C) Modified Newtonian dynamics

October 12th, 2006

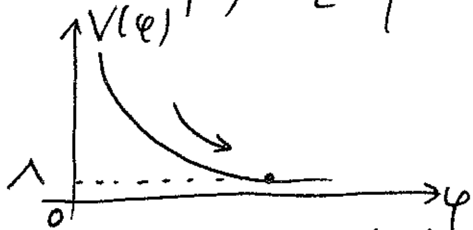
C.1: Dark matter

* Cosmology tells us that the energy content of the Universe is composed of about

$\begin{pmatrix} \Omega_\Lambda \simeq 0.7 \\ \Omega_m \simeq 0.3 \end{pmatrix}$

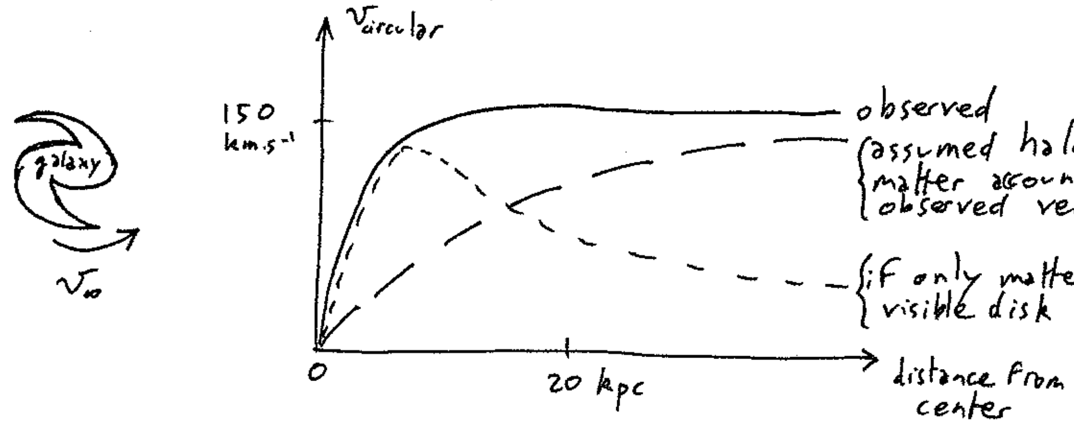
- 72% dark energy : fluid with pressure $\simeq -\varepsilon_{nergy}$
  $\Rightarrow$ accelerated expansion of universe
- \+ 24% dark matter : noninteracting fluid with $p \simeq 0$
  $\Rightarrow$ flat rotation curves of galaxies & clusters, notably (*)
- \+ 4% baryonic matter (only part of it emitting light!)

* Dark energy may be understood as a cosmological constant $\Lambda \simeq 3 \times 10^{-122} \frac{c^3}{\hbar G}$, or as the present value of a scalar field potential $V(\varphi_0)$ ["quintessence"].



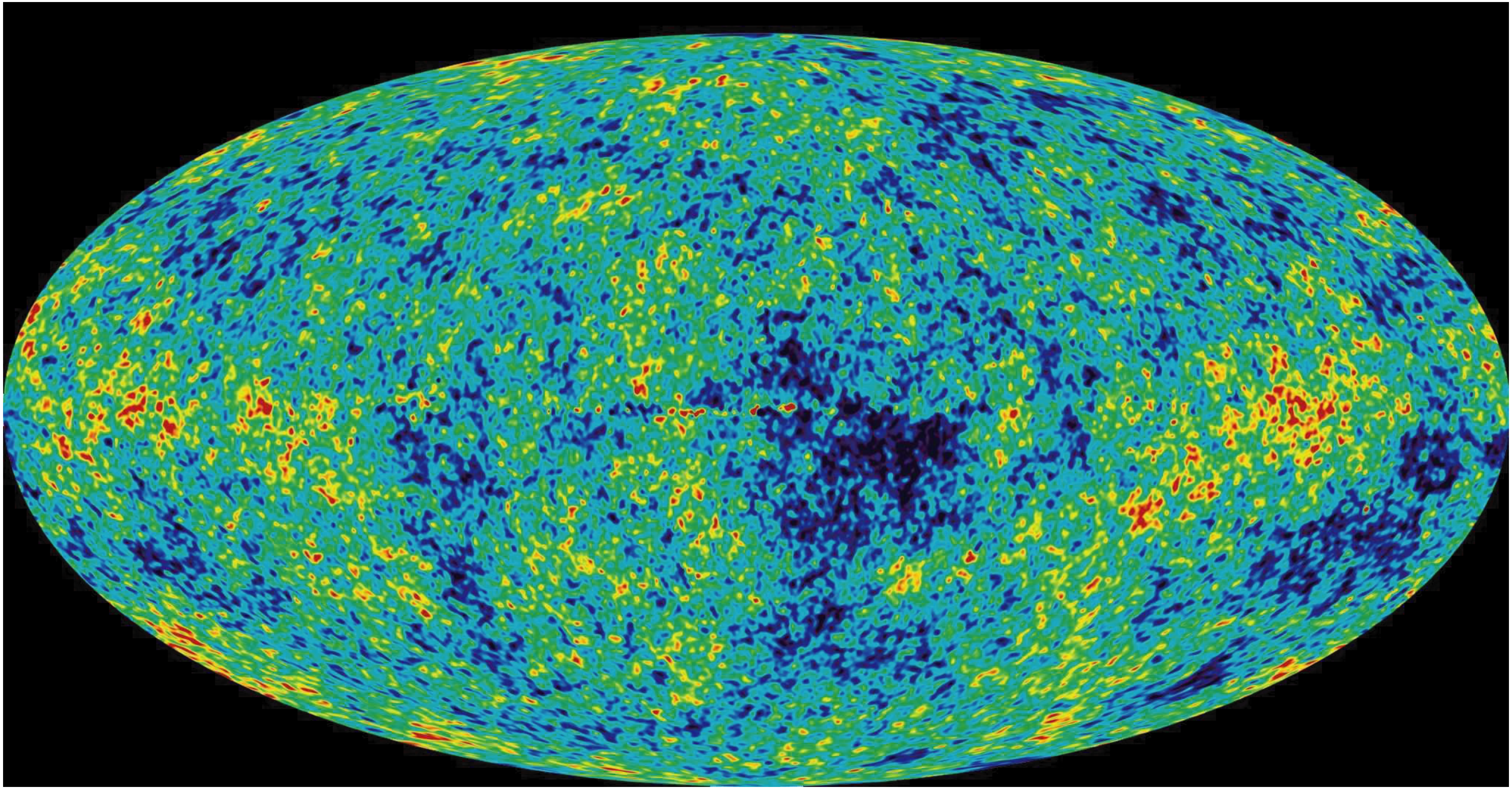
* Dark matter : ∃ many candidates (cf. "neutralinos" is super-symmetric theories), and numerical simulations of structure formation are quite successful when incorporating such a dark matter.

(*) Flat rotation curves of galaxies & clusters:



Combination of various cosmological observations:

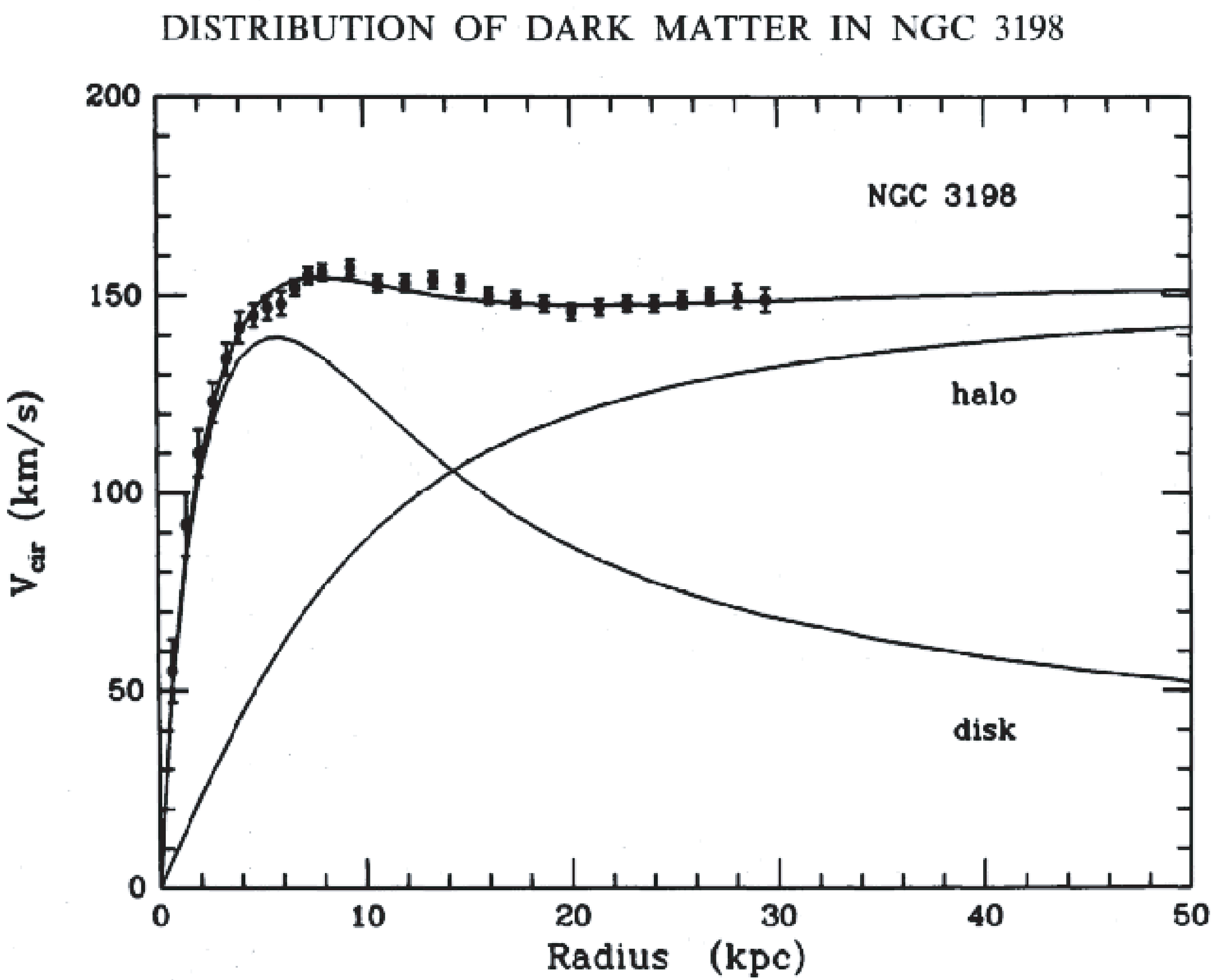
- 72% of "dark energy"
- 24% of "dark matter"
- 4% of baryonic matter



snapshot of Universe at 380 000 years (now 13.7 billion years)

■ Dark matter = pressureless and noninteracting component of matter

- Imposed notably by rotation curves of galaxies and clusters:



  really some dark matter (many theoretical candidates notably from SUperSYmmetry), or modification of Newton's law at large distances?

C.2 : Milgrom's MOND phenomenology

* Instead of assuming such a dark matter, one may think that the law of gravity might be changed at large distances.
* In 1983, M. Milgrom showed that observed rotation curves can be very well fitted by a <u>Modified Newtonian Dynamics</u> involving a characteristic **acceleration scale** $a_0 \simeq 1.2 \times 10^{-10}\ m.s^{-2}$ (but no mass nor distance scale !).
* Basic idea:

$$\begin{cases} a = a_{Newton} = \dfrac{GM}{r^2} & \text{if } a > a_0 \\ a = \sqrt{a_0 a_N} = \dfrac{\sqrt{GMa_0}}{r} & \text{if } a < a_0 \end{cases}$$

* Automatically recovers the Tully-Fisher law [1977]:

$$v_\infty^4 \propto M_{baryonic}$$

(and not v_∞^2 as in Newtonian theory without dark matter!)

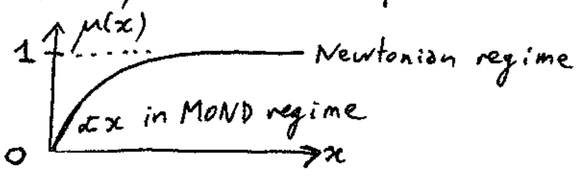
* This simple law superbly accounts for galaxy rotation curves even today [Sanders & McGaugh, Ann. Rev. Astron. Astrophys., **40** (2002) 263]

[but <u>galaxy clusters</u> still require some amount of dark matter!]

* Classical field-theoretical interpretation:

$$\vec{\nabla}\left(\mu(|\vec{\nabla} U|)\, \vec{\nabla} U\right) = -4\pi G \rho \qquad \text{instead of standard } \Delta U = -4\pi G\rho$$

with



C.3.: Various theoretical attempts

* The problem is to devise a <u>consistent relativistic field theory reproducing the MOND phenomenology</u> without being – mathematically ill defined
  – unstable
  or – inconsistent with some experiments.

* Some papers write field equations which do <u>not</u> derive from an action (⇒ inconsistent field theory).

[For instance, assuming that a massive scalar field is generated by the dark matter density φ —● (Dark) only, without any coupling to baryonic matter φ —○ (baryon) $= 0$, but simultaneously assuming that ordinary baryonic matter does feel this potential ⇒ φ —○ (baryon) $\neq 0$. <u>Cannot</u> be a field theory, although the phenomenology may anyway be interesting.]

* Many papers write models involving ghost degrees of freedom ⇒ unstable field theories.

* Several do write consistent actions, but they involve the mass $M_{baryonic}$ of the considered galaxy as a parameter: maybe interesting <u>fits</u> of data, but not predictive field theories, since ≠ action for different galaxies! One should be able to <u>prove</u> the relation $M_{DM} \propto \sqrt{M_{bary}}$ and not assume it.

[Note that obtaining a $\ln r$ Newtonian force is rather easy with a potential unbounded by below: $V(\varphi) = -2a^2 e^{-b\varphi}$
$\Rightarrow \varphi = \frac{2}{b}\ln(abr)$ is a solution of $\Delta\varphi = V'(\varphi)$.
However, $\frac{2}{b}$ is here independent of the source instead of $\propto \sqrt{M_{bar}}$]

* One of the most consistent models proposed in the literature (no ghost, well-posed Cauchy problem, etc.) derives an extra force $\propto k \frac{M^2}{r}$ instead of $\frac{\sqrt{M}}{r}$. The author then <u>assumes</u> that $k \propto M^{-3/2}$ to obtain the right MOND behavior, hoping for an internal mechanism to impose it. Maybe, but should be proven to be a predictive theory. In a more recent version, he manages to prove $k \propto M^{-3/2}$ by introducing a specific potential in the model. <u>But</u> this potential depends on $M \Rightarrow$ we are back to the models fitted for each galaxy, and this is thus not yet a consistent field theory.

* Attempts using higher-order gravity: we saw in § A.2 that they involve a ghost (spin-2) degree of freedom if they are not mere scalar-tensor theories.
- Actually, some authors ingeniously devised models $f(R, R_{\mu\nu}, R_{\mu\nu\rho\sigma})$ such that the <u>quadratic</u> term reduces to the Gauss-Bonnet topological invariant $G.B. = R_{\mu\nu\rho\sigma}^2 - 4R_{\mu\nu}^2 + R^2 \Rightarrow$ no excited ghost degree of freedom around a flat background. However, such models generically do not admit Minkowski as a stable solution $\Rightarrow$ terms in $(R^2_{\mu\nu\rho\sigma})^2$ <u>will</u> generate the infamous spin-2 ghost when expanded around a curved background.
- An even subtler idea is to consider $R + f(G.B.)$, so that the quadratic term remains $\propto G.B.$ even around a curved background solution $\Rightarrow$ ghost d.o.f never propagates.

- However, ∃ theorem by Ostrogradski [1850] showing that theories involving a <u>finite</u> number of derivatives [higher than 1] of the fields are unstable [Woodard 2006].

* <u>Illustration on the simplest case</u>: $\mathcal{L}(q, \dot{q}, \ddot{q})$, where $\ddot{q}$ is assumed not to be eliminable by partial integration.

[If $\ddot{q}$ appears only linearly in the Lagrangian $\mathcal{L}$, then it can be eliminated, cf. $\int dt\, q^n \dot{q}^m \ddot{q} = \frac{-n}{m+1} \int dt\, q^{n-1} \dot{q}^{m+2}$.
This is actually why the $\partial^2 g_{\mu\nu}$ involved in the Einstein-Hilbert action do not pose any problem: they may be eliminated.]

Then the equation $\boxed{p_2 \equiv \frac{\partial \mathcal{L}}{\partial \ddot{q}}}$ may be inverted as $\boxed{\ddot{q} = F(q, \dot{q}, p_2)}$

Ostrogradski proves that the initial data must be specified by 2 pairs of conjugate momenta:

$$\boxed{\begin{aligned} q_1 &\equiv q & p_1 &\equiv \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \ddot{q}}\right) \\ q_2 &\equiv \dot{q} & p_2 &\equiv \frac{\partial \mathcal{L}}{\partial \ddot{q}} \end{aligned}}$$

and that the following Hamiltonian

$$\mathcal{H} = p_1 \dot{q}_1 + p_2 \dot{q}_2 - \mathcal{L}(q, \dot{q}, \ddot{q})$$

does generate time translations, i.e.

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \quad \& \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$$

reproduce the original Euler-Lagrange equations deriving from $\mathcal{L}(q, \dot{q}, \ddot{q})$.

However, when using $\ddot{q} = F(q, \dot{q}, p_2)$, this Hamiltonian reads in terms of the conjugate momenta p_i, q_i:

$$\boxed{\mathcal{H} = p_1 q_2 + p_2 F(q_1, q_2, p_2) - \mathcal{L}(q_1, q_2, F(q_1, q_2, p_2))}$$

Linear in p_1 $\Rightarrow$ not bounded by below $\Rightarrow$ unstable.

* Even worse when considering higher derivatives (linear in several p_i's !), but not true if ∞ number of higher derivatives (nonlocal theories; cf. Woodard & Soussa).

* Therefore, even the theories $R + f(G.B.)$ are actually unstable, although a "ghost degree of freedom" cannot be identified at linear order.

---

* Totally different route for describing MOND dynamics: "modified inertia" [Milgrom 1994, 1999]: do not change the dynamics of gravity itself (i.e., $S_{grav.} = \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}\, R$ as in G.R., for instance), but assume that the matter dynamics depends on higher derivatives.

For instance $S_{\text{point particle}}(\vec{x}, \vec{v}, \vec{a}, \dot{\vec{a}}, \ldots)$ instead of G.R.'s $S_{pp} = -\int mc\sqrt{-g_{\mu\nu}(x)v^\mu v^\nu}\, dt$.

But if one wishes right Newtonian limit and satisfy Galileo invariance (or Lorentz invariance, in relativistic version)
$\Rightarrow$ must depend on infinite series of derivatives.

(+) Avoids the Ostrogradskian instability recalled above.

(−) Nonlocal $\Rightarrow$ difficult to analyze (long papers by Milgrom).

C.4: Aquadratic (k-essence) models [Bekenstein & Sanders]

* In the following, we focus on models in which the matter action takes its standard metric form $S_{matter}[\psi; \tilde{g}_{\mu\nu}]$, but review the various actions which have been derived for $\tilde{g}_{\mu\nu} = f(g^*_{\mu\nu} + \text{other fields})$ in order to reproduce the MOND phenomenology.

  ($g^*_{\mu\nu}$: spin 2; other fields: spin 0 notably)

* Aquadratic scalar-tensor theories are defined by a generalization of the scalar-tensor action (defined in § A.4, p. (11)).

$$S = \frac{c^4}{4\pi G_*} \int \frac{d^4x}{c} \sqrt{-g_*} \left\{ \frac{R^*}{4} - \frac{1}{2} f(s, \varphi) - V(\varphi) \right\} + S_{matter}[\psi; \tilde{g}_{\mu\nu} = A^2(\varphi)\, g^*_{\mu\nu}]$$

where $s \equiv g_*^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$ is the standard kinetic term.

[Note that $V(\varphi)$ may be absorbed in the definition of $f(s,\varphi)$, but we will mostly study $f(s)$ alone in the following $\Rightarrow$ better to separate a possible potential.]

* Before showing that $\exists\, f(s)$ such that the MOND potential $\sqrt{GMa_0} \ln r$ can be generated, let us discuss the conditions such a general $f(s, \varphi)$ <u>must</u> satisfy.

Let us denote as $' \equiv \frac{\partial}{\partial s}$

One must have, $\forall s \in [-\infty, +\infty]$:

$$\text{(a)} \quad f'(s,\varphi) \geq 0$$

$$\text{(b)} \quad 2s f''(s,\varphi) + f'(s,\varphi) > 0$$

- (a) is <u>necessary</u> for the Hamiltonian to be bounded by below:

$$H = 2(\partial_0\varphi)^2 f'(s) + f(s) \qquad \text{[in a locally inertial frame]}$$

If $\exists\, s_-$ such that $f'(s_-) < 0$, then $H \to -\infty$ for large enough $(\partial_0\varphi)^2$ and $(\partial_i\varphi)^2$ such that $-(\partial_0\varphi)^2 + (\partial_i\varphi)^2 = s_-$.

- (b) is necessary and sufficient [+ $f' \neq 0$] for the scalar-field equation to remain always hyperbolic. Indeed, it reads

$$G^{\mu\nu} D^*_\mu D^*_\nu \varphi = \frac{1}{2}\frac{\partial f}{\partial\varphi} - s\frac{\partial f'}{\partial\varphi} + \frac{\partial V}{\partial\varphi} - \frac{4\pi G}{c^3\sqrt{-g_*}}\frac{\delta S_{matter}}{\delta\varphi}$$

where $G^{\mu\nu} = f' g_*^{\mu\nu} + 2f''\partial^\mu\varphi\,\partial^\nu\varphi$ plays the role of an effective metric in which φ propagates. Easy to check that the lowest eigenvalue of $G^{\mu\nu}$ is strictly negative if (b) and $f' \neq 0$ are satisfied, and that the other 3 eigenvalues are then positive.

[<u>Quick check</u>: consider $\partial_i\varphi = 0$ in $g^*_{\mu\nu} = \eta_{\mu\nu}$. Then $G^{00} = -f' + 2f'' \times (-s) < 0$ only if (b) is satisfied.]

- Finally, (a) and (b) <u>suffice</u> for the Hamiltonian to be bounded by below for any φ, provided $f(s,\varphi)$ is analytic and $f(s=0,\varphi)$ is itself bounded by below (kind of potential for φ).

- Indeed, $s \geq g_*^{\mu 0}\partial_\mu\varphi\,\partial_0\varphi$ since $g_*^{\mu\nu}$ is of signature $-+++$.
  Using (a) $\Rightarrow H = -2g_*^{\mu 0}\partial_\mu\varphi\,\partial_0\varphi\, f' + f \geq -2sf' + f$, decreasing function of s because of (b). Therefore, $\forall s \leq 0$, $H \geq f(s=0,\varphi)$.
- If $s \geq 0$, we know $H \geq f \geq f(s=0,\varphi)$ because $-g_*^{\mu 0}\partial_\mu\varphi\,\partial_0\varphi \geq 0$ and because (a) $\Rightarrow$ f increases with s.

- Surprizingly (b) is <u>not implied</u> by the boundedness by below of the Hamiltonian, although we used it to prove this boundedness, and although it is linked with the stability of small perturbations: Indeed their speed c_s is such that $c_s^2 = \frac{f'}{2sf''+f'}$ [Damour & Mukhanov, notably] ⇒ (a) & $c_s^2 > 0$ for stability implies (b). But the energy of perturbations is only <u>part</u> of the total Hamiltonian, and H bounded by below does not suffice to guarantee their stability. The hyperbolicity condition (b) is <u>also</u> crucial.

- In [Aharonov, Komar, Susskind 1969], a third condition was imposed on $f(s)$:

  (c) $f''(s) \leq 0$

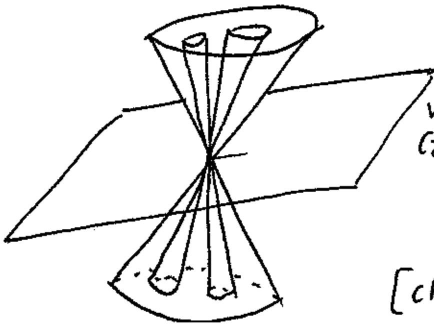
  in order for the causal cone of the scalar field to remain inside the graviton causal cone defined by $g^*_{\mu\nu}$.

  [Indeed, if k^μ is null with respect to $G_{\mu\nu} = \frac{1}{f'}\left[g^*_{\mu\nu} - 2f''\partial_\mu\varphi\partial_\nu\varphi/(2sf''+f')\right]$, inverse of $G^{\mu\nu}$ of p. (65), then $G_{\mu\nu}k^\mu k^\nu = 0$ ⇒
  $g^*_{\mu\nu}k^\mu k^\nu = 2f''(k^\mu\partial_\mu\varphi)^2/(2sf''+f') \leq 0$ if (b) & (c) are satisfied.]

  They conclude that monomials $f(s) = s^n$ are excluded since it is impossible to satisfy (a) & (c) simultaneously for any s.

- However, this condition (c) is actually <u>not</u> required for the consistency of the model. If (b) is satisfied, then the causal cone of the scalar field never fully opens (i.e., it remains a <u>cone</u>), and no closed Timelike Curve can exist.

- More generally, if ∃ several causal cones for ≠ fields, but that their <u>union</u> remains embedded in a wider cone, no causality problem, contrary to some claims in the literature.



well defined Cauchy surface ⇒ <u>Local field equations are</u> well defined

[cf. J-Ph. Bruneton 2006]

- If one chooses to specify initial data on a surface lying <u>between</u> two cones, i.e. spacelike with respect to one of them but <u>timelike</u> with respect to the second one, one obviously finds causal paradoxes: this surface is <u>not</u> a good Cauchy surface for the second cone. Most of the problems discussed in the literature rely on the use of such improper Cauchy surfaces.
- N.B.: Physicists working on "k-essence" (Mukhanov et al.) have well understood from the beginning that condition (c) has no reason to be imposed.

---

* MOND as an aquadratic model

Let us assume that $\begin{cases} A(\varphi) = e^{\alpha\varphi} \\ V(\varphi) = 0 \end{cases}$ as in Brans-Dicke theory $\left(\alpha^2 = \frac{1}{2\omega_{BD}+3}\right)$.

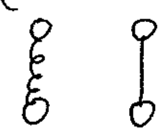
Then the scalar-field equation reads

$$\boxed{\nabla^*_\mu \left[F'(s)\, \nabla^\mu_* \varphi \right] = -\frac{4\pi G_*}{c^4}\, \alpha\, T^*_{\text{matter}}}$$

- For large enough accelerations of a test particle (i.e. for small enough distance r from a gravitational source), one should recover the Newtonian limit $\Rightarrow$ $\boxed{F'(s) \to \text{const.} = 1 \text{ (choice) when } s \to +\infty}$ (Newtonian limit)

Then $\varphi \simeq -\alpha GM/rc^2$ as in scalar-tensor theories, and a test particle feels an extra potential $\alpha\varphi c^2 = -\alpha^2 GM/r$, in addition to the standard $-GM/r$ mediated by $g^*_{\mu\nu}$:

$$V_{\text{total}} = -\frac{GM}{r}\left(1 + \alpha^2\right)$$



[In the following, I usually write G instead of G_*, but explicitly underline G_{eff} when I talk about the measured gravitational constant by Cavendish-type experiments.]

- For small & positive values of $s \equiv (\partial_\mu \varphi)^2$, i.e. when φ varies spatially ($\partial_r \varphi \neq 0$, other $\partial_\mu \varphi = 0$) and creates a <u>small</u> acceleration $a < a_0$ on a test particle, one wishes to predict a $\frac{\sqrt{GMa_0}}{r}$ force ("MOND") and thereby a potential $\alpha \varphi c^2 = \sqrt{GMa_0}\,\ln r$.

It suffices to choose $f'(s) \simeq \sqrt{\bar{s}}$ for $s \to 0$

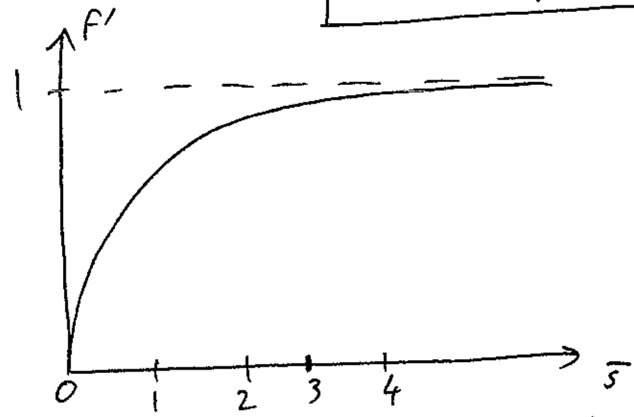
where $\boxed{\bar{s} \equiv \frac{\alpha^6 c^4 s}{a_0^2}} \simeq \left(\frac{\alpha^3 c^2 \partial_r \varphi}{a_0}\right)^2$.

(Indeed, the field equation for φ gives

$$r^2 f'(s)\, \partial_r \varphi = \alpha GM/c^2$$

$$\Rightarrow \quad (\partial_r \varphi)^2 = \frac{GMa_0}{\alpha^2 r^2 c^2} \Rightarrow \quad \varphi = \frac{\sqrt{GMa_0}}{\alpha c^2} \ln r \;)$$

- A simple way to connect the two (Newton & MOND) regimes is to impose for instance $\boxed{f'(s) = \frac{\sqrt{\bar{s}}}{\sqrt{1+\bar{s}}}}$



[cf p. 59 with $\mu(x) = f(x^2)$]

- However, Bekenstein & Sanders realized that $f''(s) \propto \sqrt{\bar{s}} > 0$ in the MOND regime, contradicting condition (c) p. 66: <u>the scalar field propagates quicker than gravitons</u> (and matter, in this model), and these authors considered it was deadly. Actually, we underlined above that causality is not threatened provided the hyperbolicity condition (b) p. 65 is satisfied.

[And this is the case: (b) $\frac{\sqrt{\bar{s}}\,(2+\bar{s})}{(1+\bar{s})^{3/2}} > 0$ for $\bar{s} > 0$]

They thus discarded such a simple model, but for the wrong reason.

- Actually, ∃ problems in this simple model, but not the one of superluminal propagation.
  - First, the above function $f'(s)$ is not defined for negative values of s, and does not satisfy the strict inequality (b) for $s=0$ ⇒ Cauchy problem ill posed on any surface where s vanishes (between local physics of galaxies, $s>0$ and cosmological behavior, where $s<0$).
    Easily solved by adding a small positive constant ε to $f'(s)$, i.e. a standard kinetic term for the scalar field in addition to the "k-essence" one $f(s)$. [J.P. Bruneton 2006].
  - A more serious problem is that the MOND regime starts manifesting at too small distances. We know that the acceleration caused by φ is numerically small with respect to the Newtonian force $\frac{GM}{r^2}$ in the solar system, because $a_0 \simeq 1.2\times10^{-10}\, m\cdot s^{-2}$ is needed to fit galaxy rotation curves. But post-Newtonian tests of gravity tightly constrain deviations from G.R. in the solar system: we know that $\boxed{\alpha^2 < 10^{-5}}$ to pass the Cassini test of the γ^{PPN} parameter, notably.
    Let us compute at which radius r from the sun the scalar field φ takes its MOND $\ln r$ behavior:

$$f'(s) \simeq \sqrt{s} \quad \text{for} \quad s \lesssim 1, \text{ where } s = (\partial_r\varphi)^2 \simeq \frac{GMa_0}{\alpha^2 r^2 c^4}$$

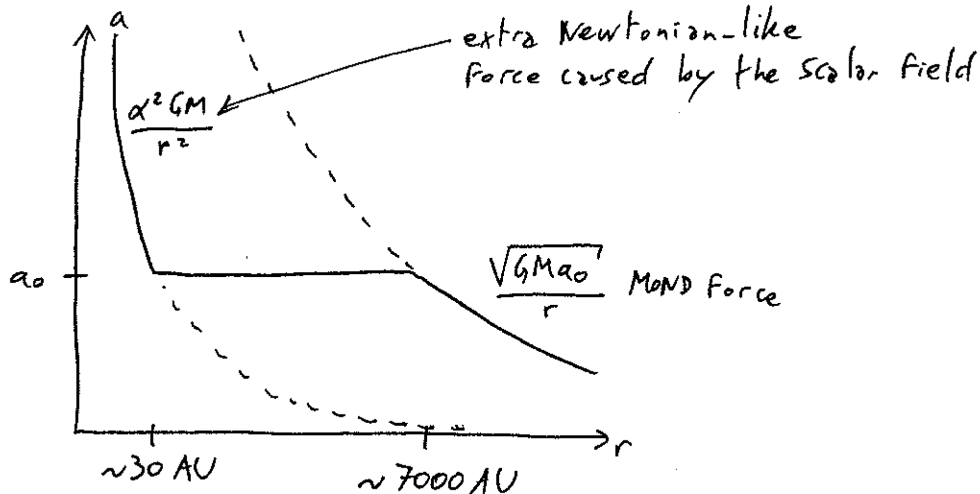
$$\Leftrightarrow \quad \text{MOND force} \quad \text{for} \quad \frac{\alpha^4 GM}{a_0 r^2} \lesssim 1$$

$$\Leftrightarrow \quad \text{''} \quad r \gtrsim \underbrace{\alpha^2}_{<10^{-5}} \underbrace{\sqrt{\frac{GM}{a_0}}}_{\sim 7000\,AU}$$

Therefore, test particles should feel an extra MOND force $\propto \frac{\sqrt{GMa_0}}{r}$ already at distances $r \sim 0.1\,AU <$ ☿'s orbit ⚠
This is ruled out by tests of Kepler's third law.
[J.P. Bruneton & G.E.F 2006]

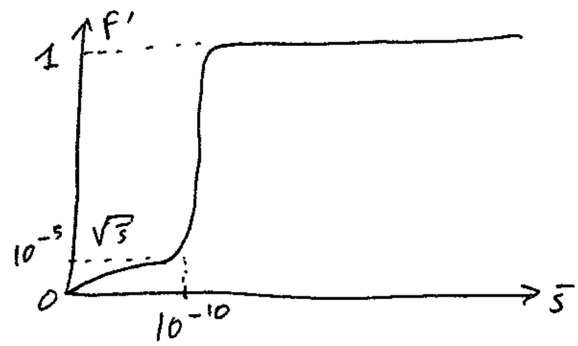
- 2 ways to save it:

— Fine tune $f(s)$ so that the MOND force appears at larger distances?



One finds that it would _just_ be possible to pass present post-Newtonian solar-system tests ($\alpha^2 < 10^{-5}$) and recover the right MOND force at radii $\gtrsim \sqrt{\frac{GM}{a_0}} \sim 7000$ AU (needed for galaxy rotation curves), provided the extra acceleration caused by the scalar field remains $\simeq$ constant between 30 and 7000 AU. [Looks like the Pioneer anomaly, but a_0 is too small by a factor 7. See § III.8 below.].

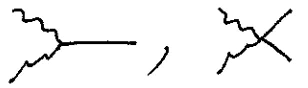
This would need a function



extremely fine-tuned ⚠

— Relax the post-Newtonian constraint on α^2 by coupling φ to matter in a different way than the above $A(\varphi) = e^{\alpha\varphi}$: cf. next sections, notably C.6.

C.5: Light deflection

* As underlined in § A.5 & A.6 above, the electromagnetic action is conformal invariant in 4 dimensions ⇒ no coupling between φ and photons:  ... $= 0$

$$\Rightarrow \underset{\text{light deflection}}{\Delta\theta} = \frac{4 G_* M}{b c^2} \quad \text{as in G.R.}$$

but we measure $G_{eff} M = G_* (1+\alpha^2) M$ with a test particle

$$\Rightarrow \Delta\theta = \frac{4 G_{eff} M}{(1+\alpha^2) b c^2} \leq \text{G.R.'s prediction}$$, at $G_{eff} M$ fixed.

* This has been immediately recognized by Bekenstein & Sanders as a serious experimental difficulty, since weak lensing experiments confirm the amount of dark matter deduced from rotation curves.

(Note however that there seems to exist a confusion about which masses are measured, in the literature. The above inequality, valid at first post-Newtonian order, shows that

$$M_{\substack{\text{deduced from}\\ \text{weak lensing}}} \leq M_{\substack{\text{deduced from}\\ \text{rotation curves}}}$$

But none of them is a priori equal to M_{baryon}, since $T_{\mu\nu}(\varphi)$ can contribute to the total mass. [Easy to build an explicit example, for instance usual dark matter described by a massive φ, non-interacting with ordinary matter!]
[Some claims that $M_{weak\ lensing} \not\leq M_{baryon}$ are thus erroneous, and one can locate where the reasonings fail.])

C.6 : Disformal coupling and vector-tensor theories

[Bekenstein & Sanders, inspired by Ni's "stratified theory of gravity"]

* ∃ trick to increase by hand light deflection in scalar-tensor or "k-essence" theories: couple φ differently to g^*_{00} and g^*_{ij}. [A priori non-covariant, but always possible to introduce a vector field u^μ to write it covariantly.]

Idea: couple matter to $\tilde{g}_{00} = e^{2\alpha\varphi}\, g^*_{00}$

but $\tilde{g}_{ij} = e^{-2\alpha\varphi}\, g^*_{ij}$

Indeed, if the factors are the same, we know that photons do not feel them (conformal invariance of S_{EM}).
On the other hand, if the factors are inverse one another, then φ mimics the behavior of the Schwarzschild solution, with

$$\begin{cases} g^*_{00} = -1 + \dfrac{2GM}{rc^2} + O\left(\dfrac{1}{c^4}\right) \\ g^*_{ij} = \delta_{ij}\left(1 + \dfrac{2GM}{rc^2}\right) + O\left(\dfrac{1}{c^4}\right) \end{cases}$$

* If ∃ preferred frame where a vector u^μ takes the form

$u^\mu = (1,0,0,0)$, then [in locally inertial coordinates where $g^*_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$]

$$\begin{cases} -u_\mu u_\nu = \mathrm{diag}(-1,0,0,0) & \text{behaves like } g^*_{00} \\ \text{and } g^*_{\mu\nu} + u_\mu u_\nu = \mathrm{diag}(0,1,1,1) & \text{" " } g^*_{ij} \end{cases}$$

⇒ [Bekenstein & Sanders 2004-2005] decide to couple matter to the following physical metric:

$$\tilde{g}_{\mu\nu} = -e^{2\alpha\varphi} u^*_\mu u^*_\nu + e^{-2\alpha\varphi}(g^*_{\mu\nu} + u^*_\mu u^*_\nu)$$
$$= e^{-2\alpha\varphi} g^*_{\mu\nu} - 2 u^*_\mu u^*_\nu \sinh(2\alpha\varphi)$$

[This is a <u>choice</u> to mimic G.R.'s behavior with the scalar field, and thereby reproduce the observed light deflection (in weak-lensing experiments) ≠ real "prediction" of the model.]

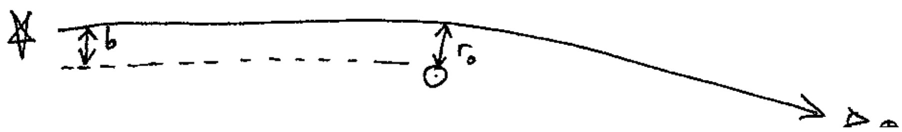
* To compute the predicted light deflection, the simplest method is to use the exact integral

$$\Delta\theta = -\pi + 2\int_{r_0}^{\infty} \frac{dr}{r^2}\left(\frac{\mathcal{A}(r)\,\mathcal{B}(r)}{\mathcal{B}(r_0)/r_0^2 - \mathcal{B}(r)/r^2}\right)^{1/2}$$

valid for any metric of the form ("Schwarzschild coords"):

$$ds^2 = -\mathcal{B}(r)\,c^2dt^2 + \mathcal{A}(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

[Easy to derive from geodesics equation, cf. Weinberg's book for instance. If p is the affine parameter chosen to measure the light geodesics, then the Eq. for $\frac{d^2\phi}{dp^2}$ can be integrated once to give a first constant of motion $r^2 d\phi/dp$. The Eq. for d^2t/dp^2 can also be integrated and gives $c\,dt = dp/\mathcal{B}$. Finally, the Eq. for d^2r/dp^2 can also be integrated onces and gives a second constant of motion (related to energy). Replacing t by ϕ thanks to the above constants of motion immediately yields the integral for $\Delta\theta$ (light deflection) above.]



For a scalar field φ in the MOND regime, i.e.

$\alpha \varphi c^2 = \sqrt{GMa_0} \ln r + O\left(\frac{1}{c^2}\right)$, one then gets:

$$\Delta\theta = \Delta\theta_{GR} + \frac{4\sqrt{GMa_0}}{c^2} \int_{r_0}^{\infty} \frac{\ln(r/r_0)}{\left(1 - r_0^2/r^2\right)^{3/2}} \frac{r_0 \, dr}{r^2} + O\left(\frac{1}{c^4}\right)$$

$$= \Delta\theta_{GR} + \frac{4\sqrt{GMa_0}}{c^2} \left[\arcsin\left(\frac{r_0}{r}\right) - \frac{(r_0/r) \ln(r_0/r)}{\sqrt{1 - r_0^2/r^2}} \right]_{r_0}^{\infty} + O\left(\frac{1}{c^4}\right)$$

$$= \Delta\theta_{GR} + \boxed{\frac{2\pi \sqrt{GMa_0}}{c^2}} + O\left(\frac{1}{c^4}\right)$$

This is exactly the light deflection angle predicted by G.R. in presence of a halo of dark matter (isothermal sphere) reproducing the $\frac{\sqrt{GMa_0}}{r}$ force.

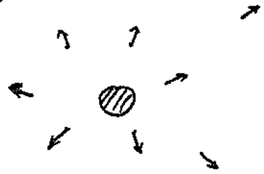
$\Rightarrow$ consistent with observation (as much as GR + dark matter)

[But note that if observation tells us one day that there is a discrepancy with this value, it would be easy to change $\tilde{g}_{\mu\nu}$ to reproduce it...]

* Note in passing that the extra contribution to $\Delta\theta$ does not depend on b (← impact parameter), contrary to the standard deflection $\frac{4GM}{bc^2}$ by a point-mass. Indeed, the $\frac{\sqrt{GMa_0}}{r}$ force is generated by a dark matter distribution $T_{00}/c^2 = \sqrt{GMa_0} / 4\pi G r^2$ with large extension.

[Note also that $\exists$ anyway weak lensing, since the angle is constant but not its direction!

$\Rightarrow \exists$ deformations of images of distant objects]



* In the solar system, the assumed physical metric $\tilde{g}_{\mu\nu}$ p. (73) reproduces the Schwarzschild metric at $O(\frac{1}{c^2})$ $\Rightarrow$ light deflection now reads $\Delta\theta = \dfrac{4\, G_*(1+\alpha^2)M}{bc^2} = \dfrac{4\, G_{eff} M}{bc^2}$ = G.R.'s result.

(Therefore, we donot have any longer any constraint on the matter-scalar coupling strength α o——
$\Rightarrow$ one may choose it large enough for the MOND force to start manifesting at distances > all planets, cf. p. (70).

---

* Actually, a somewhat more natural model was proposed earlier by the same authors [Bekenstein & Sanders]: use $\partial_\mu\varphi$ as a vector, instead of introducing the new field u^μ as above.

- In local physics, $\partial_\mu\varphi$ will basically depend only on space ($\partial_r\varphi \neq 0$, all other $\partial_\mu\varphi = 0$), therefore it does not allow us to separate g^*_{00} and g^*_{ij} as simply as in p. (73).
- However, it suffices to tune g_{rr} to obtain the right light deflection, without modifying g_{00} nor $g_{\phi\phi}$.

Let

$$\boxed{\tilde{g}_{\mu\nu} \equiv A^2(s,\varphi)\, g^*_{\mu\nu} + B(s,\varphi)\, \partial_\mu\varphi\, \partial_\nu\varphi}$$

and tune the model so that $(B/A^2)(\partial_r\varphi)^2 = 4\sqrt{GMa_0}/c^2$.

[Easy to do by choosing the appropriate functions of $(\partial_\mu\varphi)^2 = s$ and φ in the two independent factors $A(s,\varphi)$ and $B(s,\varphi)$].

Then one can easily compute

$$\Delta\theta = \Delta\theta_{GR} + \int_{r_0}^{\infty} \frac{dr/r}{\sqrt{r^2/r_0^2 - 1}}\, \frac{B(\partial_r\varphi)^2}{A^2} + O\left(\frac{1}{c^4}\right)$$

$$= \Delta\theta_{GR} + \frac{2\pi\sqrt{GM a_0}}{c^2} + O\left(\frac{1}{c^4}\right)$$

as needed.

* Note that one must choose $B(s,\varphi) \geq 0$ to <u>increase</u> light deflection (as needed). However, Bekenstein & Sanders noticed that light rays are such that

$$\tilde{g}_{\mu\nu} dx^\mu dx^\nu = 0$$

$$\Leftrightarrow \quad g^*_{\mu\nu} dx^\mu dx^\nu = -\frac{B}{A^2}\left(\partial_\mu \varphi \, dx^\mu\right)^2 \leq 0$$

Therefore, <u>photons are timelike with respect to the Einstein metric</u> $g^*_{\mu\nu}$ [no problem ⚠].
But the above authors interpreted it as dangerous "<u>superluminal gravitons</u>", ruling out the model.
[This is the reason why they focused on the previous solution p. (73) with a vector field U^μ defining a preferred frame].
Actually, they are "superluminal", but perfectly causal (since $g^*_{\mu\nu}$ actually defines the global hyperbolicity of spacetime), and thereby not "dangerous" at all.

⇒ {possible to derive MOND-like models with "disformal couplings" of the form
$$\tilde{g}_{\mu\nu} \equiv A^2(s,\varphi)\, g^*_{\mu\nu} + B(s,\varphi)\, \partial_\mu\varphi\, \partial_\nu\varphi$$
while remaining perfectly consistent. [Bruneton & G.E-F 2006]}

(However, consistency of field equations <u>within</u> matter is difficult to analyze in full detail, and some work remains to be done.)

C.7: Experimental issues

* Stability of the Bekenstein-Sanders "TeVeS" model (Tensor-Vector-Scalar) presented p. (73) is unclear.
  Does not seem to exist any problem with the scalar field φ (besides the discontinuity of some functions assumed in the first versions $\Rightarrow$ ill-posed Cauchy problem), but probably
  $\exists$ negative-energy vector mode [cf. Foster & Jacobson 2005].
* One of the most difficult works by Bekenstein was to prove that all degrees of freedom can propagate at most with
  under some restrictive hypotheses.
  the speed of light,
  Actually, $\exists$ experimental constraints on gravitons slower
  than the maximum matter speed: Cerenkov radiation would
  forbid the existence of high-energy cosmic rays! [Moore & Nelson 2001]

  But we anyway know that there is no causality problem
  with speeds $> c_{light}$, provided there exists a
  maximum speed that no degree of freedom can exceed.
* Although the above TeVeS model is more complicated than the standard scalar-tensor models studied in § A & B in the present lectures, it behaves exactly like them at short distances $\Rightarrow$ same predictions as scalar-tensor models for binary pulsars, and constraint

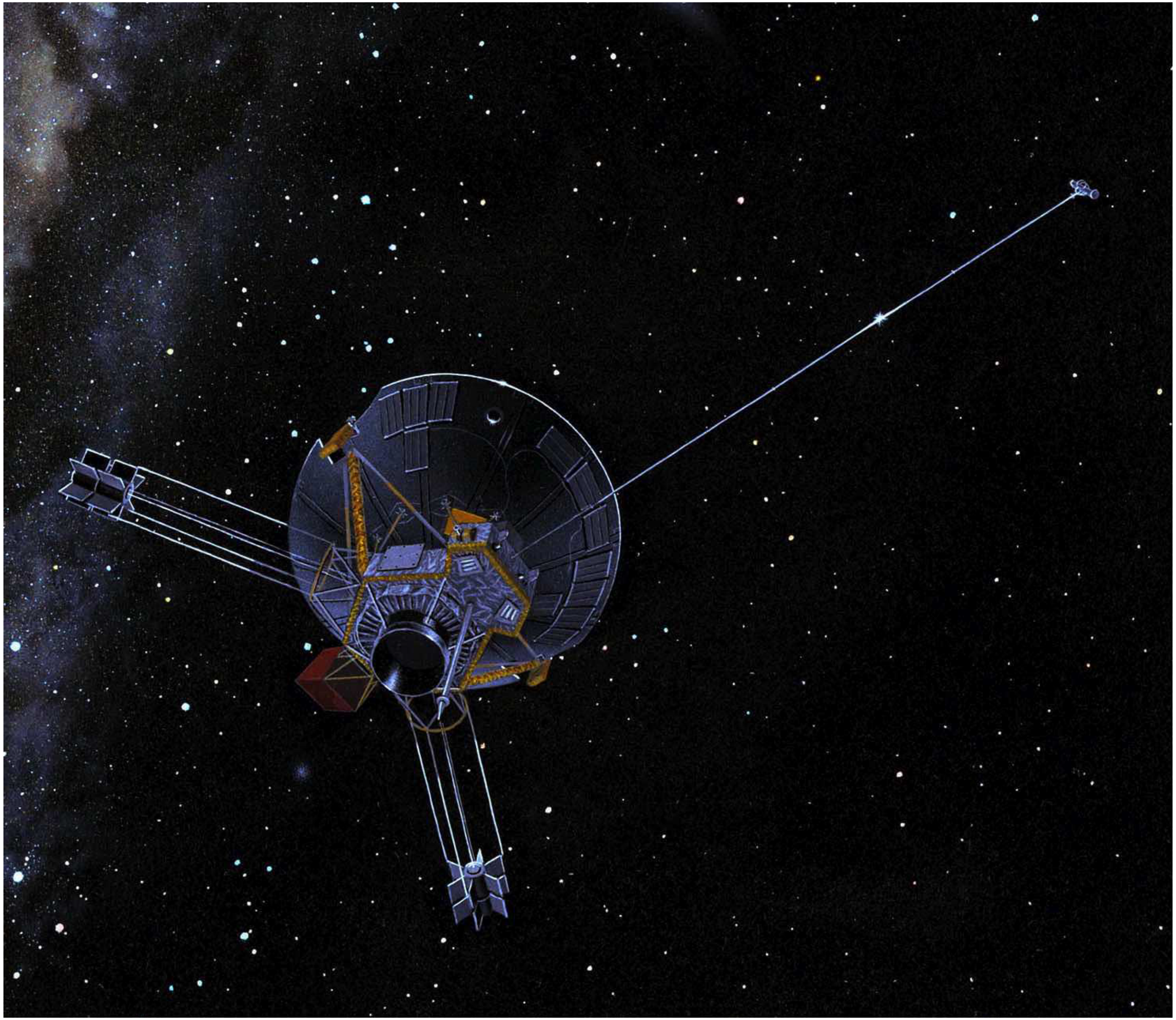
$$\boxed{\alpha^2 < 4 \times 10^{-4}}$$

imposed by dissymmetric PSR-white dwarf PSR J 1141-65 45.
(no observed dipolar radiation)

$\Rightarrow$ we face back the fine-tuning problem underlined p. (70):
we need a priori a quite unnatural function $f'(s)$ to pass binary-pulsar tests without spoiling Kepler's third law in solar system.

Pioneer 10 & 11 anomaly

Extra acceleration ~ 8.5 $\times 10^{-10}$ m s^{-2} towards the Sun, between 30 and 70 AU



* Generic problem with CMB spectrum: the heights of the odd picks vs even ones should decrease monotonically if ∄ dark matter (Silk damping).
* How can structure form without dark matter? Possible solution if $a_0 \simeq cH_0/6$ is actually a function of time

---

C.8 : Pioneer anomaly

* Pioneer 10 & 11 spacecrafts exhibited an anomalous extra acceleration ≃ directed towards the ☉, between 20 and 70 AU.

$$\boxed{\delta a \simeq 8.5 \times 10^{-10}\ m \cdot s^{-2}}$$ ($\sim \frac{1}{1000}$ × Newton's force but 1000 × larger than experimental sensitivity)

(7 × greater than the MOND constant a_0).

* impossible to explain by many ≠ sources of noise which have been carefully analysed [cf. S. Turyshev's papers].
* Impossible to explain within the PPN formalism, otherwise planets' orbits would be inconsistent with observation.
* In particular, trying to explain this extra acceleration by a modification of g_{00} (with respect to G.R.) is inconsistent with Kepler's 3rd law tests, unless this force starts manifesting beyond Saturn or Uranus ⟹ would be quite fine tuned (but who knows?).
* On the other hand, Reynaud & Jaekel [2005-2006] proved that a spatial dependence of $\gamma^{PPN}(r)$ can account for such an anomalous acceleration, without spoiling planetary orbits. [This does not fit within the standard PPN formalism, because one now needs an explicit length scale entering the model, whereas PPN assumes ∃ no scale besides G, c and the matter sources].

* The above "disformal couplings"

$$\boxed{\tilde{g}_{\mu\nu} = A^2(s,\varphi)\, g^*_{\mu\nu} + B(s,\varphi)\, \partial_\mu\varphi\, \partial_\nu\varphi}$$

are a simple way to create a varying $\gamma^{PPN}(r)$, although one would need again some fine tuning to reproduce both MOND and the Pioneer anomaly.

$\Rightarrow$ This justifies the careful study of this class of models, including the stability issues within matter [Bruneton & G. E-F 2006].

* The intuitive (and correct!) reason why $\gamma^{PPN}(r)$ can "explain" (= fit!) Pioneer data without spoiling Kepler's third law is that the Pioneer spacecrafts have hyperbolic orbits, whereas planets have elliptic (bounded) ones.

Recall the PPN form of the metric in isotropic coordinates:

$$\begin{cases} ds^2 = -\left(1 - \frac{2Gm}{rc^2} + 2\beta^{PPN}\left(\frac{Gm}{rc^2}\right)^2 + O\left(\frac{1}{c^6}\right)\right) c^2 dt^2 \\ \qquad + \left(1 + 2\gamma^{PPN}\frac{Gm}{rc^2} + O\left(\frac{1}{c^4}\right)\right) d\vec{x}^{\,2} \end{cases}$$

- If bounded orbits, then virial theorem $\Rightarrow \frac{v^2}{c^2} \sim \frac{Gm}{rc^2}$

  $\Rightarrow$ effects of γ^{PPN} & β^{PPN} appear at same (1PN) order.
- But if $\frac{v^2}{c^2} \gg \frac{Gm}{rc^2}$, for hyperbolic orbits, then

$$\left|\gamma^{PPN}\frac{Gm}{rc^2}\frac{d\vec{x}^{\,2}}{dt^2}\right| \gg \left|\beta^{PPN}\left(\frac{Gm}{rc^2}\right)^2\right|$$

and γ^{PPN} plays a larger role.

Conclusions

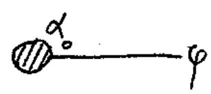
⊛ General relativity has passed all precision **tests** with flying colors.

⊛ ∃ anyway some puzzling experimental issues, notably:
- dark energy
- dark matter
- Pioneer anomaly

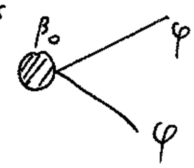
⇒ important to go on analyzing gravitational experiments, both with
- * a phenomenological viewpoint (cf. PPN)
- * a field-theoretical one (consistent relativistic theories)

⊛ ∃ <u>qualitative</u> difference between

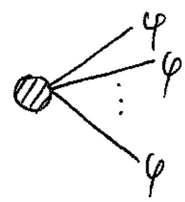
- solar-system tests [weak-field regime tested at the 10^{-5} level]



- binary-pulsar tests [nonperturbative effects in strong-field regime]



- cosmological observations: more noisy but a priori allows us to reconstruct full shape of $A(\varphi)$ and $V(\varphi)$



⊛ Lesson for Gravitational Wave data analysis:
Thanks to binary pulsars, we know that GR wave templates are secure (at least for LIGO/VIRGO)