Halphen's solution in modern perspective

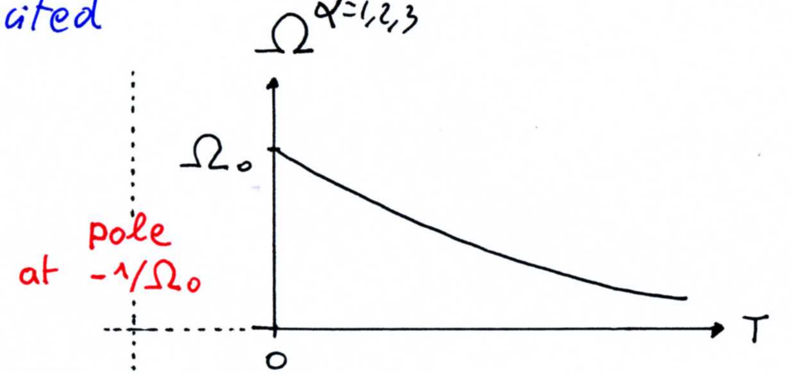
Halphen's system describes the Ricci flow of a deformed S^3 with or without torsion

→ three different types of solutions depending on the unbroken symmetry

- $\Omega^1 = \Omega^2 = \Omega^3 \rightarrow \dot{\Omega}^1 = \dot{\Omega}^2 = \dot{\Omega}^3$: round sphere with $SU(2) \times SU(2)$

  → only the breathing mode is excited

  → $$\Omega^{\alpha=1,2,3} = \frac{1}{T + 1/\Omega_0}$$



  → universal behaviour

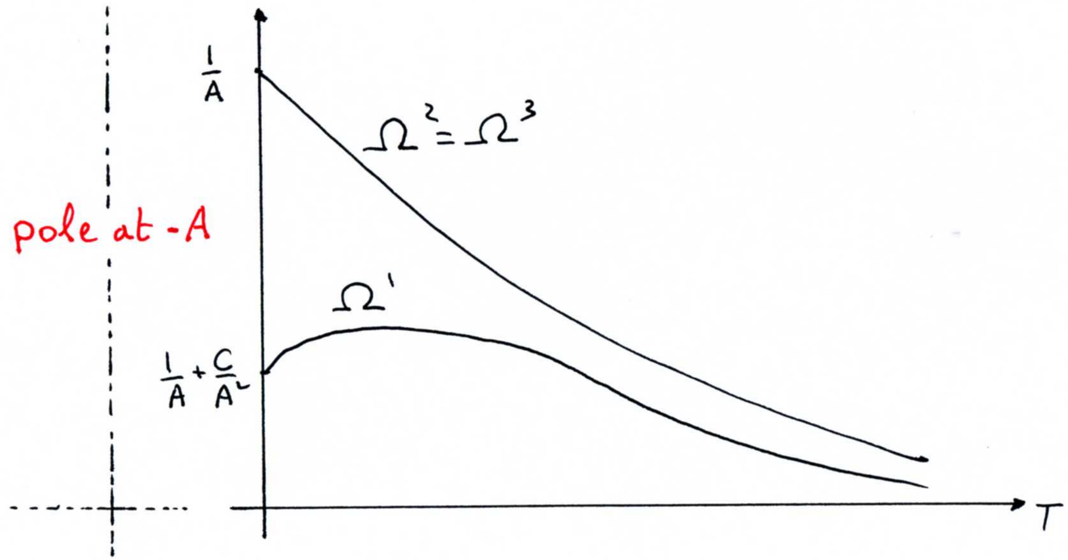
usual I-R fixed points
- without torsion $\quad \gamma_{\alpha=1,2,3} \sim \frac{1}{\sqrt{E}} \longrightarrow 0$
- with torsion $\quad \gamma_{\alpha=1,2,3} \longrightarrow 1$

- $\Omega^1 \neq \Omega^2 = \Omega^3 \rightarrow \dot{\Omega}^1 \neq \dot{\Omega}^2 = \dot{\Omega}^3$: axisymmetric deformation with $SU(2) \times U(1)$

$$\rightarrow \quad \Omega^1 = \frac{1}{T+A} + \frac{C}{(T+A)^2}$$

A, C are arbitrary constants $\leftrightarrow$ initial conditions

$$\rightarrow \quad \Omega^2 = \Omega^3 = \frac{1}{T+A}$$



$\rightarrow$ same universal behaviour with restoration of the $SU(2) \times SU(2)$

- $\Omega^1 \neq \Omega^2 \neq \Omega^3 \;\longrightarrow\; \dot{\Omega}^1 \neq \dot{\Omega}^2 \neq \dot{\Omega}^3$: anisotropic deformation with $SU(2)$

  → no algebraic first integrals

  → remarkable modular properties

- consider a solution $\omega^\alpha(z)$ $z \in \mathbb{C}$ $(\bullet \equiv d/dz)$

$$\tilde{\omega}^\alpha(z) = \frac{1}{(cz+d)^2}\,\omega^\alpha\!\left(\frac{az+b}{cz+d}\right) + \frac{c}{cz+d} \quad \text{with } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{C})$$

is another solution

- a solution reads $\omega^\alpha = -\frac{1}{2}\frac{d}{dz}\log E^\alpha$

  E^α : triplet of weight-two modular forms of $\Gamma(2) \subset PSL(2,\mathbb{Z})$

  depends on the initial conditions

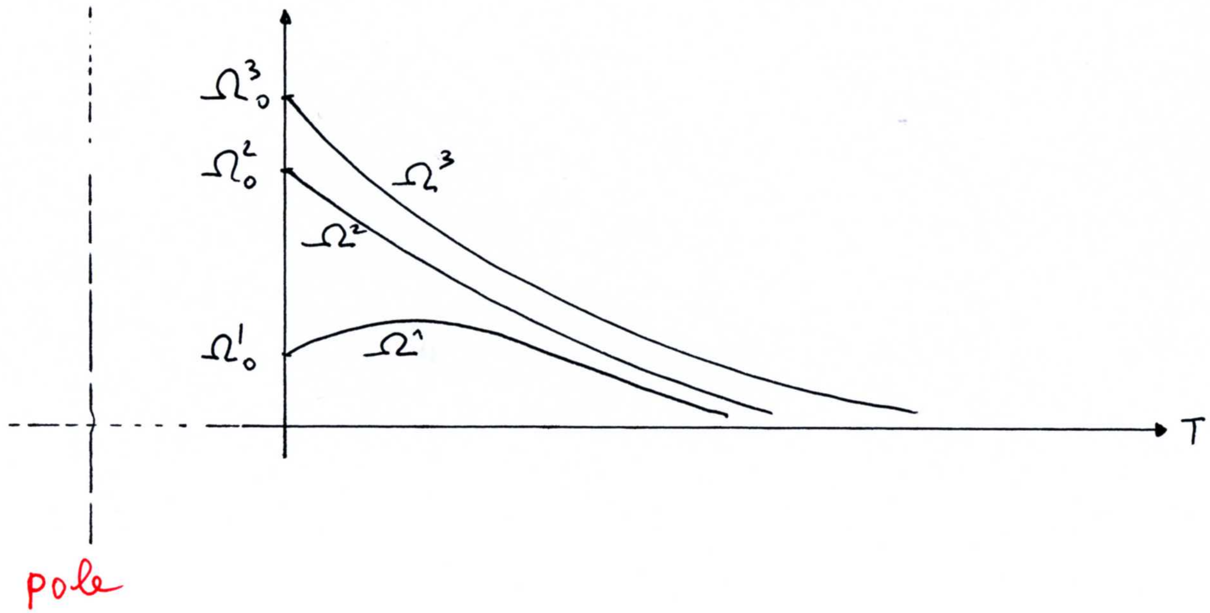
→ large-T and small-T behaviours are not independent

$$\Omega^{1,2,3}(T) = -\frac{1}{T^2}\,\Omega^{2,1,3}\left(\frac{1}{T}\right) + \frac{1}{T}$$

→ assuming Ω_0^{α} finite we recover the UNIVERSAL BEHAVIOUR

$$\Omega^{\alpha} \simeq \frac{1}{T} + \text{subleading at large } T$$

→ if $0 < \Omega_0^1 < \Omega_0^2 < \Omega_0^3 \rightarrow 0 < \Omega^1 < \Omega^2 < \Omega^3 \ \forall T$



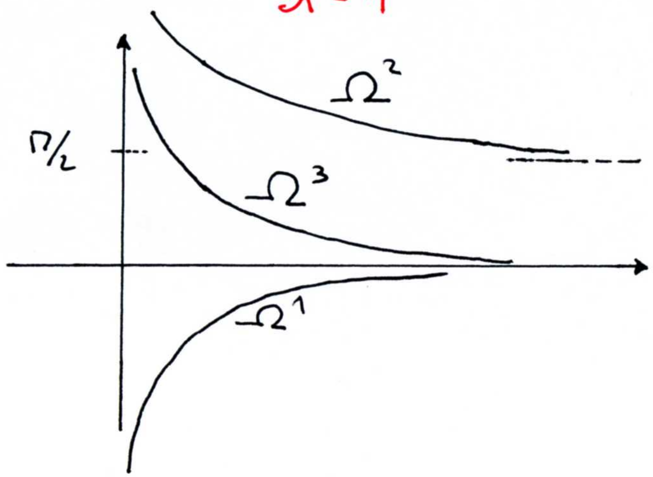
→ the pole is always present : $\omega^\alpha(z)$ are regular, univalued, holomorphic in a region with movable boundary which is a dense set of essential singularities.

→ movable according to the initial conditions i.e. E^α

The non-algebraic nature → no exact dictionary

→ example with pole at $T=0$

$$E^1 = \frac{d\lambda/dz}{\lambda} \qquad E^2 = \frac{d\lambda/dz}{\lambda - 1} \qquad E^3 = \frac{d\lambda/dz}{\lambda(\lambda-1)} \qquad \lambda = \left(\frac{\vartheta^2}{\vartheta^3}\right)^4$$



Halphen's system and 20^{th} century physics

- Gibbons and Pope met these equations in 1979 in the search for self-dual solutions of 4-dim Euclidean gravity of the Bianchi IX class

$$ds^2 = (\gamma_1 \gamma_2 \gamma_3)^2 dT^2 + (\gamma_\alpha J^\alpha)^2$$

and found $SU(2) \times U(1)$ solutions with $\gamma_2 = \gamma_3$

- Atiyah and Hitchin found the same equations in 1985 in the study of scattering of two $SU(2)$ monopoles: the configuration space is endowed with Gibbons-Pope metric. They found a particular elliptic $\to SU(2)$ solution

- The connexion with Halphen's work is more recent and points towards a general conjecture that integrability goes along with self-duality — at least in three dimensions — in Yang-Mills

Time-dependent string solutions

- $$\mathcal{L}_{\text{space-time}} = \frac{1}{2}\left(-\partial t\,\bar{\partial} t + \left(g_{ij}(t) + B_{ij}\right)\partial X^i\,\bar{\partial} X^j + R_{(2)}\,\Phi(t)\right)$$

→ isotropic $\quad g_{ij}(t) = e^{2\sigma(t)}\, g_{ij} \quad$ (g_{ij} : unperturbed Einstein)

→ anisotropic (WZW) $\quad g_{ij}(t) = \sum_{\alpha} \gamma_\alpha(t)\, J^\alpha_i\, J^\beta_j$

- Equations: $\quad \beta(g) = \beta(B) = \beta(\Phi) = 0$

$\beta(B)$ ↓ automatic

$\beta(\Phi)$ ↓ $1 + d - \mathcal{E} + c_I = c_{\text{critical}}$

- due to the curvature R of d-space
- due to the $\Phi(t)$

⚠ keep $\sigma(\alpha')$ sensible ←

the breathing mode

- Equations : $Q(t) \overset{\text{def}}{=} -\dot{\Phi}(t) + \frac{d}{2}\dot{\sigma}(t)$

$$
\begin{cases}
\dot{Q} = -\frac{d}{2}\dot{\sigma}^2 \\
\ddot{\sigma} = \underbrace{-\frac{R}{d} e^{-\sigma}\left(1 - e^{-4\sigma}\right)}_{-V'(\sigma) = \beta(\sigma(\mu))} - 2\dot{\sigma} Q
\end{cases}
$$

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- properties :
  - these are not R-G-flow eqs.
  - particle in a potential with friction driven by the dilaton via $Q(t)$ [Tseytlin '92]
  - for « large enough friction » the kinetic energy is subleading
    - → R-G-flow eqs (Ricci-flow) with $\log \mu = -\frac{2\pi t}{Q}$
    - → transmutation of time to RG scale : role of Liouville in non-critical strings

Around the "unwarped" conformal point

- "unwarped" conformal point: $\sigma = 0$, $\dot{\sigma} = 0$, $Q = \bar{Q}$, $\Phi = \Phi_0 - \bar{Q}t$

  → $\bar{Q}$ is a background charge for t → drives a linear dilaton $\left(c_t = 1 - 3\bar{Q}^2\right)$

  [Antoniadis, Bachas, Ellis, Nanopoulos '88]

- linearization: $Q = \bar{Q}$ and σ satisfies $\ddot{\sigma} = -\frac{4R}{d}\sigma - 2\bar{Q}\dot{\sigma}$

  → $\sigma(t) = C_1 \exp - \left(\bar{Q} + \sqrt{\bar{Q}^2 - \frac{4R}{d}}\right)t + C_2 \exp - \left(\bar{Q} - \sqrt{\bar{Q}^2 - \frac{4R}{d}}\right)t$

  → $\Phi(t) = \Phi_0 - \bar{Q}t + \frac{d}{2} C_1 \exp - \left(\bar{Q} + \sqrt{\bar{Q}^2 - \frac{4R}{d}}\right)t + \frac{d}{2} C_2 \exp - \left(\bar{Q} - \sqrt{\bar{Q}^2 - \frac{4R}{d}}\right)t$

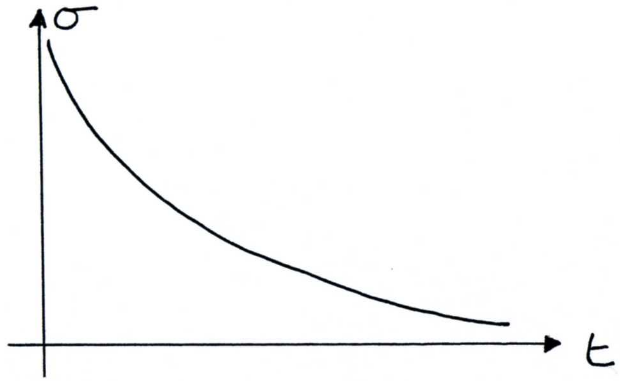
  - always stays within the perturbative regime
  - always converges towards the unwarped conformal point
  - competition between $\bar{Q} \lessgtr 4R/d$

    how much the dilaton drives c_t ↲ ↳ how much c_d is away from d

R-G flow versus conformal regimes

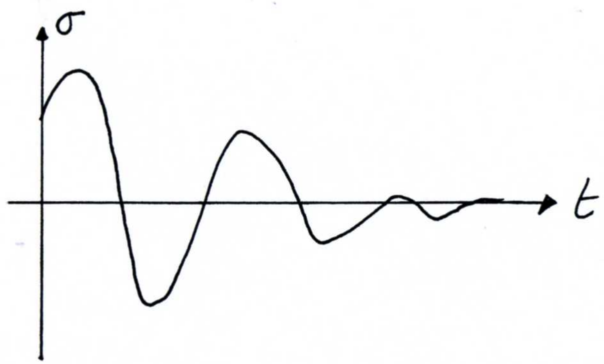
- $\bar{Q}^2 > 4R/d$: friction dominates

  → d-dimensional "R-G-flow regime"



- $\bar{Q}^2 < 4R/d$: d-curvature dominates

  → (d+1)-dimensional "conformal regime"



- The same conclusion holds for the anisotropic deformation : initial anisotropies are washed out by time evolution as they were by R-G-flowing (in the absence of genuine time).

the Einstein frame and cosmological interpretation

- We must reabsorb the dilaton $\Phi(t)$

$$ds^2_{\text{space-time, Einstein}} = -d\tau^2 + a(\tau)\underbrace{g_{ij}\,dx^i\,dx^j}_{d\Omega^2}$$

- $a(\tau)$ can be determined

→ unwarped conformal point [linear dilaton solution, $\bar{Q}, R, d$]

$$a(\tau) = \bar{Q}^2\,\tau^2$$

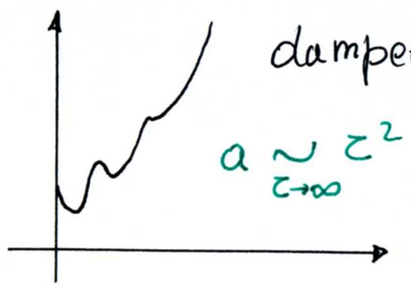
→ linearized general evolution

- $\bar{Q}^2 > 4R/d$ friction dominated

power-like

$$a(\tau) \underset{\tau\to 0}{\sim} \tau^{2\left(1+\frac{16+d}{d+4\bar{Q}}\right)} \qquad a \underset{\tau\to\infty}{\sim} \tau^2$$

- $\bar{Q}^2 < 4R/d$ d-curvature dominated



damped oscillatory

$$a \underset{\tau\to\infty}{\sim} \tau^2$$

Summary

- FRW universes as axion - dilaton string cosmology

  stabilizes the space with constant curvature R ↲ ↳ introduces a time-dependent warping

- arand the simple linear-dilaton solution with background charge $\overline{Q}$

  $\longrightarrow$ $\overline{Q}^2 > 4R/d$: "R-G-flow" regime – Power expansion

  time ≡ worldsheet scale ~ Liouville

  $\longrightarrow$ $\overline{Q}^2 < 4R/d$: "genuine" time-dependence – oscillatory plus power expansion

  RG-like at late times ←

- breathing-mode or anisotropic-perturbation RG flows are Ricci flows with torsion and converge to the unique IR fixed point – isotropic with finite curvature
- For SU(2) the Ricci flow is integrable – Darboux-Halphen

- Beyond α' ?

- Important question : what is time in string theory?

→ Is any time-evolution becoming an RG flow eventually?
And time becoming a 2-D scale with the dilaton being the interplay between target space and worldsheet?

[Polyakov]

→ Does Thurston's geometrization conjecture translate in string theory as a kind of universal behaviour of three-dimensional target space that would converge with time to a collection of homogeneous spaces?