Geometric configurations and E_{10} subalgebras of cosmological inspiration

M. Henneaux, M. Leston, D. Persson, Ph. S.

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Summary: We re-examine previously found cosmological solutions to eleven-dimensional supergravity in the light of the E_{10}-approach to M-theory. We focus on the solutions with non zero electric field determined by geometric configurations (n_m, g_3), $n \leq 10$. We show that these solutions are associated with rank g regular subalgebras of E_{10}, the Dynkin diagrams of which are the (line) incidence diagrams of the geometric configurations. Our analysis provides as a byproduct an interesting class of rank-10 Coxeter subgroups of the Weyl group of E_{10}.

Talk based on :

- J. Demaret, J.-L. Hanquin, M. Henneaux, Ph. S.
  Cosmological models in Eleven-dimensional Supergravity
  Nucl. Phus. B **252**, 538 (1985)
- M. Henneaux, M. Leston, D. Persson, Ph. S.
  Geometric Configurations, Regular Subalgebras of E_{10} *and M-Theory Cosmology*
  JHEP 0610 (2006) 021 (hep-th/0606123)
- M. Henneaux, M. Leston, D. Persson, Ph. S.
  A special Class of Rank 10 and 11 of Coxeter groups
  (hep-th/0610278)

11 - D, Binachi I supergravity solutions

Field configurations

$$
\begin{aligned}
ds^2 &= -\mathrm{N}^2[t]dt^2 + \mathrm{g}_{ij}[t]dx^i dx^j \\
F_{\alpha\beta\gamma\delta} &= \mathrm{F}_{\alpha\beta\gamma\delta}[t]
\end{aligned}
$$

Field equations

- dynamical equations

$$
\begin{aligned}
\frac{d\left(K^a{}_b\sqrt{g}\right)}{dt} &= -\frac{N}{2}\sqrt{g}F^{a\rho\sigma\tau}F_{b\rho\sigma\tau} + \frac{N}{144}\sqrt{g}F^{\lambda\rho\sigma\tau}F_{\lambda\rho\sigma\tau}\delta^a_b \\
\frac{d\left(F^{0abc}N\sqrt{g}\right)}{dt} &= \frac{1}{144}\eta^{0abcd_1d_2d_3e_1e_2e_3e_4}F_{0d_1d_2d_3}F_{e_1e_2e_3e_4} \\
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- Constraint equations

Hamiltonian C. $$K^a{}_b K^b{}_a - K^2 + \frac{1}{12} F_{\perp abc} F_\perp{}^{abc} + \frac{1}{48} F_{abcd} F^{abcd} = 0$$

Momentum C. $$\frac{1}{6} N F^{0bcd} F_{abcd} = 0$$

Gauss law $$\varepsilon^{0abc_1c_2c_3c_4d_1d_2d_3d_4} F_{c_1c_2c_3c_4} F_{d_1d_2d_3d_4} = 0$$

where

$$K_{ab} = (-1/2N)\dot{g}_{ab} \text{ and } F_{\perp abc} = (1/N) F_{0abc} \qquad .$$

Bianchi I configurations

Diagonal field configurations

Diagonal metric implies diagonal extrinsic curvature K_{ab}
Evolution and constraint equations imply diagonal energy-momentum tensor: $F^{a\rho\sigma\tau}F_{b\rho\sigma\tau} \propto \delta^a_b$

• Freund-Rubin ansatz: 10=3+7

$ds^2_{11} = -N^2 dt^2 + ds^2_3 + ds^2_7$

$F^{0abc} \propto \frac{1}{\sqrt{g}N}\varepsilon^{0abc} \qquad (a,b,c = 1,2,3)$

[P.G.O. Freund, M.A. Rubin, Phys. Lett. 97B (1980) 233]

• Different splittings: $10 = n + (10-n)$, $n \geq 0$

$ds^2_{11} = -N^2 dt^2 + R^2[t]\sum_{a\leq n}(dx^a)^2 + S^2[t]\sum_{\overline{a}\geq n}(dx^{\overline{a}})^2$

Only $F^{0abc} \neq 0$

Einstein-Maxwell equations imply:

$$F^{0abc} = \frac{1}{N\sqrt{g}} E^{abc}, \qquad E^{apq} E_{bpq} = f^2 \, \delta^a_b$$

- n=1, 2

No non-trivial three-index tensor

- n=3

$E^{abc} = f \, \varepsilon^{abc}$: solution proportional to the Levi-Civita tensor

- n=4

Let $A^a = \varepsilon^{abcd} E_{bcd}$: $A^a A_b \propto \delta^a_b$ *i.e.* $A^a = 0$

- n=5

Let $B^{ab} = \varepsilon_{abcde} E^{cde}$, $B^{ac} B^{cb} \propto \delta^a_b$

i.e. $B^2 = \mu^2 Id$ in matrix notations,

but B is antisymmetric and the dimension odd: $B = 0$

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In dimensions greater than five Special solutions are obtained by imposing the following conditions:

1. given a pair of indices (a, b), there is at most one c such that $E^{abc} \neq 0$
2. for each index a there are exactly m pairs (b, c) such that $E^{abc} \neq 0$,
3. all non-vanishing E^{abc} are equal up to sign : $E^{abc} = \pm h$

Condition 1 implies $E^{apq} E_{bpq} = 0$ when $a \neq b$; conditions 2 and 3 imply $E^{apq} E_{bpq} = m h^2 \delta^a_b$

The first two conditions can be reformulated in terms of geometric configurations (n_m, g_3) *i.e.* set of n points with g distinguished subsets, called lines, such that

0. Each line contains exactly three points and defines an E^{abc} component
1. Two points determine at most one line (condition 1)
2. Each point belongs to m lines (condition 2)

[S. Kantor, "Die configurationen $(3,3)_{10}$", K. Academie der Wissenschaften, Vienna, Sitzungsbereichte der matematisch naturewissenshaftlichen classe, 84 II, 1291-1314 (1881).
D. Hilbert and S. Cohn-Vossen, "Geometry and the Imagination",(Chelsea, New York, 1952)
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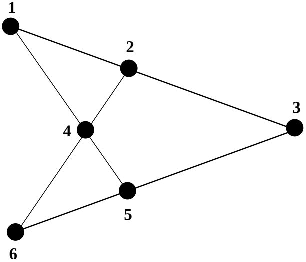

Figure: $(6_2, 4_3)$: The first configuration with intersecting lines.

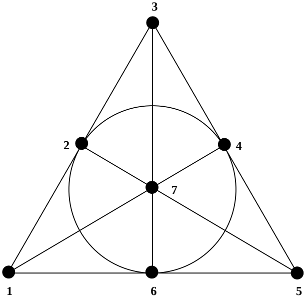

Figure: $(7_3, 7_3)$: The Fano plane; the multiplication table of the octonions.

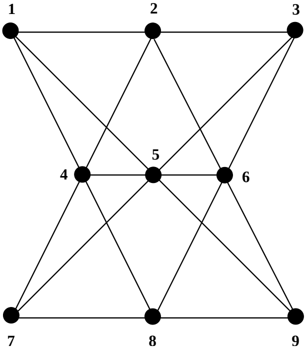

FIGURE: $(9_3, 9_3)_1$: The so-called *Pappus configuration*.

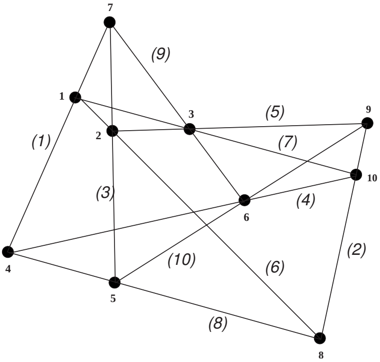

FIGURE: $(10_3, 10_3)_3$: The Desargues configuration, dual to the Petersen graph.

- The Kac-Moody algebra : E_{10}

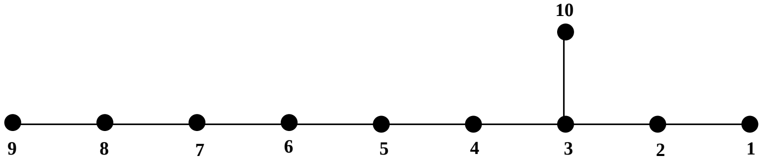

Figure: The Dynkin diagram of E_{10}. Labels $i = 1, \ldots, 9$ enumerate the nodes corresponding to simple roots, α_i, of the A_9 subalgebra and the exceptional node, labeled "10", is associated to the root α_{10} that defines the level decomposition.

$$[h_i, h_j] = 0 \quad , \quad [h_i, e_j] = A_{ij} e_j \quad , \quad [h_i, f_j] = -A_{ij} f_j \quad , \quad [e_i, f_j] = \delta_{ij} h_j$$
$$(ad\, e_i)^{(1-A_{ij})} e_j = 0 \qquad , \qquad (ad\, f_i)^{(1-A_{ij})} f_j = 0 \qquad .$$

[V. Kac, "Infinite dimensional Lie algebras", 3rd Ed., Cambridge University Press (1990).]

- The Kac-Moody "group" : $\mathcal{E}_{10} = Exp[E_{10}]$
- The compact subalgebra : $\mathcal{K}(\mathcal{E}_{10})$

The subalgebra fixed by the Chevalley involution:

$$\tau(h_i) = -h_i \quad , \quad \tau(e_i) = -f_i \quad , \quad \tau(f_i) = -e_i \quad .$$

Hidden symmetries of M-theory

The dynamics of eleven-dimensional supergravity can be formulated as "geodesics" on the coset space": $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$

[B. Julia, "Kac-Moody Symmetry Of Gravitation And Supergravity Theories," LPTENS 82/22
T. Damour and M. Henneaux, "E(10), BE(10) and arithmetical chaos in superstring cosmology," Phys. Rev. Lett. **86**, 4749 (2001) [arXiv:hep-th/0012172].
P. C. West, "E(11) and M theory," Class. Quant. Grav. **18**, 4443 (2001) [arXiv:hep-th/0104081].
T. Damour, M. Henneaux and H. Nicolai, "E(10) and a 'small tension expansion' of M theory," Phys. Rev. Lett. **89**, 221601 (2002) [arXiv:hep-th/0207267].]

Consistent Truncations

Truncations to a sub-model that provides solutions of the full model.

•Level truncation : set equal to zero the momenta conjugate to the σ-model variables above a given level.

[T. Damour, M. Henneaux and H. Nicolai, “Cosmological billiards,” Class. Quant. Grav. **20**, R145 (2003) [arXiv:hep-th/0212256].]

•Subgroup truncation : restrict the equations of motion to a well chosen subgroup (subgroups obtained from the exponentiation of regular subalgebras)

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REGULAR SUBALGEBRAS

- Definition

Let $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+$ be a Kac-Moody subalgebra of $\mathfrak{g}$, with triangular decomposition.
Assume $\bar{\mathfrak{g}}$ canonically embedded in $\mathfrak{g}$, i.e., that the Cartan subalgebra $\bar{\mathfrak{h}}$ of $\bar{\mathfrak{g}}$ is a subalgebra of the Cartan subalgebra of $\mathfrak{g}$:$\bar{\mathfrak{h}} = \bar{\mathfrak{g}} \cap \mathfrak{h}$.
Then $\bar{\mathfrak{g}}$ is a regular subalgebra iff :

1. the step operators of $\bar{\mathfrak{g}}$ are step operators of $\mathfrak{g}$
2. the simple roots of $\bar{\mathfrak{g}}$ are real roots of $\mathfrak{g}$

It follows that the Weyl group of $\bar{\mathfrak{g}}$ is a subgroup of the Weyl group of $\mathfrak{g}$ and that the root lattice of $\bar{\mathfrak{g}}$ is a sublattice of the root lattice of $\mathfrak{g}$.

•Theorem

Let Φ^+_{real} be the set of positive real roots of a Kac-Moody algebra $\mathcal{A}$. Let $\beta_1, \cdots, \beta_n \in \Phi^+_{real}$ be chosen such that none of the differences $\beta_i - \beta_j$ is a root of $\mathcal{A}$. Assume furthermore that the β_i's are such that the matrix $C = [C_{ij}] = [2 \langle \beta_i | \beta_j \rangle / \langle \beta_i | \beta_i \rangle]$ has non-vanishing determinant. For each $1 \leq i \leq n$, choose non-zero root vectors E_i and F_i in the one-dimensional root spaces corresponding to the positive real roots β_i and the negative real roots $-\beta_i$, respectively, and let $H_i = [E_i, F_i]$ be the corresponding element in the Cartan subalgebra of $\mathcal{A}$. Then, the (regular) subalgebra of $\mathcal{A}$ generated by $\{E_i, F_i, H_i\}$, $i = 1, \cdots, n$, is a Kac-Moody algebra with Cartan matrix $[C_{ij}]$.

[A. J. Feingold and H. Nicolai, "Subalgebras of Hyperbolic Kac-Moody Algebras," [arXiv:math.qa/0303179].]

•Comments

- We obtain subalgebras by defining simple roots within the root lattice of the larger algebra. But there are consistency conditions to be satisfied in order that the Chevalley-Serre relations can be fulfilled. For instance for the simple roots β_i and β_j, $\beta_i - \beta_j$ cannot be a root otherwise the relation $[E_i, F_j] = \delta_{ij} H_i$ will be violated.
- When the Cartan matrix is degenerate, the corresponding Kac-Moody algebra has non trivial ideals. Verifying that the Chevalley-Serre relations are fulfilled is not sufficient to guarantee that one gets the Kac-Moody algebra corresponding to the Cartan matrix $[C_{ij}]$ since there might be non trivial quotients.

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- If the matrix $[C_{ij}]$ is decomposable, say $C = D \oplus E$ with D and E indecomposable, then the Kac-Moody algebra $\mathbb{KM}(C)$ generated by C is the direct sum of the Kac-Moody algebra $\mathbb{KM}(D)$ generated by D and the Kac-Moody algebra $\mathbb{KM}(E)$ generated by E. The subalgebras $\mathbb{KM}(D)$ and $\mathbb{KM}(E)$ are ideals. If C has non-vanishing determinant, then both D and E have non-vanishing determinant. Accordingly, $\mathbb{KM}(D)$ and $\mathbb{KM}(E)$ are simple and hence, either occur faithfully or trivially. Because the generators E_i are linearly independent, both $\mathbb{KM}(D)$ and $\mathbb{KM}(E)$ occur faithfully.

[V. Kac, "Infinite dimensional Lie algebras", 3rd Ed., Cambridge University Press (1990).]

The link

• Level zero elements: $\mathfrak{gl}(10,\mathbb{R})$ with commutation relations:

$$[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b. \quad (a,\, b = 1, \ldots, 10)$$

• Level ± 1 generators: E^{abc} at level 1 and their "transposes" $F_{abc} = -\tau(E^{abc})$ at level -1; they transform contravariantly and covariantly with respect to $\mathfrak{gl}(10,\mathbb{R})$:

$$[K^a{}_b, E^{cde}] = 3\delta^{[c}_b E^{de]a}\ ,\ [K^a{}_b, F_{cde}] = -3\delta^a{}_{[c}F_{de]b}.$$

• Diagonal metric : $K^a_b = 0$ if $a \neq b$. *i.e.*no level zero root.
• Electric regular subalgebra : all the simple roots, $\alpha_{i_1 i_2 i_3}$, are at level one ($\alpha_{123} \equiv \alpha_{10}$).
From $[E^{abc}, F_{def}] = 18\delta^{[ab}_{[de}K^{c]}{}_{f]} - 2\delta^{abc}_{def}\sum_{a=1}^{10} K^a{}_a$ we obtain $\alpha_{i_1 i_2 i_3} - \alpha_{i'_1 i'_2 i'_3} \in \Phi_{E_{10}}$ if and only if the sets $\{i_1, i_2, i_3\}$ and $\{i'_1, i'_2, i'_3\}$ have exactly two points in common.

The rules

• One must choose the set of simple roots of the electric regular subalgebra S in such a way that given a pair of indices (i_1, i_2), there is at most one i_3 such that the root α_{ijk} is a simple root of S, with (i, j, k) the re-ordering of (i_1, i_2, i_3) such that $i < j < k$.

• To each of the simple roots $\alpha_{i_1 i_2 i_3}$ of S, one can associate the line (i_1, i_2, i_3) connecting the three points i_1, i_2 and i_3 *i.e.* the set of points and lines associated with the simple roots must fulfill the third rule defining a geometric configuration, namely, that two points determine at most one line.

•The first rule, which states that each line contains 3 points, is a consequence of the fact that the E_{10}-generators at level one are the components of a 3-index antisymmetric tensor.

• The second rule, that each point is on m lines, is less fundamental from the algebraic point of view; it was imposed in order to allow for solutions isotropic in the directions that support the electric field. It implies that each node of the Dynkin diagram has the same number of nodes.

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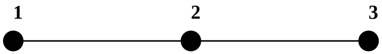

Figure: $(3_1, 1_3)$: The only allowed configuration for $n = 3$.

• Only one generator E^{123}; the diagonal metric components correspond to the Cartan generator $h = [E^{123}, F_{123}]$.
• A_1 regular subalgebra $\{e, f, h\}$ with $e \equiv E^{123}$, $f \equiv F_{123}$ and $h = [e, f] = -\frac{1}{3}\sum_{a\neq 1,2,3} K^a{}_a + \frac{2}{3}(K^1{}_1 + K^2{}_2 + K^3{}_3)$.
• Cartan matrix : (2) not degenerate.

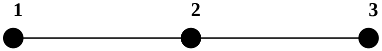

FIGURE: $(3_1, 1_3)$: The only allowed configuration for $n = 3$.

- Only one generator E^{123}; the diagonal metric components correspond to the Cartan generator $h = [E^{123}, F_{123}]$.
- A_1 regular subalgebra $\{e, f, h\}$ with $e \equiv E^{123}$, $f \equiv F_{123}$ and $h = [e, f] = -\frac{1}{3}\sum_{a\neq 1,2,3} K^a{}_a + \frac{2}{3}(K^1{}_1 + K^2{}_2 + K^3{}_3)$.
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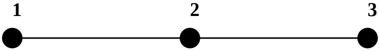

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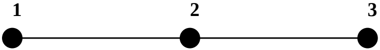

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Conclusion

The appropriate regular electric subalgebra of E_{10} corresponding to the geometric configuration $(3_1, 1_3)$ is $A_1 \oplus \mathbb{R}l$. (An "SM2-brane" solution describing two asymptotic Kasner regimes separated by a collision against an electric wall).

[A. Kleinschmidt and H. Nicolai, "E(10) cosmology," JHEP **0601**, 137 (2006) [arXiv:hep-th/0511290].]

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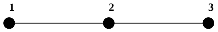
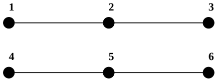
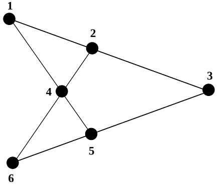
Configuration		Dynkin diagram	Comments
$(3_1, 1_3)$			$A_1 \oplus \mathbb{R}\,\ell$ $\ell = \sum_{i=4}^{10} K^i_i$
$(6_1, 2_3)$			$A_2 \oplus \mathbb{R}\,\ell$ $\ell = \sum_{i=7}^{10} K^i_i$ level 2 : mag.fields
$(6_2, 4_3)$			$A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus \mathbb{R}\,\ell$ $\ell = \sum_{i=7}^{10} K^i_i$ Four SM2-branes

TABLE: All configurations for $n \leq 6$ and their dual finite dimensional Lie algebras.

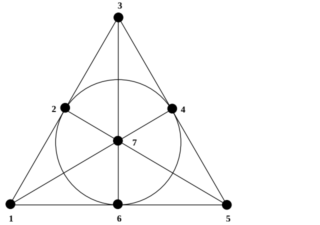
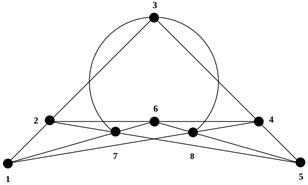
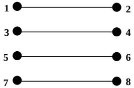
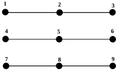
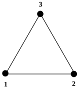
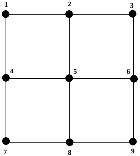
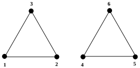
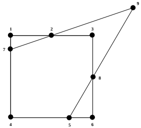
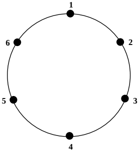
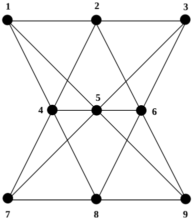
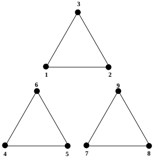
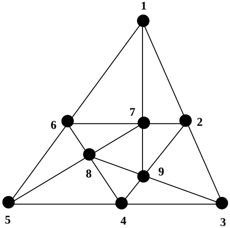
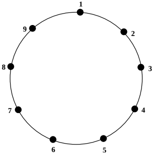
Configuration		Dynkin diagram	Comments
$(7_3, 7_3)$			$\mathfrak{g}_{(7_3,7_3)} = A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \subset A_1 \oplus A_1 \oplus A_1 \oplus D_4 \subset A_1 \oplus D_6 \subset E_7$ $\ell = \sum_{i=8}^{10} K_i^i$
$(8_3, 8_3)$			$\mathfrak{g}_{(8_3,8_3)} = A_2 \oplus A_2 \oplus A_2 \oplus A_2 \subset A_2 \oplus E_6 \subset E_8$ $\ell = K_9^9 + K_{10}^{10}$

TABLE: All configurations for $n = 7,\ 8$ and their dual finite dimensional Lie algebras.

Infinite affine subalgebras — $n = 9$

Configuration		Dynkin diagram	Lie algebra
$(9_1, 3_3)$			$\mathfrak{g}_{(9_1,3_3)} = A_2^J$ $c = -K_{10}^{10}$
$(9_2, 6_3)_1$			$\mathfrak{g}_{(9_2,6_3)_1} =$ $(A_2 \oplus A_2)^J$ $c_1 = c_2$
$(9_2, 6_3)_2$			$\mathfrak{g}_{(9_2,6_3)_2} = A_5^J$

Configuration		Dynkin diagram	Lie algebra
$(9_3, 9_3)_1$			$\mathfrak{g}_{(9_3,9_3)_1} = (A_2 \oplus A_2 \oplus A_2)^J$
$(9_3, 9_3)_2$			$\mathfrak{g}_{(9_3,9_3)_2} = A_8^J$

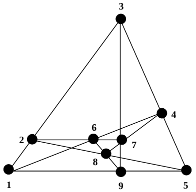
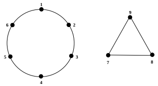
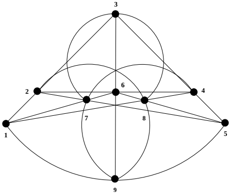
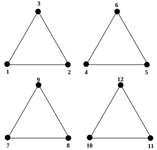
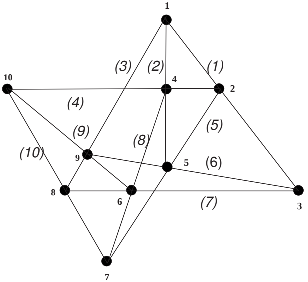
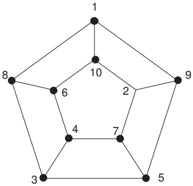
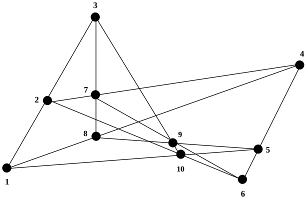
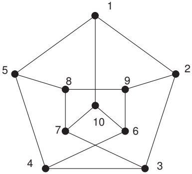
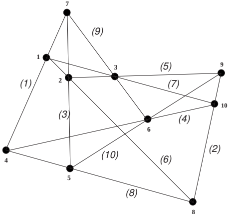
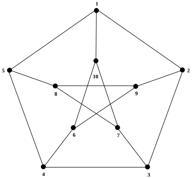
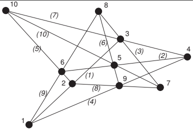
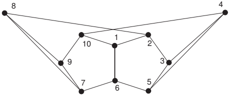
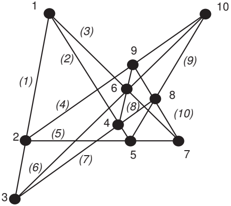
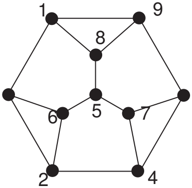
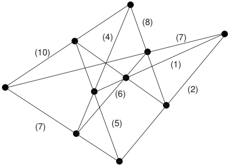
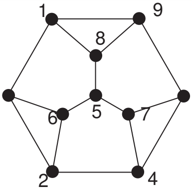
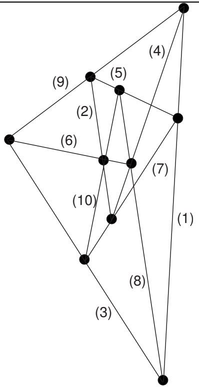
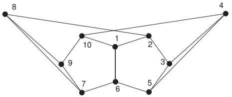
Configuration		Dynkin diagram	Lie algebra
$(9_3, 9_3)_3$			$\mathfrak{g}_{(9_3,9_3)_3} = (A_5 \oplus A_2)^J$
$(9_4, 12_3)$			$\mathfrak{g}_{(9_4,12_3)} = (A_2 \oplus A_2 \oplus A_2 \oplus A_2)^J$

TABLE: $n = 9$ configurations and their dual affine Kac-Moody algebras.

Lorentzian Kac-Moody algebras

Configuration $n = 10$		Dynkin diagram	Det. of A
$(10_3, 10_3)_1$			-121
$(10_3, 10_3)_2$			-256

<table>
<tr><td>$(10_3, 10_3)_3$</td><td>
</td><td>
</td><td>-256</td></tr>
<tr><td>$(10_3, 10_3)_4$</td><td>
</td><td>
</td><td>$= 0$</td></tr>
<tr><td>$(10_3, 10_3)_5$</td><td>
</td><td>
</td><td>$= -16$</td></tr>
</table>

$(10_3, 10_3)_6$			$= -16$
$(10_3, 10_3)_7$			$= 0$

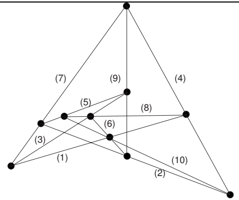
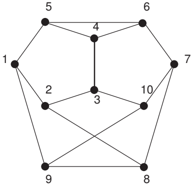
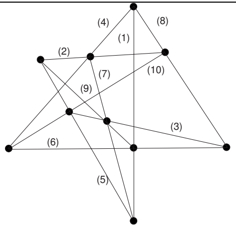
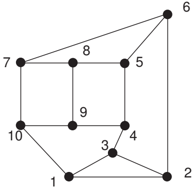
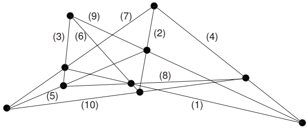
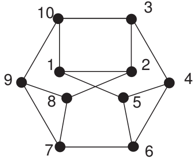
$(10_3, 10_3)_8$			$= -64$
$(10_3, 10_3)_9$			$= -49$
$(10_3, 10_3)_{10}$			-25

TABLE: $n = 10$ configurations and their dual Lorentzian Kac-Moody algebras. Note that some of the configurations give rise to equivalent Dynkin diagrams. Here, we have ceased to number the points of the geometrical configurations as this information is not needed in order to draw the Dynkin diagram.

Conclusions

• Each geometric configuration (n_m, g_3) appears as the Dynkin diagram of an associated regular subalgebra of E_n
• Possible explicit new solutions are available
• Magnetic solutions also
• Relaxing of some rule, we still have supergravity solutions :

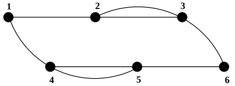


FIGURE: This set of six points, four lines containing three points each, with two lines through each point, is not a geometric configuration because it violates Rule **3**: two points may determine more than one line.

• Seven rank-10 Coxeter subgroups of the Weyl group of E_{10} have been obtained.Configurations with $n > 10$: it exists 31 $(11_3, 11_3)$ configurations from which we obtain 28 Coxeter subgroups of the Weyl group of E_{11} (among the 252 rank 11-$\mathcal{I} = 4$ Coxeter groups). They provide several interesting mathematical questions.

Le mot de la fin

”...there was a time when the study of configurations was considered the most important branch of all geometry.”

- David Hilbert