matter is minimally coupled to the "metric" $g_{\mu\nu}$ ($S_{mat}[\psi; g_{\mu\nu}]$) is called the "Jordan frame"

This Jordan metric $g_{\mu\nu}$ is the one defining lengths and times as measured by rods and clocks (made of matter) ⇒ this frame is usually more intuitive for the interpretation of observations

* ⚠ But the theory is strictly the same: this is a mere change of variables. Even in the Einstein frame, the full action defines what is observed. [Beware that the literature sometimes spends long discussions on "what is the physical metric"? By definition, this is the one coupled to matter, i.e. $g_{\mu\nu}$, but all calculations may anyway be performed in the Einstein frame if one wishes!]

A.4: Scalar-tensor theories

They are defined by the above action

$$S = \frac{c^4}{4\pi G_*} \int \frac{d^4x}{c} \sqrt{-g^*} \left\{ \frac{R^*}{4} - \frac{1}{2} g_*^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right\} + S_{matter}\left[\psi \, ; \, g_{\mu\nu} \equiv A^2(\varphi)\, g^*_{\mu\nu}\right]$$

[Bergmann 1968, Nordtvedt 1970, Wagoner 1970]

where $A(\varphi)$ is any (nonvanishing) function of φ (instead of $e^{\frac{\varphi}{\sqrt{3}}}$ above)

and $V(\varphi)$ is any scalar-field potential (bounded by below to ensure the stability of the theory).

- Note that we write G_* instead of G in the action, to underline that this constant is actually not the observed Newton's constant (see below).
- It is often useful to use heavy notation to distinguish even further the physical metric $\tilde{g}_{\mu\nu}$ (Jordan frame) $= g_{\mu\nu}$ and the Einstein one $g^*_{\mu\nu}$ (Einstein frame).

Many justifications for considering this class of theories:

* Natural generalizations of the above $f(R)$ models, themselves suggested by quantum loops. [* Scalar field nonminimally coupled $-\partial_\mu\varphi^2 + \xi R\varphi^2$ equivalent to a scalar-tensor theory.]
* Scalar partners to graviton arise in all extra-dimensional (unified) theories.
  For instance, supersymmetry imposes that a dilaton enters the graviton's supermultiplet in 10 dimensions, in string theory.
  Moreover, many other scalar degrees of freedom (called "moduli") arise when performing a dimensional reduction to 4 dim.

For instance, Kaluza proposed (immediately after GR's publication, but published in 1921) a 5-dimensional generalization of GR, where

$$\underbrace{g_{mn}}_{5\text{-dim}} \simeq \left(\begin{array}{c|c} g_{\mu\nu} & A_\mu \\ \hline A_\nu & \textcircled{\varphi} \end{array}\right)$$

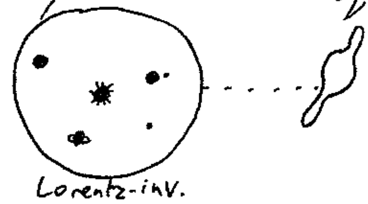
↑ $\textcircled{g_{55}}$ behaves as a scalar field when interpreted in 4 dim.

[Actually, a better definition is $g^{(5)}_{mn} = \left(\begin{array}{c|c} g_{\mu\nu} + e^{2\varphi} A_\mu A_\nu & A_\mu e^{2\varphi} \\ \hline A_\nu e^{2\varphi} & e^{2\varphi} \end{array}\right)$

and one finds $\int \textcircled{d^5x} \sqrt{-g_{(5)}}\, \mathbb{R} \propto \int d^4x \sqrt{-g_{(4)}}\, e^{\varphi}\left(R - \frac{e^{2\varphi}}{4} F^2_{\mu\nu}\right)$, and in the Einstein frame $g^*_{\mu\nu} = e^{\varphi} g_{\mu\nu}$, one finds $\int d^4x \sqrt{-g^*}\left(R^* - \frac{3}{2}\underbrace{[\partial_\mu \varphi]^2 - \frac{e^{3\varphi}}{4} F^2_{\mu\nu}}_{\text{indices contracted with } g_*^{\mu\nu}\ \triangle!}\right)$]

More generally $g^{(D\text{-dimensional})}_{MN} = \left(\begin{array}{c|c} g_{\mu\nu} & A_\mu{}^a \\ \hline A_\nu{}^b & \phi^{ab} \end{array}\right)$, where $A_\mu{}^a$ are Yang-Mills type vector fields, and ϕ^{ab} is a $(D-4)\times(D-4)$ symmetric matrix of <u>scalar fields</u>.

* Scalar-tensor respect most of GR's symmetries: conservation laws, constancy of (non-gravitational) constants, local Lorentz invariance [not true for vector partner] of all physics (even if a subsystem is influenced by external masses) and satisfy exactly Einstein Equiv. Principle even if the scalar field is massless $(V(\varphi)=0)$ [whereas this is impossible for vector or tensor partners to the usual graviton]



Lorentz-inv.

* They anyway describe many possible deviations from GR, and are simple enough for their predictions to be computed in many ≠ domains (solar-system, binary-pulsars, grav. waves, cosmology ⟨→ inflation, → quintessence)

Field equations : $\delta S = 0 \Rightarrow$

$$R^*_{\mu\nu} - \frac{1}{2} g^*_{\mu\nu} R^* = \frac{8\pi G_*}{c^4} \left(T_{* \, \text{matter}}^{\mu\nu} + \tau_*^{\mu\nu}(\varphi) \right),$$

$$\Box^* \varphi = - \frac{4\pi G_*}{c^4} \, \alpha(\varphi) \, T_*^{\text{matter}} + \frac{dV(\varphi)}{d\varphi},$$

$$\frac{\delta S_{\text{matter}}[\psi; A^2(\varphi) g^*_{\mu\nu}]}{\delta \psi} = 0 .$$

where $T_{* \, \text{matter}}^{\mu\nu} \equiv \frac{2c}{\sqrt{-g^*}} \frac{\delta S_{\text{matter}}[\psi; A^2(\varphi) g^*_{\mu\nu}]}{\delta g^*_{\mu\nu}}$,

$$\frac{4\pi G_*}{c^4} \tau_*^{\mu\nu}(\varphi) \equiv \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g^*_{\mu\nu} (\partial_\lambda \varphi)^2 - V(\varphi) g^*_{\mu\nu},$$

($(\partial_\lambda \varphi)^2$ contracted with $g_*^{\lambda\rho}$ ⚠)

and $\alpha(\varphi) \equiv \frac{d \ln A(\varphi)}{d\varphi}$ = matter-scalar coupling strength.

N.B.1 : • The observed matter stress-energy tensor is

$$\tilde{T}^{\mu\nu} = \frac{2c}{\sqrt{-\tilde{g}}} \frac{\delta S_{\text{matter}}[\psi; \tilde{g}_{\mu\nu}]}{\delta \tilde{g}_{\mu\nu}} = A^{-6}(\varphi) T^{*\mu\nu} \quad ⚠$$

(↑ simple calculation [compare defs. of $\tilde{T}^{\mu\nu}$ and $T_*^{\mu\nu}$])

• Einstein's Eq. $\Rightarrow \nabla^*_\mu \left(T_{* \, \text{matter}}^{\mu\nu} + \tau_*^{\mu\nu} \right) = 0$:
conservation in Einstein frame of the sum of matter and φ's energy-momentum, but not independently (since coupled!):

$$\nabla^*_\mu T_{* \, \text{matter}}^{\mu\nu} = \alpha(\varphi) \, T^*_{\text{matter}} \nabla_*^\nu \varphi \qquad (1)$$

• But since $S_{\text{matter}}[\psi; \tilde{g}_{\mu\nu}]$ is diffeomorphism-invariant, we also know that

$$\tilde{\nabla}_\mu \left(\tilde{T}^{\mu\nu} \right) = 0 \quad \text{alone}$$

Actually, this is equivalent to (1) above.

N.B.2: • $\alpha(\varphi)$, the logarithmic derivative of $A(\varphi)$, enters the scalar field's equation because

$$\frac{\delta S_{mat}}{\delta\varphi} = \frac{d\left[A^2(\varphi)\, g^*_{\mu\nu}\right]}{d\varphi} \quad \frac{\delta S_{mat}[\psi;\tilde{g}_{\mu\nu}]}{\delta \tilde{g}_{\mu\nu}}$$

$$= 2\alpha(\varphi)\, A^2(\varphi)\, g^*_{\mu\nu} \; \frac{\sqrt{-\tilde{g}}}{2c}\; \tilde{T}^{\mu\nu}_{matter} = \frac{\sqrt{-g^*}}{c}\,\alpha(\varphi)\, T^*_{matter}$$

$$\cancel{A^2} \qquad \cancel{A^4} \qquad \cancel{A^{-6}}$$

• Other simple way to see it:
Consider a (constant-observed-mass) point particle:

$$S_{pp} = -\int \tilde{m}\, c\, d\tilde{s} \qquad = -\int \tilde{m}\, c \sqrt{-\tilde{g}_{\mu\nu}\, dz^\mu dz^\nu}$$

($\tilde{m}$: const.; $d\tilde{s}$: because matter universally coupled to $\tilde{g}_{\mu\nu} = A^2(\varphi)\, g^*_{\mu\nu}$)

$$= -\int \underbrace{\left[\tilde{m}\, A(\varphi)\right]}_{\equiv\, m^*(\varphi)} c\, ds^*$$

In the Einstein frame, "masses" are functions of φ (even for laboratory-size, non self-gravitating, bodies) ⚠

Now the source term for $\Box^*\varphi$ involves

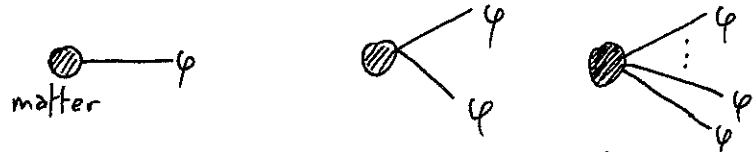
$$\frac{\delta S_{pp}}{\delta\varphi} \propto \tilde{m}\, \frac{dA(\varphi)}{d\varphi} = \alpha(\varphi)\; m^*(\varphi)$$

($\alpha(\varphi)$: coupling strength; $m^*(\varphi)$: Einstein-frame mass)

Diagrammatic interpretation of $A(\varphi)$:

Expand $\ln A(\varphi)$ around a background value φ_0 (imposed by boundary conditions at ∞, say because of the cosmological evolution of the Universe).

$$\ln A(\varphi) = \underset{(\text{const.})}{\ln A(\varphi_0)} + \alpha_0 \varphi + \frac{1}{2}\beta_0 \varphi^2 + \cdots$$



(The coupling strength $\alpha(\varphi) = \alpha_0 + \beta_0 \varphi + \cdots$ takes thus into account nonlinearities.)

A.5: Nordström, Brans–Dicke and generalizations

* Nordström 1913 [before G.R.!]: particular case where $\exists$ scalar but no spin-2 usual graviton

$$S = -\frac{c^4}{8\pi G}\int \frac{d^4x}{c}\, \eta^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi + S_{\text{matter}}\left[\psi;\ \tilde{g}_{\mu\nu} \equiv A^2(\varphi)\,\eta_{\mu\nu}\right]$$

[1912 : $A(\varphi) = e^{\varphi}$; 1913 : $A(\varphi) = \varphi$]

$$\frac{\delta S}{\delta\varphi} = 0 \Rightarrow \quad \Box_{\text{flat}}\,\varphi = -\frac{4\pi G}{c^4}\,\alpha(\varphi)\,A^4(\varphi)\,\tilde{T}_{\text{matter}}$$

$$\Leftrightarrow \quad \tilde{R} = \frac{24\pi G}{c^4} A'^2(\varphi)\,\tilde{T}_{\text{matter}} - 6\frac{A''}{A}\,\tilde{g}^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi$$

(explicit dependence on φ)

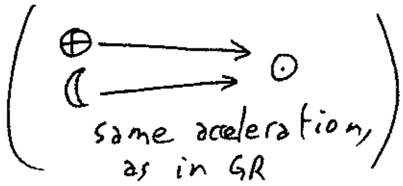
Therefore, if $A(\varphi) = \varphi$, the field equation may be written as

$$\boxed{\tilde{R} = \frac{24\pi G}{c^4}\,\tilde{T}}$$

and $\boxed{\tilde{C}^{\lambda}{}_{\mu\nu\rho} = 0} \Rightarrow \tilde{g}_{\mu\nu}$ conformally flat

This rewriting [Einstein–Fokker 1914] without any explicit dependence on φ proves that the <u>strong</u> equivalence principle also holds (same reasoning as in C. Will's lectures: always possible to choose $\tilde{g}_{\mu\nu} \to \eta_{\mu\nu}$ on a sphere around system, up to tidal effects).

Therefore, gravitational binding energy falls in the same way as any other form of energy $\Rightarrow$ no Nordtvedt effect



$\Rightarrow$ { $\exists$ ② theories satisfying the strong equivalence principle: Nordström's and GR. And Nordström's is technically <u>much</u> simpler.

<u>What's the problem with it?</u>

$$S_{EM} = \frac{-1}{4}\int d^4x \sqrt{-\tilde{g}}\; \tilde{g}^{\mu\rho}\, \tilde{g}^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}$$

$$= -\frac{1}{4}\int d^4x\; \cancel{A^4}\, \cancel{A^{-2}}\eta^{\mu\rho}\, \cancel{A^{-2}}\eta^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \qquad \text{"conformal invariance of } S_{EM}\text{"}$$

$\Rightarrow$ photons do not feel at all the scalar field!

$\Rightarrow$ they propagate like in flat space, and there cannot exist any light deflection: ruled-out experimentally.

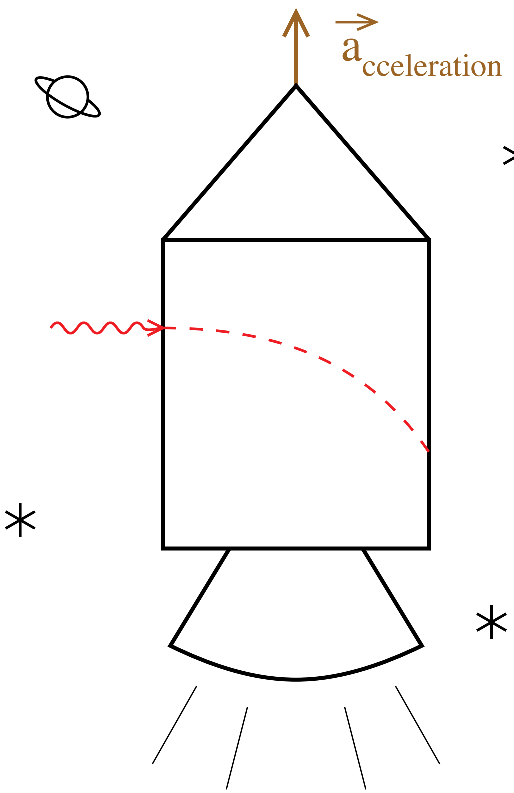
[cf. also $ds^2 = 0 \Leftrightarrow \eta_{\mu\nu} dx^\mu dx^\nu = 0 \Rightarrow$ flat-space geodesics if null]

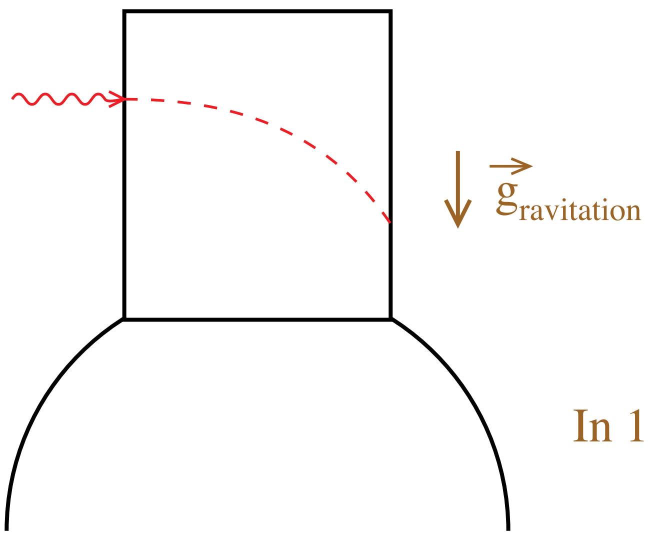
<u>N.B.</u>:
- No light deflection $\Rightarrow 1+\gamma^{PPN} = 0 \Rightarrow \gamma^{PPN} = -1$
- No Nordtvedt effect $\Rightarrow 4\beta^{PPN} - \gamma^{PPN} - 3 = 0 \Rightarrow \beta^{PPN} = \frac{1}{2}$

  [Easy to prove directly by solving the static & spherically symmetric solution $\Rightarrow \tilde{g}_{\mu\nu} = \left(1 - \frac{GM}{rc^2}\right)^2 \eta_{\mu\nu}$.]
- Therefore ☿'s perihelion $\propto \frac{2\gamma^{PPN} - \beta^{PPN} + 2}{3} = -\frac{1}{6} \times$ RG's result

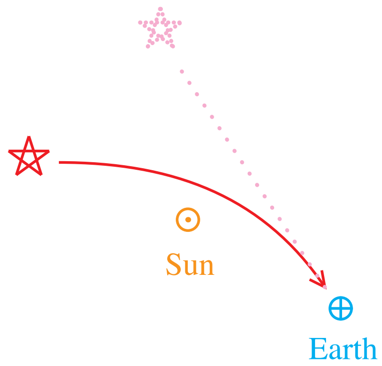
  Also <u>ruled out</u> experimentally.

■ Light deflection and the equivalence principle

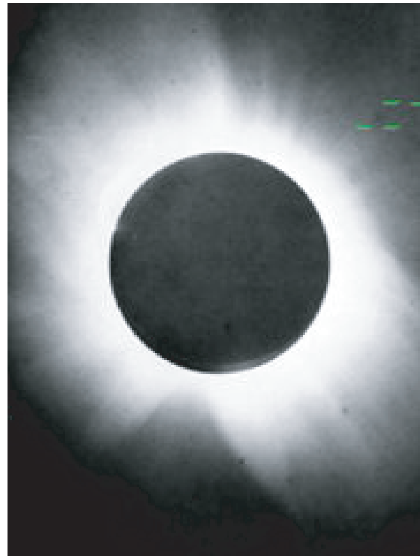




□ Modification of the stars' apparent position



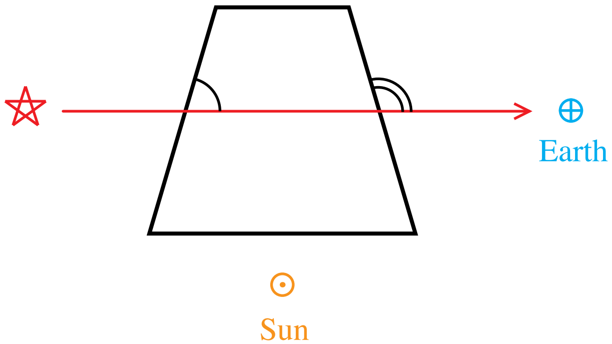
In 1911–14, Einstein predicts **half** the correct value

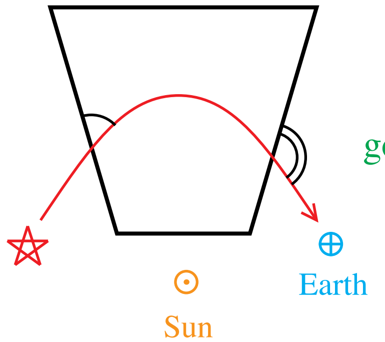


[Eddington 1919]

This is because □ also a deformation of *space*:

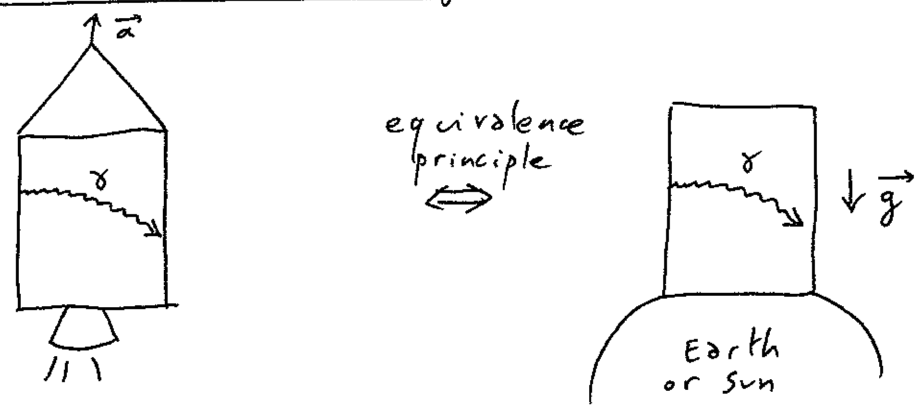
Nordström's theory 1913





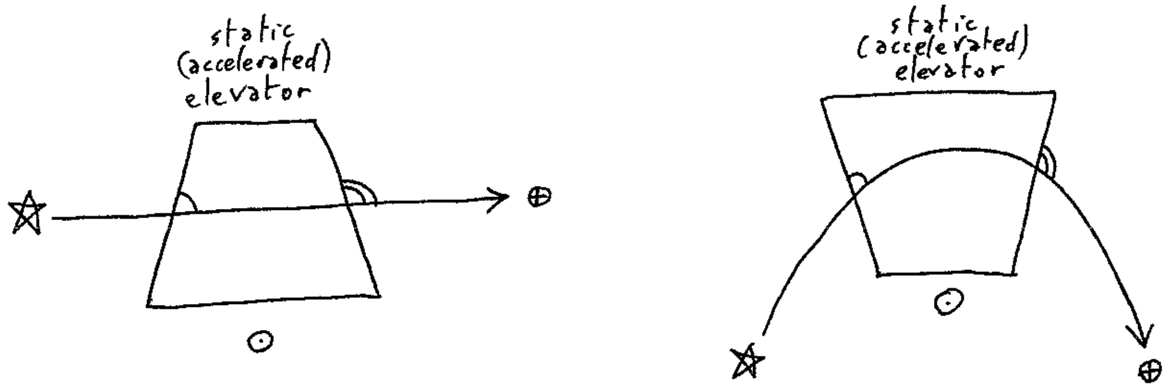
Einstein's general relativity 1915

Equivalence principle without light deflection??



Light must be locally attracted by a massive body!

But as shown in C. Will's lectures, ∃ also a global effect due to the spatial curvature:



Nordström 1913

$$\Delta\theta = \frac{2\, GM_\odot}{bc^2}\left(1+\gamma^{PPN}\right), \quad 1+\gamma^{PPN} = 0$$

Equivalence principle (From Newton's potential in g_{00}) — contribution of g_{ij} at order $1/c^2$

Einstein 1915

$$\Delta\theta = \frac{2\, GM_\odot}{bc^2}\left(1+\gamma^{PPN}\right), \quad 1+\gamma^{PPN} = 2$$

From g_{00} — From g_{ij}

= twice the naive Newtonian result

[cf. Soldner 1803 & Einstein 1911]

* P. Jordan {1949, 1955}:

$$\boxed{S = \frac{c^4}{16\pi G_*}\int \frac{\sqrt{-g}\,d^4x}{c}\; e^{\eta\varphi}\left[R - \omega(\partial_\mu\varphi)^2 + e^{\varphi}\mathcal{L}_{matter}\right]}$$

where η and ω are constants.

Example of matter (to simplify): $\mathcal{L}_{matter} = -k_1(\partial_\mu \pi^0)^2 - k_2 F_{\mu\nu}^2$

$$\Rightarrow S_{matter} \propto \int \sqrt{-g}\,d^4x \left[-k_1\, e^{(\eta+1)\varphi} g^{\mu\nu}\partial_\mu\pi^0\partial_\nu\pi^0 - k_2\, e^{(\eta+1)\varphi} g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\right]$$

Let $\tilde{g}_{\mu\nu} \equiv e^{(\eta+1)\varphi} g_{\mu\nu}$

$$\Rightarrow \sqrt{-\tilde{g}} = e^{2(\eta+1)\varphi}\sqrt{-g} \quad \text{and} \quad \tilde{g}^{\mu\nu} = e^{-(\eta+1)\varphi} g^{\mu\nu}$$

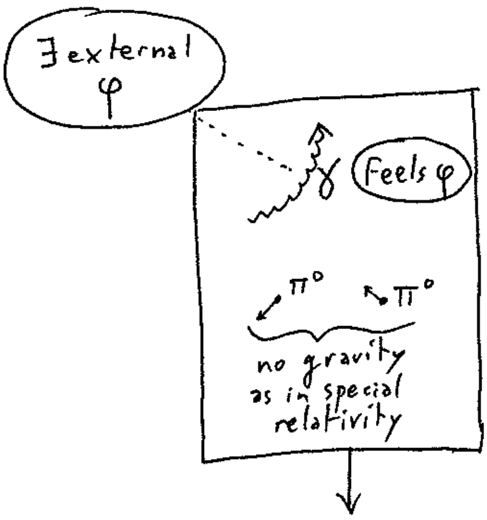
$$\Rightarrow \sqrt{-g}\, g^{\mu\nu} = e^{-(\eta+1)\varphi}\sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu}$$

Therefore

$$S_{matter} \propto \int \underbrace{\sqrt{-\tilde{g}}\,d^4x\left[-k_1\,\tilde{g}^{\mu\nu}\partial_\mu\pi^0\partial_\nu\pi^0\right.}_{\text{metric coupling}} - k_2\, \underset{\uparrow}{e^{(\eta+1)\varphi}}\,\tilde{g}^{\mu\rho}\tilde{g}^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\Big]$$

⚠ not changed because of conformal invariance of S_{EM} in 4 dimensions (shown above)

non-metric coupling ("dilatonic")

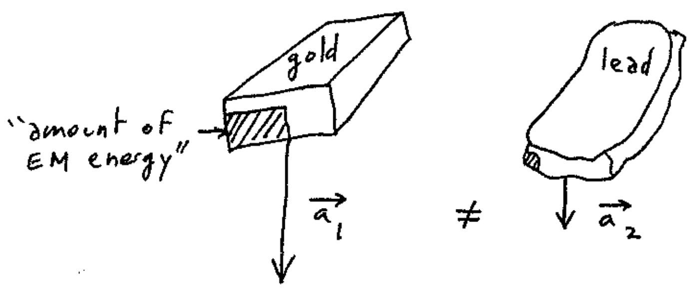


Freely falling elevator with respect to $\tilde{g}_{\mu\nu}$ (i.e. elevator made of π^0's, for instance!)

Photons Feel the scalar field, whatever the definition of $\tilde{g}_{\mu\nu}$!

N.B. : • Actually, an electromagnetic action $-\int \sqrt{-g}\, d^4x\, B^2(\varphi) F_{\mu\nu}^2$, although "non-metric", does not change the light trajectory in the eikonal approximation (cf. C. Will's lectures). It can also be shown that the polarization is not affected either by $B^2(\varphi)$ in this approximation [private communication with B. Bertotti]. This explicit coupling $B^2(\varphi)$ to a scalar field actually only changes the amplitude of electromagnetic waves — difficult to measure accurately anyway.

• BUT this $B^2(\varphi)$ factor, i.e. $e^{(\eta+1)\varphi}$ in the Jordan framework, does change the amount of electromagnetic energy contained in any material body:



because EM energy feels the external scalar φ.

Since universality of free-fall is presently tested at the $\sim 3\times 10^{-13}$ level, this imposes $|\eta+1| \lesssim 10^{-5}$

$\Rightarrow$

* Fierz 1956 & Jordan 1959: Same class of models with $\eta = -1$:

$$S = \frac{c^4}{16\pi G_*}\int \frac{\sqrt{-g}\, d^4x}{c}\left[e^{-\varphi} R - \omega e^{-\varphi} (\partial_\mu \varphi)^2 + \mathcal{L}_{\text{matter}} \right]$$

* Brans & Dicke 1961: Let $\Phi \equiv e^{-\varphi} \Rightarrow$

$$S = \frac{c^4}{16\pi G_*}\int \frac{\sqrt{-g}\, d^4x}{c}\left[\Phi R - \frac{\omega}{\Phi}(\partial_\mu \Phi)^2 \right] + S_{\text{matter}}[\psi; g_{\mu\nu}]$$

* Bergmann – Nordtvedt – Wagoner ~1970: generalization to any $\omega(\Phi)$ and with a possible potential $-2V(\Phi)$.

* Translation in Einstein frame:

- Use the exercise p. (9): define $\boxed{g^*_{\mu\nu} \equiv \Phi\, g_{\mu\nu}}$

$$\Rightarrow \int \sqrt{-g}\,\Phi R = \int \sqrt{-g^*}\left[R^* - \frac{3}{2} g_*^{\mu\nu} \frac{\partial_\mu \Phi\, \partial_\nu \Phi}{\Phi^2} + \text{tot. div.}\right]$$

and
$$-\int \sqrt{-g}\,\frac{\omega}{\Phi} g^{\mu\nu} \partial_\mu \Phi\, \partial_\nu \Phi = -\int \sqrt{-g^*}\;\omega\, g_*^{\mu\nu} \frac{\partial_\mu \Phi\, \partial_\nu \Phi}{\Phi^2}$$

$$\Rightarrow \begin{cases} S = \dfrac{c^4}{16\pi G_*}\displaystyle\int \frac{\sqrt{-g^*}\,d^4x}{c}\left[R^* - \frac{2\omega+3}{2} g_*^{\mu\nu} \partial_\mu \ln\Phi\, \partial_\nu \ln\Phi - 2\frac{U(\Phi)}{\Phi^2}\right] \\ \quad + S_{\text{matter}}\left[\psi;\; g_{\mu\nu} = \dfrac{g^*_{\mu\nu}}{\Phi}\right] \end{cases}$$

- Let $\boxed{2\,d\varphi = \pm\sqrt{2\omega+3}\; d\ln\Phi}$ $\Leftrightarrow$
$$\varphi - \varphi_0 = \pm\int \frac{\sqrt{2\omega(\Phi)+3}\; d\Phi}{2\Phi} = \pm\frac{\sqrt{2\omega+3}}{2}\ln\Phi$$
if $\omega = \text{const.}$ [Brans–Dicke]

$\Rightarrow$ S takes now the general scalar–tensor form of p. (11):

$$\boxed{S = \frac{c^4}{4\pi G_*}\int \frac{d^4x}{c}\sqrt{-g^*}\left\{\frac{R^*}{4} - \frac{1}{2} g_*^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - V(\varphi)\right\} + S_{\text{matter}}\left[\psi;\, A^2(\varphi)\, g^*_{\mu\nu}\right]}$$

with $A^2(\varphi) \equiv \dfrac{1}{\Phi}$ and $2V(\varphi) \equiv \dfrac{U(\Phi)}{\Phi^2}$.

- Matter–scalar coupling strength:

$$\alpha(\varphi) \equiv \frac{d\ln A(\varphi)}{d\varphi} = \frac{\Phi\, d\,\Phi^{-1}}{\pm\sqrt{2\omega+3}\; d\ln\Phi} = \mp\frac{1}{\sqrt{2\omega+3}}$$

$$\Rightarrow \boxed{2\omega + 3 = \frac{1}{\alpha^2(\varphi)}}$$

Proves that $\boxed{2\omega+3 > 0}$ is necessary, <u>otherwise</u> the kinetic term for φ would have the wrong sign in Einstein frame $\Rightarrow$ <u>ghost</u> scalar degree of freedom ⚠

* Further generalizations

$$S = \int \sqrt{-g}\, d^4x \; f(R, \Box R, \Box^2 R, \dots, \Box^m R)$$

Same kind of reasoning as for $f(R)$ in p. (6)

⇒ generically equivalent to a tensor + (m+1) scalar fields theory. [Gottlöber, Schmidt & Starobinsky, Class. Quantum Grav. 7, 893 (1990); D. Wands, Class. Quantum Grav. 11, 269 (1994)].

$$S = \frac{c^4}{4\pi G_*} \int \frac{d^4x}{c} \sqrt{-g_*} \left\{ \frac{R^*}{4} - \frac{1}{2} g_*^{\mu\nu} \boxed{\gamma_{ab}(\varphi^c)}\, \partial_\mu \varphi^a \partial_\nu \varphi^c - V(\varphi^a) \right\} + S_{matter}\left[\psi ; A^2(\varphi^a)\, g^*_{\mu\nu}\right]$$

where $\gamma_{ab}(\varphi^c)$ is a $n \times n$ symmetric matrix possibly dependent on the n scalar fields φ^a [i.e. a "σ-model" metric].

[general study in T. Damour & G.E-F, Class. Quantum Grav 9 (1992) 2093]

* ∃ even more general scalar-tensor models, involving functions of the kinetic term $\boxed{f[(\partial_\mu\varphi)^2]}$, called "k-inflation", "k-essence" or "RAQUAL" models. We will examine them in part (C) of these lectures.

A.6. : Weak-field predictions

* Effect of $V(\varphi) = \frac{1}{2} m_\varphi^2 \varphi^2 + \dots$

$$\Box^* \varphi = -\frac{4\pi G_*}{c^4} \alpha(\varphi)\, T_*^{\text{matter}} + V'(\varphi)$$

with $\Box^*\varphi \to \Delta_{\text{flat}}\varphi + O\left(\frac{1}{c^2}\right)\varphi$, $\alpha(\varphi) \to \alpha_0 + \beta_0\varphi + \dots$, $T_*^{\text{matter}} \to -\rho_* c^2 + \dots$, $V'(\varphi) \to m_\varphi^2 \varphi$

$$\Rightarrow \quad \Delta_{\text{flat}}\varphi = 4\pi \frac{G_*}{c^2} \alpha_0 \rho_* + m_\varphi^2 \varphi + O\left(\frac{1}{c^4}\right)$$

$$\Rightarrow \quad \boxed{\varphi = \varphi_0 - \frac{\alpha_0 G M_*}{r c^2} e^{-m_\varphi r} + O\left(\frac{1}{c^4}\right)} \quad \text{of the Yukawa type}$$

- Either $m_\varphi \gg \frac{1}{\text{A.U.}} \sim$ distances in the solar system and φ is thus exponentially small $\Rightarrow$ theory $\simeq$ G.R. to a high accuracy.
- Or $m_\varphi \ll \frac{1}{\text{A.U.}}$ and φ behaves (almost) as if it were massless : we will study this case ($\underline{\underline{V(\varphi)=0}}$) in the following.

- ⚠ Beware that $V(\varphi)$ plays a crucial rôle in cosmology (cf. inflation & quintessence notably).
- ⚠ Beware also that $V(\varphi)$ must be taken into account if $m_\varphi \sim \frac{1}{\text{distances}}$.

Cf. notably the interesting case of the "chameleon" field [J. Khoury & A. Weltman], in which ∃ competition between matter-scalar coupling function $A(\varphi)$ and potential $V(\varphi)$
$\Rightarrow$ effective m_φ large within matter (e.g. atmosphere) but small in vacuum.

* At the 1st post-Newtonian (1PN) level, i.e. $\frac{1}{c^2}$ smaller than Newton's force, scalar-tensor theories (with $V(\varphi)=0$) can be described in terms of the PPN parameters.

- Simplest way to obtain the only ones which differ from GR's values (but does not prove (here) that they are the only ones):

We already know that $\boxed{\varphi = \varphi_0 - \frac{\alpha_0 G M_*}{r c^2} + O\left(\frac{1}{c^4}\right)}$ if $V(\varphi)=0$

Einstein's equations $\Rightarrow$

$$\boxed{R^*_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi + \frac{8\pi G_*}{c^4}\left(T^*_{\mu\nu} - \frac{1}{2}T^* g^*_{\mu\nu}\right)}$$

(the last term crossed out: in vacuum)

Therefore, only $R^*_{rr} = 2(\partial_r\varphi)^2$ is modified with respect to G.R. in a static & spherically symmetric situation.

Let us choose isotropic coordinates

$$ds^{*2} = -e^\nu c^2 dt^2 + e^\lambda\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\right)$$

Then $R^*_{00} = 0 \Rightarrow \Delta\nu + \frac{1}{2}\nu'(\nu'+\lambda') = 0$ ($' = d/dr$)

(cf. exponential parametrization of g_{00} underlined in Damour's lectures)

we wish to know g^*_{00} at order $\frac{1}{c^4}$ included
$\Rightarrow \lambda'$ at first order $\left(\frac{1}{c^2}\right)$ suffices here.

and $\left.\begin{array}{l} R^*_{rr}=0 \\ R^*_{00}=0 \end{array}\right\} \Rightarrow \Delta\lambda + O(\lambda',\nu') = -2\varphi'^2$

[or $R^*_{\theta\theta} = R^*_{\phi\phi} = 0$]

$-2\varphi'^2$: $O(1/c^4)$ $\Rightarrow$ negligible!

$\Delta\lambda$: needed at order $\frac{1}{c^2}$ only for computing 1PN effects

Therefore, at 1PN order, $g^*_{\mu\nu}$ takes strictly the <u>same</u> form as G.R.:

$$ds_*^2 = -\left(1 - \frac{2G_* M_*}{rc^2} + 2\left(\frac{G_* M_*}{rc^2}\right)^2 + O\left(\frac{1}{c^6}\right)\right) c^2 dt^2 + \left(1 + 2\frac{G_* M_*}{rc^2} + O\left(\frac{1}{c^4}\right)\right) d\vec{x}^2$$

<u>Conclusion</u>: no deviation from G.R. at $\frac{1}{c^2}$ order (??)

- No ⚠ because matter is coupled to $\tilde{g}_{\mu\nu} = A^2(\varphi)\, g^*_{\mu\nu}$ and <u>not</u> to $g^*_{\mu\nu}$ alone!

⇒ <u>Let us compute the expansion of the conformal factor $A^2(\varphi)$ up to $O\left(\frac{1}{c^4}\right)$ included:</u>

$$A^2(\varphi) = A^2(\varphi_0) \exp\left[2\alpha_0(\varphi-\varphi_0) + \beta_0(\varphi-\varphi_0)^2 + O\left(\frac{1}{c^6}\right)\right]$$

Now we need φ up to $O\left(\frac{1}{c^4}\right)$ included. Its field equation reads, in a static situation and in vacuum:

$$\Box^* \varphi = 0 \quad \Rightarrow \quad \frac{1}{\sqrt{-g^*}}\partial_r\left(\sqrt{-g^*}\, g_*^{rr}\, \partial_r \varphi\right) = 0$$

$$\Rightarrow \partial_r\left\{ r^2 \left[1 + \cancel{\frac{2G_* M_*}{rc^2}} + O\left(\frac{1}{c^4}\right)\right]\left[1 - \cancel{\frac{2G_* M_*}{rc^2}} + O\left(\frac{1}{c^4}\right)\right] \partial_r \varphi \right\} = 0$$

$$\Rightarrow \quad \partial_r \varphi = \frac{\text{const.}}{c^2 r^2} + O\left(\frac{1}{c^6}\right)$$

↑ no term in $\frac{1}{c^4}$

$$\Rightarrow \quad \boxed{\varphi - \varphi_0 = -\frac{\alpha_A G_* M_A^*}{rc^2} + O\left(\frac{1}{c^6}\right)}$$

(just a <u>name</u> for the constant!)

[where $\alpha_A = \alpha_0 + O\left(\frac{1}{c^2}\right)$ because the source is $\alpha(\varphi)\, T_*^{\text{matter}} = (\alpha_0 + \beta_0 \varphi + \ldots)\, T_*^{\text{matter}}$

↑ depends on local value of φ where body A is located]

[redacted]

We have thus

$$\frac{A^2(\varphi)}{A^2(\varphi_0)} = 1 + 2\alpha_0(\varphi-\varphi_0) + \left[2\alpha_0^2+\beta_0\right](\varphi-\varphi_0)^2 + O\left(\frac{1}{c^6}\right)$$

$$= 1 - 2\alpha_A\alpha_0 \frac{G_* M_A^*}{rc^2} + 2\left[(\alpha_A\alpha_0)^2 + \frac{\alpha_A^2\beta_0}{2}\right]\left(\frac{G_* M_A^*}{rc^2}\right)^2 + O\left(\frac{1}{c^6}\right)$$

and the physical (Jordan-frame) line element then reads

$$\begin{cases} d\tilde{s}^2 = A^2(\varphi)\, ds_*^2 \\ \quad = -\left[1 - 2\dfrac{\tilde{G}_{eff}\tilde{M}_A}{\tilde{r}c^2} + 2\beta^{PPN}\left(\dfrac{\tilde{G}_{eff}\tilde{M}_A}{\tilde{r}c^2}\right)^2 + O\left(\dfrac{1}{c^6}\right)\right]c^2 d\tilde{t}^2 \\ \quad + \left[1 + 2\gamma^{PPN}\dfrac{\tilde{G}_{eff}\tilde{M}_A}{\tilde{r}c^2} + O\left(\dfrac{1}{c^4}\right)\right] d\vec{\tilde{x}}^2 \end{cases}$$

(standard post-Newtonian form, proposed by Eddington in 1922 and generalized by Will & Nordtvedt in 1968-1972)

where $\tilde{t} = A(\varphi_0)\,t$ and $\tilde{r} = A(\varphi_0)\,r$ to get $\eta_{\mu\nu}$ at infinity
$\tilde{M} = A^{-1}(\varphi_0)M_*$ consistent rescaling (compare with p. (14))

(One may choose $A(\varphi_0)=1$ from the beginning to simplify)

$\tilde{G}_{eff} = G_* A^2(\varphi_0)\left[1+\alpha_A\alpha_0\right]$	effective Newton's constant	
$\gamma^{PPN} = 1 - 2\dfrac{\alpha_A\alpha_0}{1+\alpha_A\alpha_0}$	$\neq 1$	in G.R.
$\beta^{PPN} = 1 + \dfrac{1}{2}\dfrac{\alpha_A^2\beta_0}{(1+\alpha_A\alpha_0)^2}$	$\neq 1$	in G.R.

For laboratory-size objects, negligible self-gravity $\Rightarrow$ α_A may be replaced by α_0

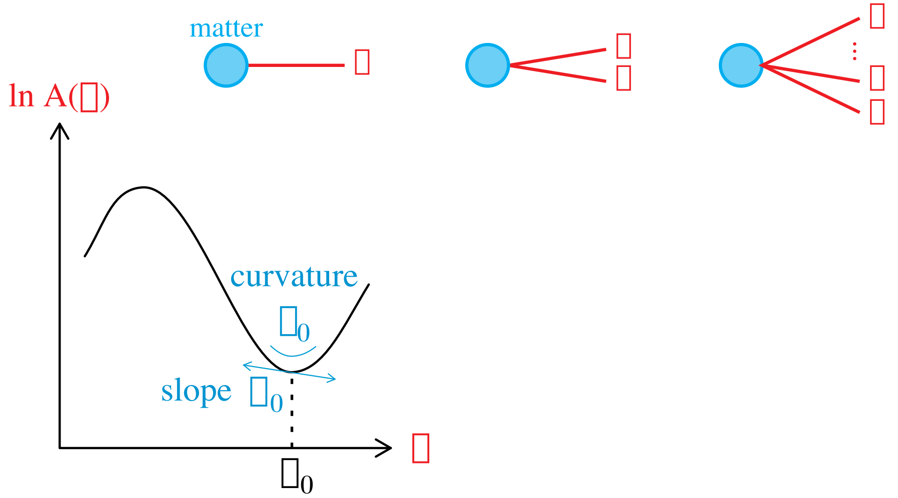
$$\begin{aligned} \tilde{G}_{eff} &= G_* A_0^2(1+\alpha_0^2) \\ \gamma^{PPN} &= 1 - 2\frac{\alpha_0^2}{1+\alpha_0^2} \\ \beta^{PPN} &= 1 + \frac{\alpha_0^2\beta_0}{2(1+\alpha_0^2)^2} \end{aligned} \quad \Leftrightarrow \quad \begin{cases} \tilde{G}_{eff} = \dfrac{G_*}{\Phi_0}\dfrac{2\omega+4}{2\omega+3} \quad \text{in Brans-Dicke} \\ \gamma^{PPN} = \dfrac{1+\omega}{2+\omega} \\ \beta^{PPN} = \dfrac{\Phi_0\, d\omega/d\Phi}{(2\omega+3)(2\omega+4)^2} \end{cases}$$

Tensor–scalar theories

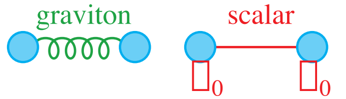
$$S = \frac{1}{16\pi G}\int d^4x \sqrt{-g_*}\left\{R^* - 2\left(\partial_\mu \varphi\right)^2\right\} + S_{matter}\left[matter,\, g_{\mu\nu} \equiv A^2(\varphi)\, g^*_{\mu\nu}\right]$$

spin 2 — spin 0 — physical metric

$$\ln A(\varphi) = \alpha_0\,(\varphi-\varphi_0) + \frac{1}{2}\beta_0\,(\varphi-\varphi_0)^2 + \dots$$

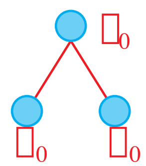


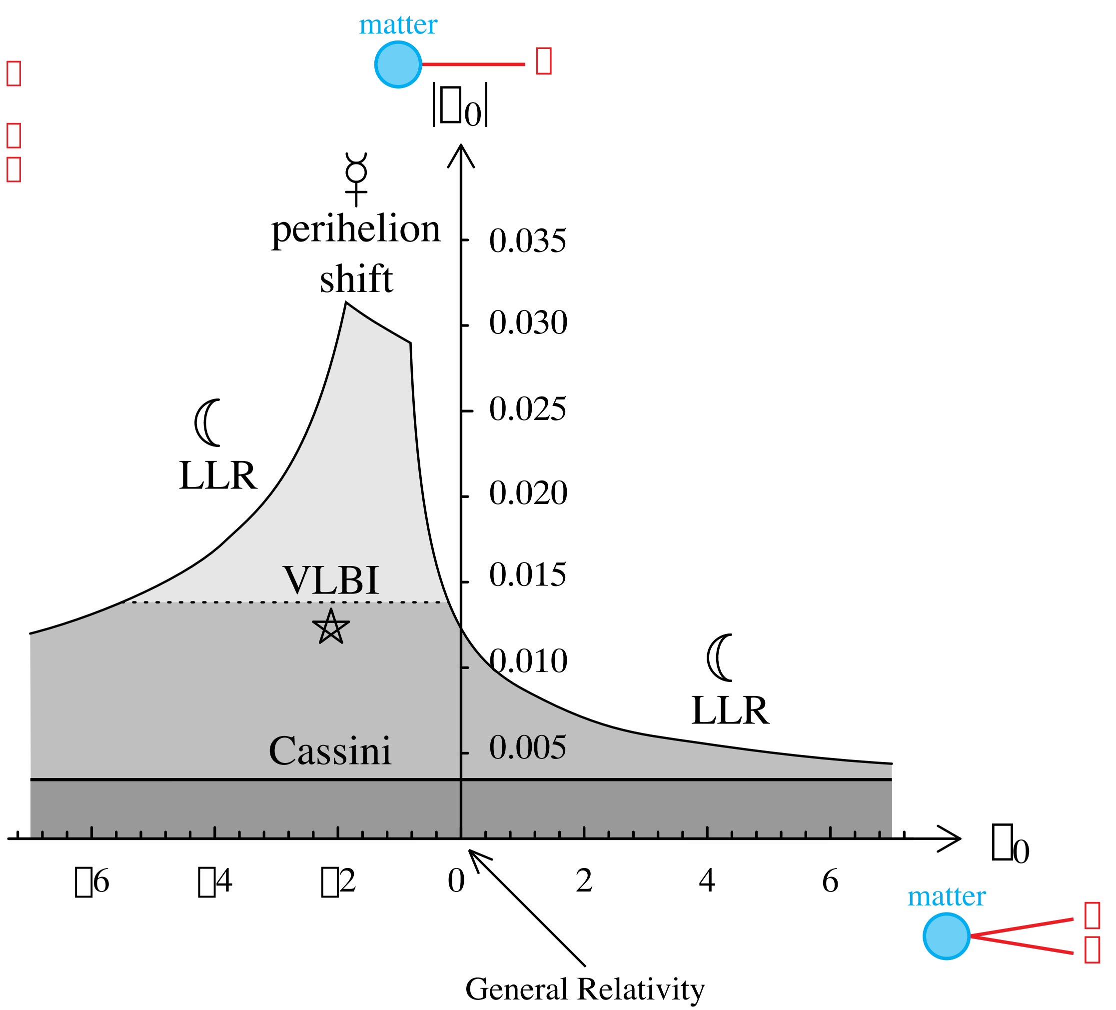
$$G_{eff} = G\,(1 + \alpha_0^{\,2})$$



$$\gamma^{PPN} - 1 \propto \alpha_0^{\,2}$$

$$\beta^{PPN} - 1 \propto \alpha_0^{\,2}\,\beta_0$$



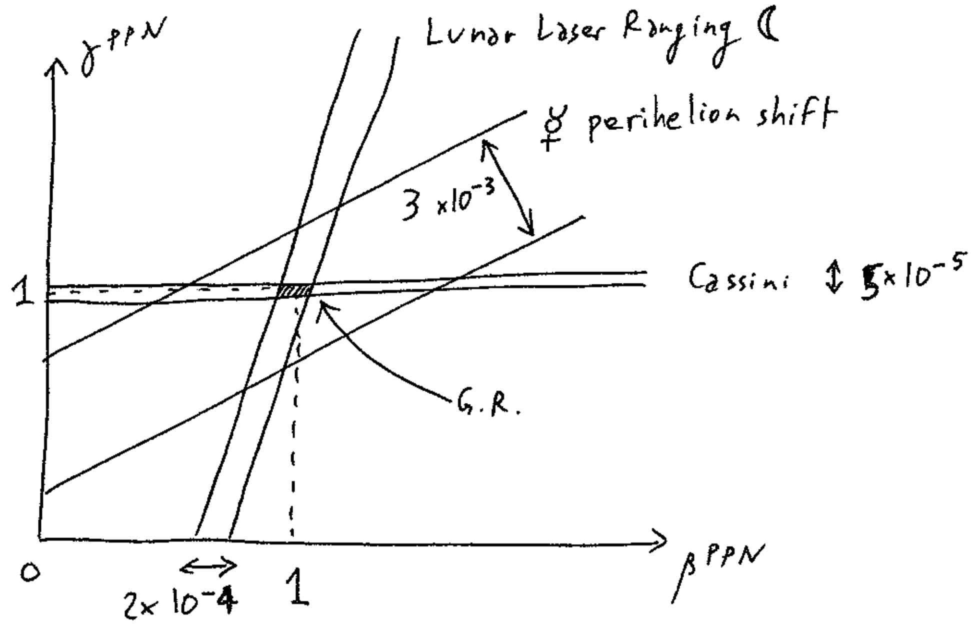


Vertical axis ($\beta_0 = 0$) : Jordan–Fierz–Brans–Dicke theory $\alpha_0^2 = \frac{1}{2\,\omega_{BD} + 3}$

Horizontal axis ($\alpha_0 = 0$) : perturbatively equivalent to G.R.

[N.B.: Beware that $\tilde{G}_{eff}$ is set to 1 in Will's book, so that Φ_0 is replaced by $\frac{2\omega+4}{2\omega+3}$ and β^{PPN} becomes $\beta^{PPN} = \frac{d\omega/d\Phi}{(2\omega+3)^2(2\omega+4)}$!]

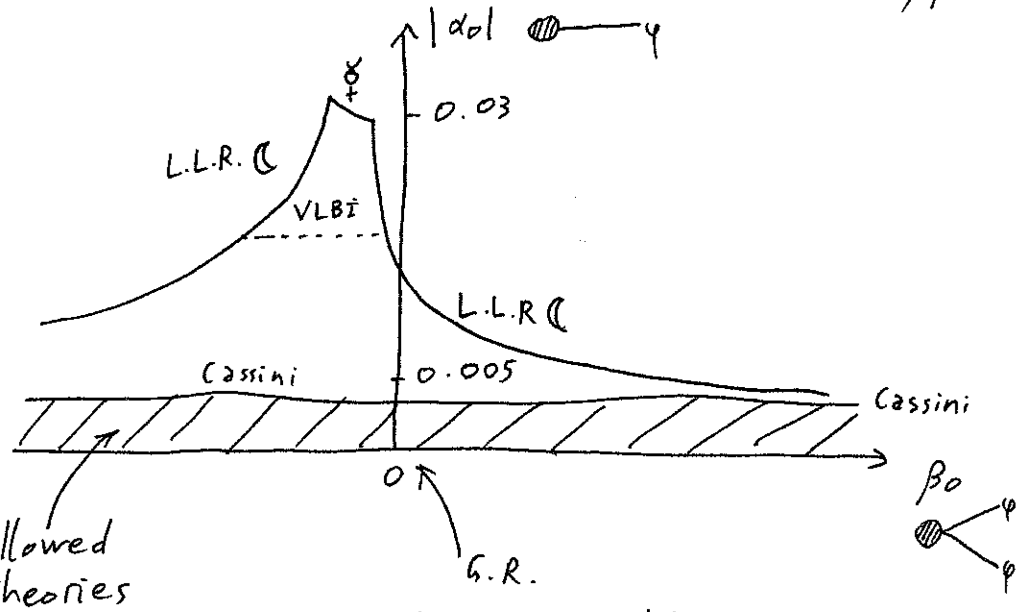
- Solar-system constraints



Best constraint (Cassini) : $|\gamma-1| < 2\times10^{-5} \Rightarrow$ $\boxed{|\alpha_0| < 3\times10^{-3}}$

The scalar field must be weakly linearly coupled to matter

On the other hand, other constraints on $|\alpha_0^2\beta_0|$ do not tell us much. Plot of the constraints in the (α_0, β_0) plane:



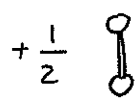
[vertical axis ($\beta_0 = 0$) : "Brans-Dicke" theory with $\alpha_0^2 = \frac{1}{2\omega_{BD}+3}$]

- ∃ many other ways to derive the 1PN limit of scalar-tensor theories. For instance, one may follow Will's lectures in this framework and compute which potentials enter the metric when only a tensor (spin-2 graviton) and a scalar (spin-0) field mediate gravity.
- A powerful method has been illustrated in Damour's lectures: the use of classical Feynman-like diagrams, computed in x space (instead of momentum space as in QFT).

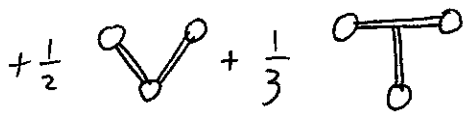
[T. Damour & G.E.F, Phys. Rev. D 53 (1996) 5541]

By replacing in the action the fields ($g^*_{\mu\nu}$ and φ) by their expressions in terms of the matter sources, one gets a so-called "Fokker" action describing the dynamics of the bodies in terms of their positions, velocities, accelerations (... at higher PN orders):

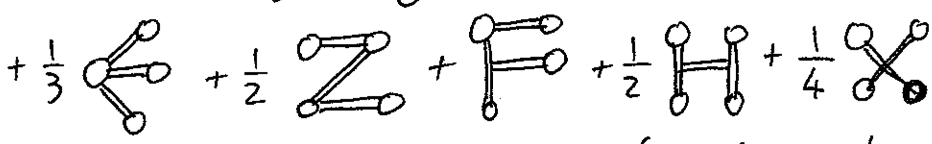
$$S_{N\,bodies} = -\sum_A m_A c^2 \sqrt{1-\vec{v}_A^{\,2}/c^2}$$ (free point particles)



(Newtonian interaction)



(1PN corrections)



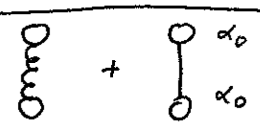
(2PN corrections)

$+ \cdots$

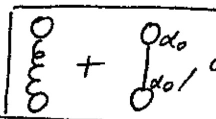
where the numerical factors are related to the symmetries of the various diagrams, and

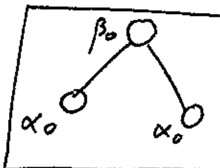
stands for either graviton or scalar field

* For instance, the Newtonian interaction reads

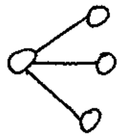
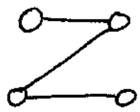


cf. $G_{eff} = G_*(1 + \alpha_0^2)$

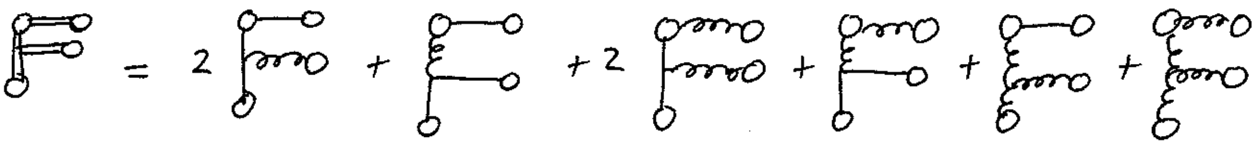
* γ^{PPN} is related to a $\frac{\vec{v}^2}{c^2}$ correction to the Newton $\frac{1}{r^2}$ force and thus to relativistic effects in  , cf. $\gamma^{PPN}-1 \propto \alpha_0^2$

* $\beta^{PPN}-1$ is related to  , cf. $\beta^{PPN}-1 \propto \alpha_0^2 \beta_0$

* Even without any calculation, such diagrams can be useful to identify new possible deviations from G.R. at higher orders.
For instance, the 2PN diagrams

 and  are related to new

parameters ε and ζ which take 0 value in G.R. On the other hand, the other 2PN diagrams yield 2PN effects $\propto (\gamma^{PPN}-1)$ or $(\beta^{PPN}-1)$ already tightly constrained by solar-system tests.
Indeed $\nexists$ vertex with 3 scalar fields $\Rightarrow$



$\propto \alpha_0^2\beta_0$ $\quad \propto \alpha_0^4$ $\quad \propto \alpha_0^2$ $\quad \propto \alpha_0^2$ $\quad \propto \alpha_0^2$ $\quad$ G.R.

$\Rightarrow$ no new parameter entering this F diagram.

* Simple proof that any effect on light is $\propto (\gamma^{PPN}+1)$ at the 1PN order:

$$S_{EM} = -\frac{1}{4}\int d^4x \sqrt{-\tilde{g}}\, \tilde{g}^{\mu\rho}\tilde{g}^{\nu\sigma} F_{\mu\nu}F_{\rho\sigma}$$
$$= -\frac{1}{4}\int d^4x \sqrt{-g^*}\, g_*^{\mu\rho} g_*^{\nu\sigma} F_{\mu\nu}F_{\rho\sigma} \quad \text{(conformal invariance)}$$

$\Rightarrow$ there is no vertex connecting photons ~~ to a scalar line

 $= 0$

Therefore, any effect involving light will be proportional to $\frac{G_* M}{r c^2}$ as in G.R.

However, we do not measure $G_* M$ with Cavendish-type experiments or by analyzing the planets' orbits, but $G_{eff} M = G_* (1+\alpha_0^2) M \Rightarrow$ effects on light $\propto \frac{G_{eff} M}{(1+\alpha_0^2) r c^2}$

$$= \frac{G_{eff} M}{r c^2} \left(\frac{1+\gamma^{PPN}}{2} \right)$$

From the expression of γ^{PPN} derived p. (25)

* For instance, light deflection $\Delta\theta = \frac{4 G_* M}{b c^2}$ like in G.R., but in terms of the measured $G_{eff} M$, one gets

$$\begin{cases} \Delta\theta = \frac{2(1+\gamma^{PPN}) G_{eff} M}{b c^2} \\ \quad = \frac{4 G_{eff} M}{(1+\alpha_0^2) b c^2} \end{cases}$$

(smaller) than G.R.'s prediction

(Scalar-tensor theories predict thus a light deflection whose value lies <u>between</u> the extremal cases of

Nordström's theory $\Delta\theta = 0$

and General Relativity $\Delta\theta = \frac{4 G_{eff} M}{b c^2}$)

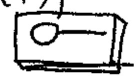
- **Example of a diagrammatic calculation:**

Action of a point particle $S_{pp} = -\int m(\varphi)\, c\, ds^*$

where $m(\varphi) \to A(\varphi)\, \tilde{m}_{const}$ and $ds^* \to \sqrt{-g^*_{\mu\nu} dz^\mu dz^\nu}$

* Expanding to first order in $(\varphi-\varphi_0)$ and $(g^*_{\mu\nu} - \eta_{\mu\nu})$, one finds the source terms

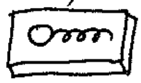
- for the scalar field: $\sigma(x) = -\sum_{part. A} \int d\tau_A\, m_A^\circ c^2 \alpha_A\, \delta^{(4)}(x - z_A(\tau))$ (with α_A ↑ (α_0))



- for the graviton:

$$\sigma^{\alpha\beta}(x) = \frac{1}{2} \sum_A \int d\tau_A\, m_A^\circ c^2 u_A^\alpha u_A^\beta\, \delta^{(4)}(x - z_A(\tau))$$

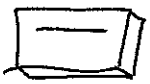
(u_A: unit 4-velocity of particle A.)



* At quadratic order, $S_{grav.}$ defines the propagators 〰 and —:

$$S_\varphi = -\frac{c^3}{4\pi G_*} \int d^4x\, \frac{1}{2} \eta^{\mu\nu} \partial_\mu\varphi\, \partial_\nu\varphi + O(h\varphi^2, \varphi^4)$$

$$= \int d^4x \left(-\frac{1}{2}\varphi\, \mathcal{P}_\varphi^{-1} \varphi\right)$$



where $\boxed{\mathcal{P}_\varphi(x,y) = \frac{G_*}{c^3}\, \mathcal{G}(x-y)}$ such that $\boxed{\Box_{flat}\, \mathcal{G}(x) = -4\pi\delta^{(4)}(x)}$ Green's function

and

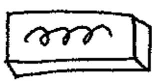
$$S_{spin2} = \frac{c^3}{16\pi G_*} \int d^4x \left[-\frac{1}{2}(\partial_\mu h_{\alpha\beta})\, Q^{\alpha\beta\gamma\delta} (\partial^\mu h_{\gamma\delta}) + \underbrace{\frac{1}{2}\left(\partial_\nu h^\nu_\mu - \frac{1}{2}\partial_\mu h^\nu_\nu\right)^2}_{\text{gauge-fixing term}}\right] + O(h^3)$$

(otherwise the propagator is non-invertible)

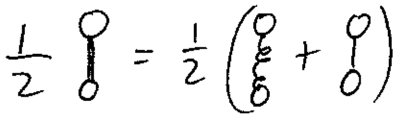
where $Q^{\alpha\beta\gamma\delta} = \frac{1}{4}\left(\eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma} - \eta^{\alpha\beta}\eta^{\gamma\delta}\right) \left[= \frac{1}{4} P^{\alpha\beta\gamma\delta}\right]$

Therefore $S_{spin2} = \int d^4x \left(-\frac{1}{2} h_{\alpha\beta}\, \mathcal{P}_h^{-1\,\alpha\beta\gamma\delta}\, h_{\gamma\delta}\right)$

with $\boxed{\mathcal{P}^h_{\alpha\beta\gamma\delta}(x,y) = \frac{4G_*}{c^3} P_{\alpha\beta\gamma\delta}\, \mathcal{G}(x-y)}$

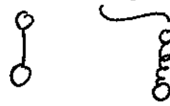


* One can thus compute the  diagram:



$$= \frac{1}{2} \iint dx\, dy \left[\sigma(x) \mathcal{P}_\varphi(x,y) \sigma(y) + \sigma^{\alpha\beta}(x) \mathcal{P}^h_{\alpha\beta\gamma\delta}(x,y)\, \sigma^{\gamma\delta}(y) \right]$$

$$= \frac{1}{2} \sum_{A \neq B} \int d\tau_A \int d\tau_B\, (m_A^\circ c^2)(m_B^\circ c^2) \frac{G_*}{c^3} \mathcal{G}(z_A - z_B) \Big[\alpha_A \alpha_B + 2 (u_A u_B)^2 - 1 \Big]$$



$$= \frac{1}{2} \sum_{A \neq B} \int d\tau_A \int d\tau_B\, \underbrace{G_* (1 + \alpha_A \alpha_B)}_{\equiv G_{AB}}\, m_A^\circ m_B^\circ \left[1 + (1 + \gamma_{AB}^{PPN}) \left((u_A u_B)^2 - 1 \right) \right] c\, \mathcal{G}(z_A-$$

This calculation gives the $O(G)$ interaction to all orders in $\left(\frac{v}{c}\right)^n$!
← "1st post-Minkowskian"

* Now one can perform a post-Newtonian expansion:

$$(u_A u_B)^2 - 1 = \frac{(\vec{v}_A - \vec{v}_B)^2}{c^2} + O\left(\frac{1}{c^4}\right)$$

$$c\, \mathcal{G}_{sym}(z_A - z_B) = \frac{\delta(t_A - t_B - \frac{r_{AB}}{c}) + \delta(t_A - t_B + \frac{r_{AB}}{c})}{2\, r_{AB}}$$

↑ for conservative part of the N-body action

$$= \frac{\delta(t_A - t_B)}{r_{AB}} + \frac{|\vec{z}_A - \vec{z}_B(t_B)|}{2c^2} \frac{\partial^2 \delta(t_A - t_B)}{\partial t_B^2} + O\left(\frac{1}{c^4}\right)$$

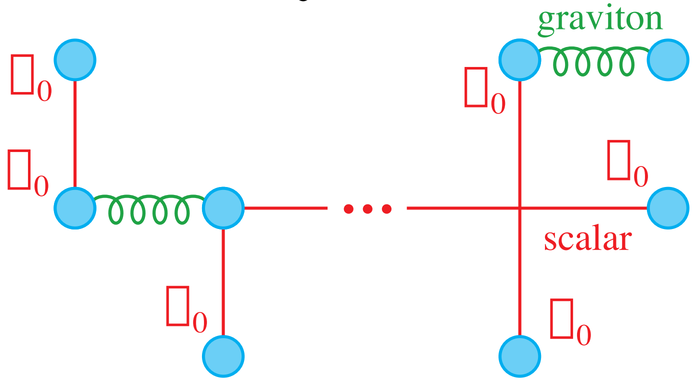
↑ not even necessary for computing the scalar-field effect, since multiplied by $(u_A u_B)^2 - 1 = O(1/c^2)$

⇒ 2-body interaction

$$\boxed{\frac{1}{2} \sum_{A \neq B} \frac{G_{AB}\, m_A m_B}{r_{AB}} \left[1 + \text{all G.R. terms in } \frac{v^2}{c^2} + (\gamma_{AB}^{PPN} - 1) \frac{(\vec{v}_A - \vec{v}_B)^2}{c^2} + O(1/c^4) \right]}$$

Deviations from general relativity due to the scalar field

- At any order in $\frac{1}{c^n}$, the deviations involve at least two $□_0$ factors:



= small deviations!

- But **nonperturbative** strong-field effects may occur:

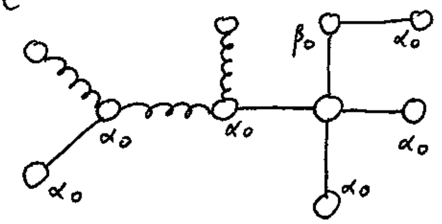
$$\text{deviations} = □_0^2 \; □ \left[a_0 + a_1 \frac{Gm}{Rc^2} + a_2 \left(\frac{Gm}{Rc^2}\right)^2 + \ \dots \right]$$

$< 10^{□5}$

LARGE for $\frac{Gm}{Rc^2}$ □ 0.2 ?

A.7 : Strong-Field predictions

* If one works perturbatively, say at the nth post-Newtonian order $\frac{1}{c^{2n}}$, deviations from G.R. come from tree diagrams like



in which $\exists$ at least one scalar line.

Each end of such a scalar line involves the linear matter-scalar coupling constant α_0. Since $\exists$ at least 2 ends:

$$\text{deviations from G.R.} = \alpha_0^2 \times \left[\lambda_0 + \lambda_1 \frac{GM}{Rc^2} + \lambda_2 \left(\frac{GM}{Rc^2}\right)^2 + \cdots \right]$$

(λ_0, λ_1: combinations of parameters entering $\ln A(\varphi) = \ln A_0 + \alpha_0 \varphi + \frac{1}{2}\beta_0 \varphi^2 + \ldots$)

where solar-system tests impose $\alpha_0^2 < 10^{-5}$.

$\Rightarrow$ All post-Newtonian deviations are already known to be small!

{ In particular, if $\alpha_0 = 0$, then the scalar-tensor model is perturbatively equivalent to G.R.

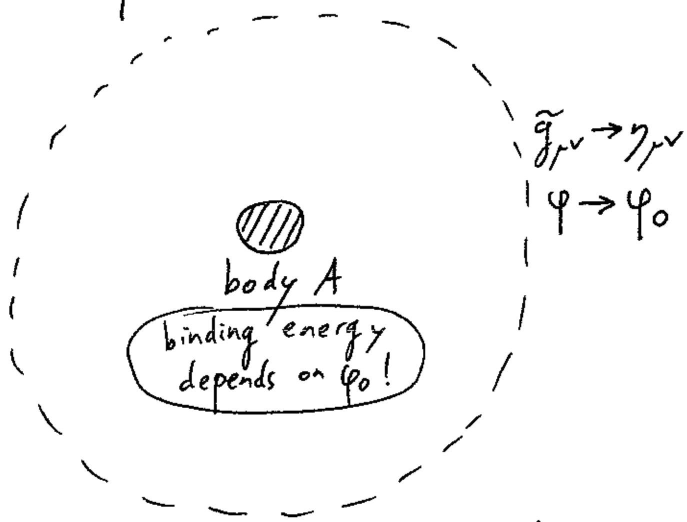
* However, the series $\left[\lambda_0 + \lambda_1 \frac{GM}{Rc^2} + \lambda_2 \left(\frac{GM}{Rc^2}\right)^2 + \cdots \right]$ may become large (or even diverge) if the compactness $\frac{GM}{Rc^2}$ of a body is large enough. Actually, we will see that it can compensate a vanishingly small factor α_0^2:

nonperturbative strong-field effects ⚠

* In scalar-tensor theories, one can (like in G.R.) impose $\tilde{g}_{\mu\nu} \to \eta_{\mu\nu}$ on a sphere surrounding a body (up to tidal effects), but the <u>gravitational physics</u> anyway depends on the boundary value φ_0 of the scalar field, notably via

$$\tilde{G}_{eff} = G_* A^2(\varphi_0) \left[1 + \alpha^2(\varphi_0) \right].$$

$\Rightarrow$ The <u>equilibrium configuration</u> of a massive body depends on φ_0!



* A simple way to take this effect into account is to allow for any <u>function</u> $m_A(\varphi)$, not necessarily $A(\varphi) \tilde{m}_A$ [const.] that we found for laboratory-size objects.

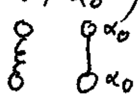
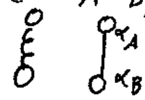
* More generally, finite-size effects (i.e. body $\neq$ point particle) can be taken into account by writing $m_A[\varphi]$ as a <u>functional</u> of φ, $g^*_{\mu\nu}$ and their (multi-) derivatives. This is an expansion in $\varepsilon = \dfrac{\text{size of body}}{\text{interbody distance}}$. If ε is small enough, a <u>function</u> $m_A(\varphi)$ suffices [this is the case in practice].

* As already shown in Sec. A.6, all the predictions remain the same as in weak-field conditions, with the only replacement of the bare coupling constants

$$\begin{cases} \alpha_0 \\ \beta_0 \\ \vdots \end{cases} \quad \text{by body-dependent ones} \quad \begin{cases} \alpha_A = \dfrac{d \ln m_A(\varphi)}{d\varphi} \\ \beta_A = \dfrac{d\alpha_A}{d\varphi} \\ \vdots \end{cases}$$

For instance

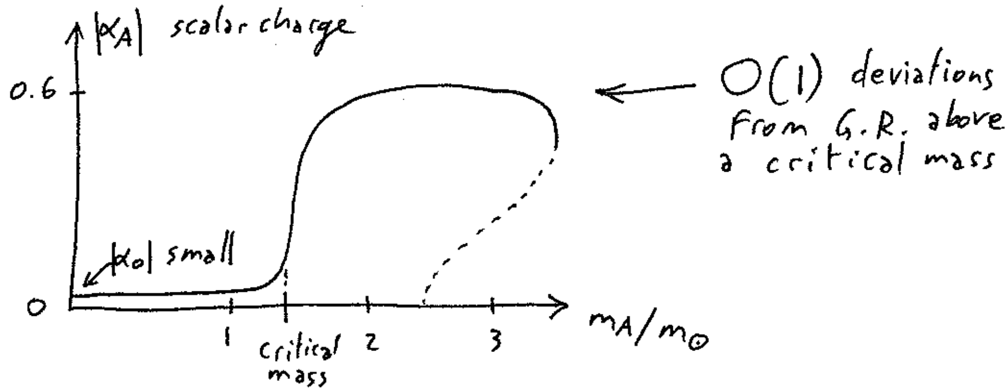
$$G_{eff} = G_*(1+\alpha_0^2) \quad \text{becomes} \quad G_{AB} = G_*(1+\alpha_A \alpha_B)$$

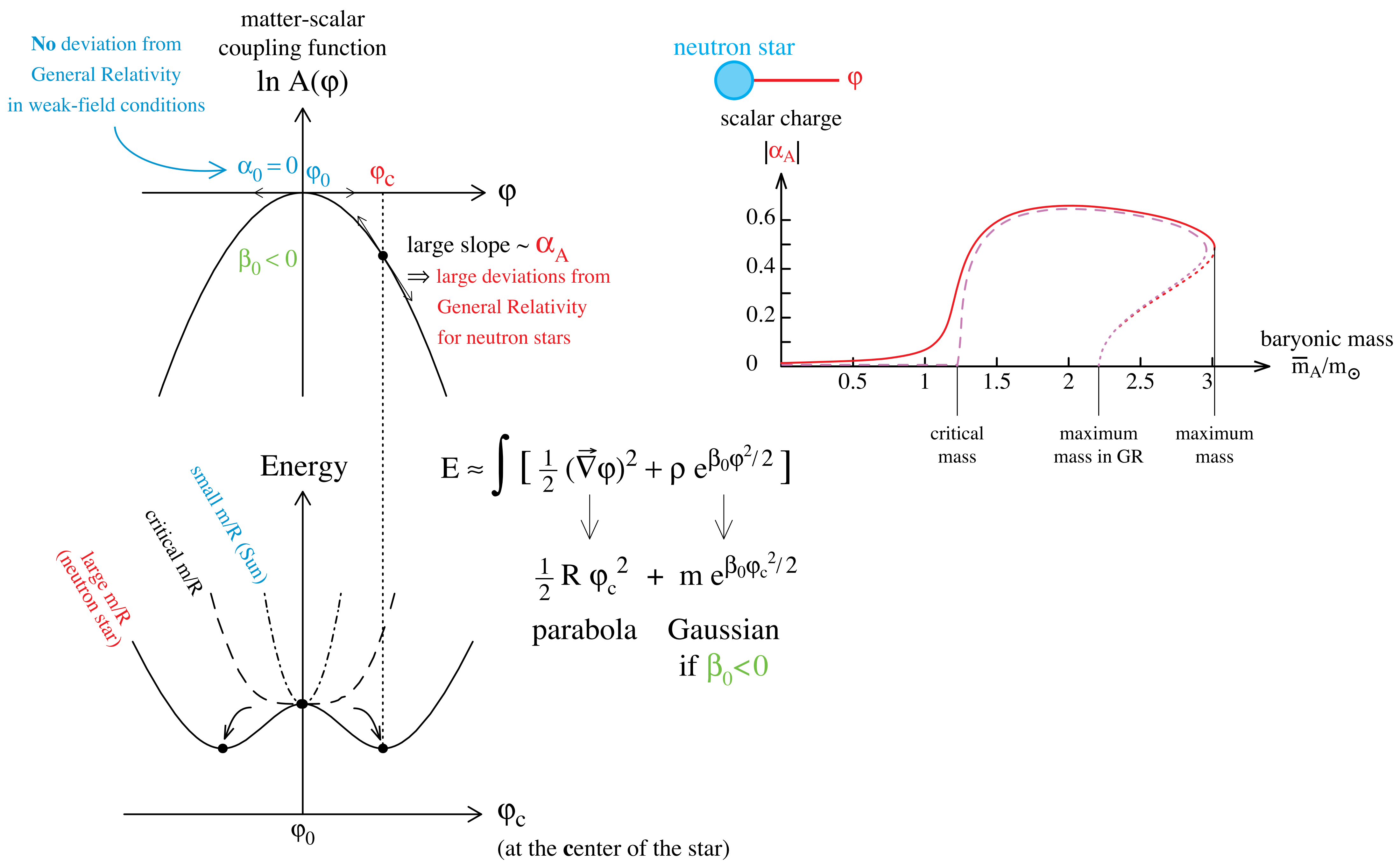



$$\gamma^{PPN} = 1 - 2\frac{\alpha_0^2}{1+\alpha_0^2} \quad \text{becomes} \quad \gamma_{AB}^{PPN} = 1 - 2\frac{\alpha_A\alpha_B}{1+\alpha_A\alpha_B}$$

$$\beta^{PPN} = 1 + \frac{1}{2}\frac{\alpha_0^2\beta_0}{(1+\alpha_0^2)^2} \quad \text{becomes} \quad \beta^A_{BC} = 1 + \frac{1}{2}\frac{\beta_A\alpha_B\alpha_C}{(1+\alpha_A\alpha_B)(1+\alpha_A\alpha_C)}$$

* The only difficulty is that one must now compute the body-dependent quantities $\alpha_A, \beta_A, \ldots$ by <u>numerically solving</u> the field equations for compact bodies, taking into account the couplings between $g^*_{\mu\nu}$, φ, and a realistic equation of state describing matter.



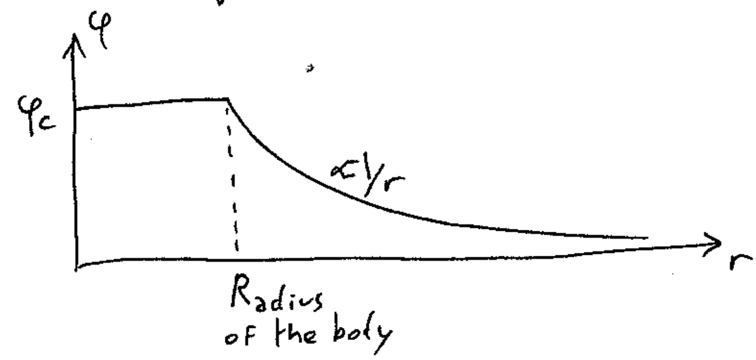


"spontaneous scalarization" [T. Damour & G.E-F 1993]

* Intuitive reason for this nonperturbative phenomenon

Let us assume $\alpha_0 = 0$ strictly, say $A(\varphi) = e^{\frac{1}{2}\beta_0 \varphi^2}$, and show that $\alpha_A \neq 0$ is possible, i.e. that a massive enough body may generate $\varphi - \varphi_0 = \frac{-\alpha_A G_* m_A}{r c^2} + O\left(\frac{1}{c^4}\right)$ outside it.

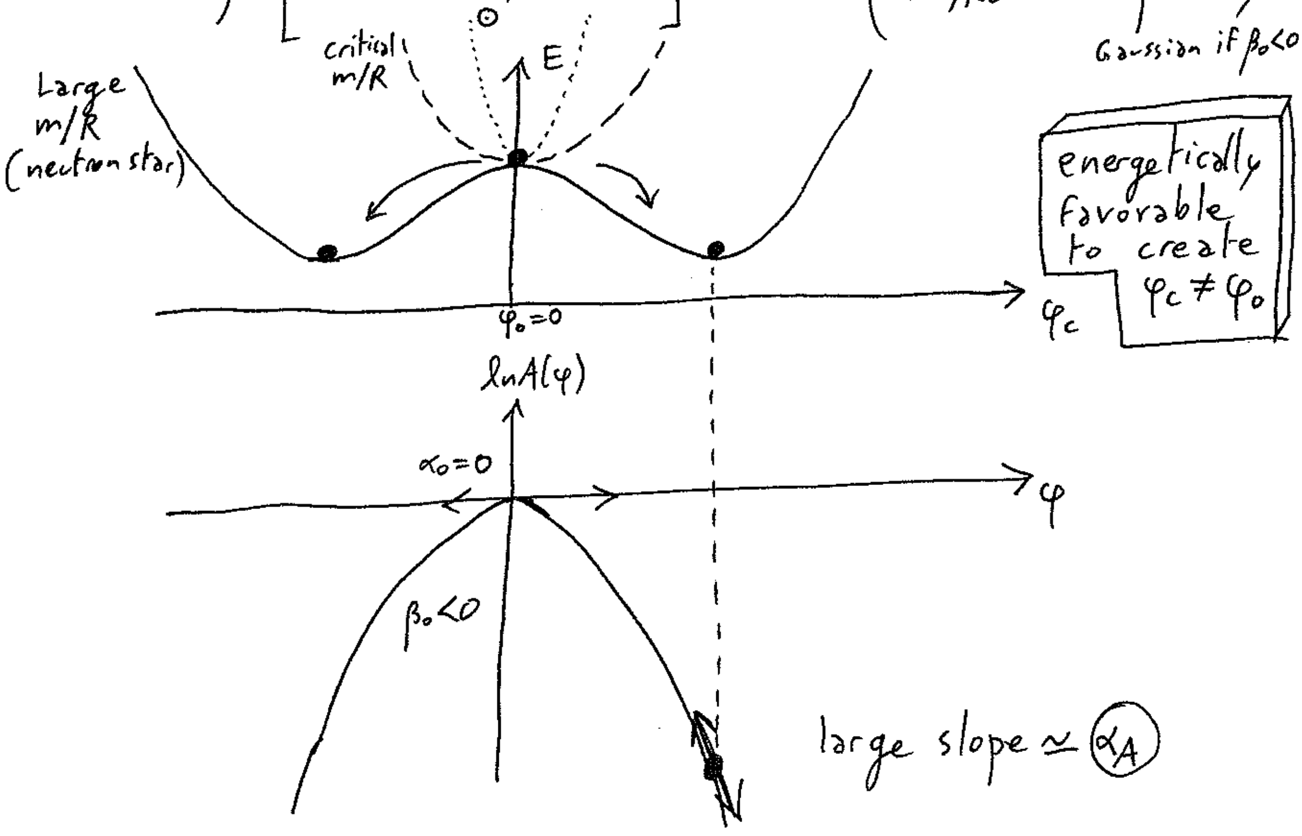
Consider a trial configuration of φ



and compute its energy:

$$E \simeq \int \left[\frac{1}{2}(\partial_i \varphi)^2 + \rho\, e^{\frac{1}{2}\beta_0 \varphi^2}\right] \simeq mc^2 \left(\frac{\varphi_c^2/2}{Gm/Rc^2} + e^{\beta_0 \varphi_c^2/2}\right)$$

↑ Gaussian if $\beta_0 < 0$



A.8. Gravitational waves

Similar calculations as in G.R. show that in a binary system (A,B):

$$\text{Energy Flux} = \left\{ \frac{\text{Quadrupole}}{c^5} + O\left(\frac{1}{c^7}\right)\right\}_{\text{spin 2}} \text{(cf. G.R.'s result)}$$

$$+ \left\{ \underbrace{\frac{\text{Monopole}}{c}\left(\underbrace{\frac{d(m_A\alpha_A)}{dt}}_{=0 \text{ at equilibrium}} + \frac{1}{c^2}\right)^2}_{O(1/c^5)} + \boxed{\frac{\text{Dipole}}{c^3}(\alpha_A-\alpha_B)^2} + \frac{\text{Quadrupole}}{c^5} + O\left(\frac{1}{c^7}\right)\right\}_{\text{spin 0}}$$

↑ Largest contribution if $\alpha_A \neq \alpha_A$, i.e. dissymmetrical binary system

This energy loss causes the orbit to shrink and go <u>faster</u>.

$$\langle \dot{P} \rangle^{\text{quadrupole}}_{g*} = \frac{-192\pi}{5(1+\alpha_A\alpha_B)} \nu \left(\frac{G_{AB}Mn}{c^3}\right)^{5/3} \frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}} + O\left(\frac{1}{c^7}\right)$$

where $\begin{cases} M = m_A + m_B \\ \nu = \dfrac{m_A m_B}{M^2} \\ n = \dfrac{2\pi}{P} \end{cases}$

This is the G.R. result up to corrections entering G_{AB} and the factor $\dfrac{1}{1+\alpha_A\alpha_B}$.

<u>New</u> terms caused by emission of <u>scalar waves</u>:

$$\begin{cases} \langle \dot{P} \rangle^{\text{monopole}}_{\varphi} \propto -\dfrac{e^2}{c^5} O(\alpha_A,\alpha_B)^2 \\ \langle \dot{P} \rangle^{\text{dipole}}_{\varphi} = \dfrac{-2\pi}{1+\alpha_A\alpha_B} \nu \left(\dfrac{G_{AB}Mn}{c^3}\right) \dfrac{1+\frac{e^2}{2}}{(1-e^2)^{5/2}} (\alpha_A-\alpha_B)^2 + O\left(\dfrac{1}{c^5}\right) \\ \langle \dot{P} \rangle^{\text{quadrupole}}_{\varphi} \propto -\dfrac{O(\alpha_A,\alpha_B)^2}{c^5} \dfrac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}} \end{cases}$$

(↑ known, referring to $O(1/c^5)$; c^3 circled)