Cliff Will

Lecture 3 - PPN Framework

1. Parametrize the metric

   - effect of boosts ~~2 PPT~~ $\boxed{6 \text{ PPT}}$

   - preferred frame if $(\alpha_1, \alpha_2\, \alpha_3)$ not all zero

2. Conservation laws

   A. Baryon number

$$(n u^\alpha)_{;\alpha} = 0$$

If mass per baryon is constant,

$$(\rho u^\alpha)_{;\alpha} = 0 = \frac{1}{\sqrt{-g}} (\sqrt{-g}\, \rho u^\alpha)_{,\alpha} = 0$$

Define $\rho^* = \rho \sqrt{-g}\, u^0 \simeq \rho\left(1 + \tfrac{1}{2} v^2 + 3\gamma U\right)$

$$\frac{\partial \rho^*}{\partial t} + \nabla \cdot (\rho^* \underset{\sim}{v}) = 0$$

– "conserved" density
– exact

Note $\rho^* d^3x$ = invariant = rest mass inside comoving volume

   B. Cons. of energy

$$T^{\mu\nu}{}_{;\nu} = 0 \qquad T^{\mu\nu} = (\rho(1+\Pi) + p) u^\mu u^\nu + p g^{\mu\nu}$$

$$u_\mu T^{\mu\nu}{}_{;\nu} = 0 = u_\mu \{ (\rho + \rho\Pi + p)_{,\nu} u^\mu u^\nu +$$

$$+ (\rho + \rho\Pi + p)\left(\underbrace{u_\mu u^\mu{}_{;\nu}}_{0} u^\nu + u_\mu u^\mu u^\nu{}_{;\nu} \right)$$

$$+ p_{,\mu} u^\mu \}$$

$$= -\frac{d}{dt}(\rho + \rho\Pi) - (\rho + \rho\Pi + p) \nabla \cdot u$$

Show that $\nabla \cdot u = \frac{1}{v}\frac{dv}{d\tau}$

$$v\frac{d}{d\tau}(\rho+\rho\Pi) + (\rho+\rho\Pi)\frac{dv}{d\tau} + p\frac{dv}{d\tau} = 0$$

$$\frac{d}{d\tau}\left[(\rho+\rho\Pi)v\right] + p\frac{dv}{d\tau} = 0$$

$$dE + p\,dv = đQ = T\,dS = 0 \quad [\text{isentropic}]$$

C. Global Conservation laws

In SRT $T^{\mu\nu}{}_{,\nu} = 0 \Rightarrow \int \left(T^{\mu 0}{}_{,0} + T^{\mu j}{}_{,j}\right) d^3x$

$$= \frac{\partial}{\partial t}\int T^{\mu 0} d^3x + \oint T^{\mu j} d^2S_j$$

$T^{\mu 0} = \text{const}\ (E, P)$ — no flux

In Related E Eq. we had

$$\tau^{\mu\nu}{}_{,\nu} = 0 \Rightarrow \int \tau^{\mu 0} d^3x = \text{const} \quad \text{if no GW}$$

But $T^{\mu\nu}{}_{;\nu} = 0$ doesn't give cons. law because of Γ's

In PPN: given $T^{\mu\nu}{}_{;\nu} = 0$ can we construct a quantity $\tau^{\mu\nu} \sim (1-aU)T^{\mu\nu} + t^{\mu\nu}$ such that

$\tau^{\mu\nu}{}_{,\nu} = 0 \Rightarrow E, P$ conserved	$\alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0$
$\tau^{\mu\nu} = \tau^{\nu\mu} \Rightarrow J$, CM	$\alpha_1 = \alpha_2 = 0$

eg NGT

3. Equations of Motion - Photons

- postulate null geodesics
- can also prove from Maxwell's eqns

$$F^{\mu\nu}{}_{;\nu} = 0 \qquad A^{\mu}{}_{;\mu} = 0 \qquad \lambda \ll \{R, L\}$$

$$\boxed{\begin{aligned} k^{\nu} k^{\mu}{}_{;\nu} &= 0 \\ k^{\mu} k_{\mu} &= 0 \end{aligned}}$$

$$A^{\mu} = (a^{\mu} + \delta B^{\mu} + \ldots) e^{i\theta/\delta}$$

$$k^{\mu} = \frac{dx^{\mu}}{d\sigma} \qquad k_{\mu} = \theta_{,\mu}$$

σ affine param.

$$\frac{d^2 x^{\mu}}{d\sigma^2} + \Gamma^{\mu}_{\alpha\beta} \frac{dx^{\alpha}}{d\sigma}\frac{dx^{\beta}}{d\sigma} = 0$$

use t as parameter, $x^0 = t$

$$\frac{d^2 t}{d\sigma^2} + \Gamma^{0}_{\alpha\beta} \frac{dx^{\alpha}}{d\sigma}\frac{dx^{\beta}}{d\sigma} = 0$$

$$\Rightarrow \quad \boxed{\begin{aligned} &\frac{d^2 x^j}{dt^2} + \left(\Gamma^{j}_{\alpha\beta} - \Gamma^{0}_{\alpha\beta}\frac{dx^j}{dt}\right)\frac{dx^{\alpha}}{dt}\frac{dx^{\beta}}{dt} \\ &g_{\mu\nu}\frac{dx^{\mu}}{dt}\frac{dx^{\nu}}{dt} = 0 \end{aligned}}$$

Subst

$$x^j = \hat{n}^j (t - t_0) + x_p^j(t) \qquad \hat{n}\cdot\hat{n} = 1$$

$$\boxed{\begin{aligned} \frac{d^2 \underset{\sim}{x}_p}{dt^2} &= (1+\gamma)\left(\underline{\nabla} U - 2\underset{\sim}{n}\,\underset{\sim}{n}\cdot\underline{\nabla} U\right) \\ \underset{\sim}{n}\cdot\frac{d\underset{\sim}{x}_p}{dt} &= -(1+\gamma)U \end{aligned}}$$

4. Equations of Motion – Bodies

$$P^{\alpha}{}_{\mu} T^{\mu\nu}{}_{;\nu} = 0$$

Projection operator

$$P^{\alpha}{}_{\nu} = \delta^{\alpha}{}_{\nu} + u^{\alpha} u_{\nu}$$

$$P^{\mu}{}_{\nu} u^{\nu} = 0$$

$$= P^{\alpha}{}_{\mu} \Big\{ (\rho + p)_{,\nu} u^{\mu} u^{\nu} + (\rho + p) \underbrace{u^{\nu} u^{\mu}{}_{;\nu}}_{a^{\mu}} + (\rho + p) u^{\mu} u^{\nu}{}_{;\nu} + p_{,\nu} g^{\mu\nu} \Big\}$$

$$\boxed{(\rho + p) a^{\alpha} = - P^{\alpha\mu} p_{,\mu}}$$

j component

$$a^{j} = \frac{d^2 x^j}{d\tau^2} + \Gamma^{j}{}_{\alpha\beta} u^{\alpha} u^{\beta} \qquad v^{\alpha} = (1, \underset{\sim}{v})$$

$$= \left(\frac{dt}{d\tau}\right)^2 \left[\frac{d^2 x^j}{dt^2} - \Gamma^{0}{}_{\alpha\beta} v^{\alpha} v^{\beta} v^{j} + \Gamma^{j}{}_{\alpha\beta} v^{\alpha} v^{\beta} \right]$$

$$\therefore \quad \boxed{\frac{d^2 x^j}{dt^2} = - \Gamma^{j}{}_{\alpha\beta} v^{\alpha} v^{\beta} + \Gamma^{0}{}_{\alpha\beta} v^{\alpha} v^{\beta} v^{j} - \left(\frac{d\tau}{dt}\right)^2 \frac{P^{j\mu} p_{,\mu}}{\rho + p}}$$

Define a CM for each body

$$X^{i}_{a} = \frac{1}{m_a} \int_a \rho x^{i} d^3x \qquad m_a = \int_a \rho d^3x$$

or $$X^{i}_{a} = \frac{1}{m_a} \int_a \rho^{*} x^{i} d^3x \qquad m_a = \int_a \rho^{*} d^3x$$

or $$X^{i} = \frac{1}{m} \int \rho^{*} \left(1 + \frac{1}{2} \bar{v}^2 - \frac{1}{2} \bar{U} + \Pi\right) x^{i} d^3x \qquad M = \int \rho^{*} (\quad) d^3x$$

$$\frac{dX^i_a}{dt} = \frac{d}{dt}\left\{ \frac{1}{m_a}\int_a \rho^*(\quad) x^i d^3x \right\}$$

<u>Result</u>

$\boxed{PPT}$

$$\underset{\sim}{a}_a = (a_a)_{SELF} + (a_a)_{NEWT} + (a_a)_{NBODY}$$

<u>Self terms</u>

6 terms eg. $t^i = \int \rho^* \rho'^* \frac{\bar{v}'^2 (x-x')^i}{|x-x'|^3} d^3x$

– all depend on $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \alpha_3$ — vanish in conservative theory

<u>"Newtonian"</u>

– for spherical bodies

$$(a_a)^i_{NEWT} = \sum_b \frac{1}{m_a}(m_P)_a \left(\sum_{b\neq a} \frac{(m_A)_b}{r_{ab}} \right)_{,i}$$

[NGT: $m = m_P = m_A$]

$\boxed{PPT}$

$$\frac{(m_P)_a}{m_a} = 1 - \left(4\beta - \gamma - 3 - \tfrac{10}{3}\xi - \alpha_1 + \tfrac{2}{3}\alpha_2 - \tfrac{2}{3}\zeta_1 - \tfrac{1}{3}\zeta_2\right)\frac{\Omega_a}{m_a}$$

$$\frac{(m_A)_b}{m_b} = 1 - \left(4\beta - \gamma - 3 - \tfrac{10}{3}\xi - \tfrac{1}{2}\alpha_3 - \tfrac{1}{3}\zeta_1 - 2\zeta_2\right)\frac{\Omega_b}{m_b} + \zeta_3 \frac{E_b}{m_b} - \left(\tfrac{3}{2}\alpha_3 + \zeta_1 - 3\zeta_4\right)\frac{P_b}{m_b}$$

$$\Omega_a = -\frac{1}{2}\int_a \frac{\rho\rho'}{|x-x'|} d^3x\, d^3x'$$

$$E_a = \int_a \rho\Pi\, d^3x$$

Note $m_P \neq m_I$ - violation of WEP for massive bodies

- Nordtvedt effect

Note $m_P \neq m_A$ - violation of 3rd law. But if

$$\alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0 , \quad m_P = m_A$$

"n-body" $\boxed{PPT}$

5. Equations of Motion - Spinning bodies

Gyroscope $\vec{S}$ - spatial in comoving LFF frame

- constant (if torque free)

$$S_{\hat{0}} = 0 \qquad \vec{S}\cdot\vec{u} = 0 \qquad S_0 u^0 + S_j u^j = 0 \quad \boxed{S_0 = -S_j v^j}$$

$$\frac{dS_j}{dt} = u^\alpha S_{j;\alpha} = 0 \Rightarrow u^\alpha S_{\mu;\alpha} = \nabla_{\vec{u}} \vec{S} = 0$$

$$u^\alpha \left(S_{\mu,\alpha} - \Gamma^\beta_{\mu\alpha} S_\beta\right) = 0$$

$$\frac{dS_j}{dt} = \left(\Gamma^0_{j0} S_0 + \Gamma^0_{jk} S_0 v^k + \Gamma^\ell_{j0} S_\ell + \Gamma^\ell_{jk} S_\ell v^k\right)$$

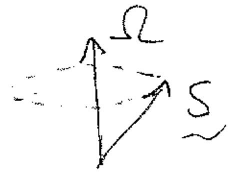
$$\Gamma^0_{j0} = -U_{,j} \qquad \Gamma^0_{jk} = \gamma \delta_{jk} \dot{U}$$

$$\Gamma^\ell_{j0} = \gamma \delta_{\ell j} \dot{U} - \tfrac{1}{2}(4\gamma + 4 + \alpha_1) V_{[\ell,j]}$$

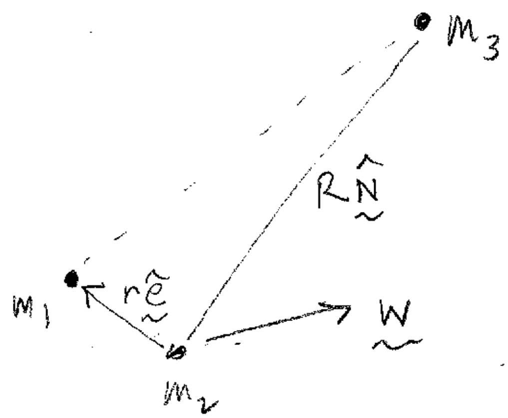
$$\Gamma^\ell_{jk} = \gamma\left(\delta^\ell_j U_{,k} + \delta^\ell_k U_{,j} - \delta_{jk} U_{,\ell}\right)$$

$$\boxed{\begin{aligned} \frac{d\mathbf{S}}{dt} &= \boldsymbol{\Omega} \times \mathbf{S} \\ \boldsymbol{\Omega} &= \tfrac{1}{4}(4\gamma + 4 + \alpha_1)\, \nabla \times \mathbf{V} + (\gamma + \tfrac{1}{2})\, \mathbf{v} \times \nabla U \end{aligned}}$$

precession



6. <u>Locally Measured G</u>



$$m_1 \frac{d^2 r_p}{d\tau^2} = - \frac{m_1 m_2}{r_p^2} G_L$$

at rest ⇒ all comes from g_{00}

$$U = \frac{m_2}{r_{12}} + \frac{m_3}{|r\mathbf{e} - R N|}$$

$$\Phi_2 = \frac{m_2}{r_{12}} \left(\frac{m_3}{R} \right) + O\left(\frac{1}{R^2}\right)$$

$$\Phi_W = \frac{m_2}{r_{12}} \frac{M_3}{R} \left(2 - (e \cdot N)^2 \right) + O\left(\frac{1}{R^2}\right)$$

$$\text{"}U\text{"} = \frac{m}{r} \Big[1 - (4\beta - 3\gamma - 1 - \zeta_2 - 3\xi) \frac{M_3}{R} + \xi \frac{M_3}{R} (\hat{e} \cdot \hat{N})^2 - \tfrac{1}{2}(\alpha_1 - \alpha_2 - \alpha_3) w^2 - \tfrac{1}{2} \alpha_2 (w \cdot e)^2 \Big]$$

convert to proper ηt,